Differential Equation Invariance Axiomatization

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Outline

1. Differential Dynamic Logic
   - Syntax
   - Axiomatization
   - Relative Completeness / ODE

2. Proofs for Differential Equations
   - Differential Invariants / Cuts / Ghosts

3. Completeness for Differential Equation Invariants
   - Darboux are Differential Ghosts
   - Derived Differential Radical Invariants
   - Real Induction
   - Derived Local Progress
   - Completeness for Invariants
   - Completeness for Noetherian Functions

4. Summary
Challenge (Hybrid Systems)

Fixed law describing state evolution with both

- **Discrete dynamics** (control decisions)
- **Continuous dynamics** (differential equations)
Hybrid Systems Analysis

Concept (Differential Dynamic Logic) (JAR’08, LICS’12)

\[ x \neq m \wedge b > 0 \rightarrow \left[ \left( (\text{init}(\text{SB}(x, m)) \ a := -b) ; \ x' = v, v' = a \right)^* \right] x \neq m \]

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Differential Equation Invariance Axiomatization

JACM’20 3 / 26
Contributions: Differential Equation Axiomatization

- Classical approach: ① Given ODE ② Solve ODE ③ Analyze solution
- Descriptive power of ODEs: ODE much easier than its solution
- Analyzing ODEs via their solutions undoes their descriptive power!
- Poincaré 1881

Now: Logical foundations of differential equation invariants

1. Identify axioms for differential equations
2. Completeness for differential equation invariants
3. Uniformly substitutable axioms, not infinite axiom schemata
4. Decide invariance by proof
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4 Summary
Hybrid Systems = Differential Equations + Discrete

Concept (Differential Dynamic Logic) (JAR’08, LICS’12)

\[ u^2 \leq v^2 + \frac{9}{2} \rightarrow u' = -v + \frac{u}{4} (1 - u^2 - v^2), \quad v' = u + \frac{v}{4} (1 - u^2 - v^2) ] \]

\[ u^2 + v^2 = 1 \rightarrow u' = -v + \frac{u}{4} (1 - u^2 - v^2), \quad v' = u + \frac{v}{4} (1 - u^2 - v^2) ] \]
**Definition (Hybrid program \( \alpha \))**

\[
\alpha, \beta ::= \; x := f(x) \; | \; ?Q \; | \; x' = f(x) & Q \; | \; \alpha \cup \beta \; | \; \alpha ; \beta \; | \; \alpha^*
\]

**Definition (dL Formula \( P \))**

\[
P, Q ::= e \geq \bar{e} \; | \; \neg P \; | \; P \land Q \; | \; \forall x \; P \; | \; \exists x \; P \; | \; [\alpha]P \; | \; \langle \alpha \rangle P
\]

---

*JAR’08, LICS’12, JAR’17*
**Differential Dynamic Logic dL: Syntax**

**Definition (Hybrid program \( \alpha \))**

\[
\alpha, \beta ::= x := f(x) \mid ?Q \mid x' = f(x) \land Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^*
\]

**Definition (dL Formula \( P \))**

\[
P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \land Q \mid \forall x P \mid \exists x P \mid [\alpha]P \mid \langle \alpha \rangle P
\]

Keywords:
- Discrete Assign
- Test Condition
- Differential Equation
- Nondet. Choice
- Seq. Compose
- Nondet. Repeat
- All Reals
- Some Reals
- All Runs
- Some Runs

JAR’08, LICS’12, JAR’17
Differential Dynamic Logic: Axiomatization

\[ \begin{align*}
[\ := ] & \quad [x := e]p(x) \leftrightarrow p(e) \\
[\ ? ] & \quad [\ ? Q]P \leftrightarrow (Q \rightarrow P) \\
[\ ' ] & \quad [x' = f(x)]p(x) \leftrightarrow \forall t \geq 0 [x := y(t)]p(x) \quad (y'(t) = f(y)) \\
[\ ∪ ] & \quad [\ α ∪ β]P \leftrightarrow [\ α]P \land [\ β]P \\
[\ ; ] & \quad [\ α; β]P \leftrightarrow [\ α][\ β]P \\
[\ * ] & \quad [\ α^*]P \leftrightarrow P \land [\ α][\ α^*]P \\
K & \quad [\ α](P \rightarrow Q) \rightarrow ([\ α]P \rightarrow [\ α]Q) \\
I & \quad [\ α^*](P \rightarrow [\ α]P) \rightarrow (P \rightarrow [\ α^*]P) \\
C & \quad [\ α^*]\forall v > 0 (P(v) \rightarrow \langle α \rangle P(v-1)) \rightarrow \forall v (P(v) \rightarrow \langle α^* \rangle \exists v \leq 0 P(v))
\end{align*} \]

equations of truth

LICS’12, JAR’17
Theorem (Sound & Complete) (JAR’08, LICS’12, JAR’17)

dL calculus is a sound & complete axiomatization of hybrid systems relative to either differential equations or relative to discrete dynamics.

Corollary (Complete Proof-theoretical Bridge)

proving continuous = proving hybrid = proving discrete
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4. Summary
Differential Invariants for Differential Equations

Differential Invariant

Differential Cut

Differential Ghost

$\frac{dx}{dt} = f(x)$ & $Q$

$\frac{dx}{dt} = f(x)$ & $Q$

$\frac{dx}{dt} = f(x)$ & $Q$

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Differential Equation Invariance Axiomatization

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Differential Invariants for Differential Equations

Differential Invariant

Differential Cut

Differential Ghost

\[ x' = f(x) & Q \]

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Differential Equation Invariance Axiomatization
Differential Invariants for Differential Equations

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Differential Invariants for Differential Equations

Differential Invariant

Differential Cut

Differential Ghost

\[ x' = f(x) \]
Differential Invariants for Differential Equations

Differential Invariant

Differential Cut

Differential Ghost

\[ x' = f(x) \]

\[ y' = g(x, y) \]

\[ x', w = f(x) \]

\[ Q \]

\[ P \]

\[ C \]

\[ w \]

\[ u \]

\[ 0 \]

\[ x \]

\[ y \]

\[ 0 \]

\[ t \]
### Differential Invariants for Differential Equations

#### Differential Invariant

- $x' = f(x) & Q$
- $x = x(t)$

#### Differential Cut

- $x' = f(x) & Q$
- $x = x(t)$

#### Differential Ghost

- $x' = f(x) & Q$
- $x = x(t)$

---

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**Differential Equation Invariance Axiomatization**  
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Differential Invariants for Differential Equations

Differential Invariant

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\[ x' = f(x) \]

\[ x' = f(x) \]

\[ x' = f(x) \]

\[ x(t) \]

\[ t \]

\[ 0 \]

\[ u \]

\[ r \]

\[ w \]

\[ Q \]

\[ P \]

\[ C \]
Differential Invariants for Differential Equations

<table>
<thead>
<tr>
<th>Differential Invariant</th>
<th>Differential Cut</th>
<th>Differential Ghost</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Differential Invariant Diagram" /></td>
<td><img src="image2" alt="Differential Cut Diagram" /></td>
<td><img src="image3" alt="Differential Ghost Diagram" /></td>
</tr>
</tbody>
</table>

\[
x' = f(x) & Q
\]

\[
y' = g(x, y)
\]

\[
x' = f(x)
\]

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Differential Invariants for Differential Equations

Differential Invariant

Differential Cut

Differential Ghost

\[ x' = f(x) & Q \]

Differential Equation Invariance Axiomatization
Differential Invariants for Differential Equations

Differential Invariant

\[ Q \rightarrow [x' := f(x)](P)' \]
\[ P \rightarrow [x' = f(x) & Q]P \]

Differential Cut

\[ P \rightarrow [x' = f(x) & Q]C \]
\[ P \rightarrow [x' = f(x) & Q \land C]P \]
\[ P \rightarrow [x' = f(x) & Q]P \]

Differential Ghost

\[ P \leftrightarrow \exists y G \]
\[ G \rightarrow [x' = f(x), y' = g(x, y) & Q]G \]
\[ P \rightarrow [x' = f(x) & Q]P \]

deductive power added \( DI \prec DI+DC \prec DI+DC+DG \)

\[ [(e)']_v = \sum_x v(x') \frac{\partial [e]}{\partial x}(v) \]

Springer’10, LMCS’12, LICS’12, JAR’17, LICS’18, JACM’20
Differential Invariant

\[ Q \rightarrow [x' := f(x)](P)' \]

\[ P \rightarrow [x' = f(x) \& Q]P \]

Differential Cut

\[ P \rightarrow [x' = f(x) \& Q]C \quad P \rightarrow [x' = f(x) \& Q \land C]P \]

\[ P \rightarrow [x' = f(x) \& Q]P \]

Differential Ghost

\[ P \leftrightarrow \exists y \; G \quad G \rightarrow [x' = f(x), y' = g(x, y) \& Q]G \]

\[ P \rightarrow [x' = f(x) \& Q]P \]

if \( g(x, y) = a(x)y + b(x) \), so has long solution!
\[ \omega^2 x^2 + y^2 \leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d \omega y & \omega \geq 0 \land d \geq 0] \omega^2 x^2 + y^2 \leq c^2 \]
\[ \omega^2 x^2 + y^2 \leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y \land \omega \geq 0 \land d \geq 0] \omega^2 x^2 + y^2 \leq c^2 \]
\[ \omega \geq 0 \land d \geq 0 \rightarrow [x' := y][y' := -\omega^2 x - 2d \omega y] 2\omega^2 x' + 2y y' \leq 0 \]

\[ \omega^2 x^2 + y^2 \leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d \omega y \& \omega \geq 0 \land d \geq 0] \omega^2 x^2 + y^2 \leq c^2 \]
Differential Invariants for Differential Equations

\[ \omega \geq 0 \land d \geq 0 \rightarrow 2 \omega^2 xy + 2y(-\omega^2 x - 2d \omega y) \leq 0 \]

\[ \omega \geq 0 \land d \geq 0 \rightarrow [x' := y] [y' := -\omega^2 x - 2d \omega y] 2\omega^2 x' + 2y y' \leq 0 \]

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damped oscillator
\[
\omega \geq 0 \land d \geq 0 \rightarrow 2 \omega^2 xy + 2y(-\omega^2 x - 2d \omega y) \leq 0
\]

\[
\omega \geq 0 \land d \geq 0 \rightarrow [x' := y, y' := -\omega^2 x - 2d \omega y] 2\omega^2 xx' + 2yy' \leq 0
\]

\[
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\[ \omega^2 x^2 + y^2 \leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y \& \omega \geq 0 \land d \geq 0] \omega^2 x^2 + y^2 \leq c^2 \]
\( \omega^2 x^2 + y^2 \leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d \omega y, d' = 7 & \omega \geq 0] \omega^2 x^2 + y^2 \leq c^2 \)
\[ \omega^2 x^2 + y^2 \leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d \omega y, \ d' = 7 \ & \omega \geq 0] \] \[ \omega^2 x^2 + y^2 \leq c^2 \]
\( \omega^2 x^2 + y^2 \leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d \omega y, d' = 7 & \omega \geq 0 \land d \geq 0] \omega^2 x^2 + y^2 \leq c^2 \)

increasingly damped oscillator
\[
\frac{\omega^2 x^2 + y^2 \leq c^2}{\omega^2 x^2 + y^2 \leq c^2} \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y, d' = 7 \& \omega \geq 0 \land d \geq 0] \frac{\omega^2 x^2 + y^2 \leq c^2}{d \geq 0} \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y, d' = 7 \& \omega \geq 0] d \geq 0
\]

increasingly damped oscillator
\[
\begin{align*}
\omega^2 x^2 + y^2 &\leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y, \quad d' = 7 \land \omega \geq 0 \land d \geq 0] \quad \omega^2 x^2 + y^2 \leq c^2 \\
\omega^2 x^2 + y^2 &\leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y, \quad d' = 7 \land \omega \geq 0] \quad \omega^2 x^2 + y^2 \leq c^2 \\
\omega \geq 0 &\rightarrow [d' := 7] \quad d' \geq 0 \\
d \geq 0 &\rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y, \quad d' = 7 \land \omega \geq 0] \quad d' \geq 0
\end{align*}
\]

increasingly damped oscillator
\( \omega^2 x^2 + y^2 \leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d \omega y, d' = 7 \& \omega \geq 0 \& d \geq 0] \omega^2 x^2 + y^2 \leq c^2 \)

\( \omega^2 x^2 + y^2 \leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d \omega y, d' = 7 \& \omega \geq 0] \omega^2 x^2 + y^2 \leq c^2 \)

\[
\begin{align*}
\omega \geq 0 & \rightarrow 7 \geq 0 \\
\omega \geq 0 & \rightarrow [d' := 7] d' \geq 0 \\
d \geq 0 & \rightarrow [x' = y, y' = -\omega^2 x - 2d \omega y, d' = 7 \& \omega \geq 0] d' \geq 0
\end{align*}
\]

increasingly damped oscillator
Differential Equation Invariance Axiomatization

\[
\begin{align*}
\omega^2 x^2 + y^2 \leq c^2 &\implies [x' = y, y' = -\omega^2 x - 2d\omega y, d' = 7 \land \omega \geq 0 \land d \geq 0] \omega^2 x^2 + y^2 \leq c^2 \\
\omega^2 x^2 + y^2 \leq c^2 &\implies [x' = y, y' = -\omega^2 x - 2d\omega y, d' = 7 \land \omega \geq 0] \omega^2 x^2 + y^2 \leq c^2
\end{align*}
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\[
\begin{align*}
\omega \geq 0 \rightarrow 7 \geq 0 \\
\omega \geq 0 \rightarrow [d' = 7] d' \geq 0 \\
d \geq 0 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y, d' = 7 \land \omega \geq 0] d' \geq 0
\end{align*}
\]

increasingly damped oscillator
\[
\omega \geq 0 \land d \geq 0 \rightarrow [x' := y][y' := -\omega^2 x - 2d \omega y] 2\omega^2 x x' + 2y y' \leq 0
\]

\[
\omega^2 x^2 + y^2 \leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d \omega y, d' := 7 \land \omega \geq 0 \land d \geq 0] \omega^2 x^2 + y^2 \leq c^2
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\]

* 

\[
\omega \geq 0 \rightarrow 7 \geq 0
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\omega \geq 0 \rightarrow [d' := 7] d' \geq 0
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d \geq 0 \rightarrow [x' = y, y' = -\omega^2 x - 2d \omega y, d' := 7 \land \omega \geq 0] d' \geq 0
\]

increasingly damped oscillator
Differential CUTS for Differential Equations

\[ \omega \geq 0 \land d \geq 0 \rightarrow 2 \omega^2 xy + 2y(-\omega^2 x - 2d \omega y) \leq 0 \]

\[ \omega \geq 0 \land d \geq 0 \rightarrow [x' := y][y' := -\omega^2 x - 2d \omega y] 2 \omega^2 xx' + 2yy' \leq 0 \]

\[ \omega^2 x^2 + y^2 \leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d \omega y, d' = 7 \land \omega \geq 0 \land d \geq 0] \omega^2 x^2 + y^2 \leq c^2 \]

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\[ \omega \geq 0 \rightarrow 7 \geq 0 \]

\[ \omega \geq 0 \rightarrow [d' := 7] d' \geq 0 \]

\[ d \geq 0 \rightarrow [x' = y, y' = -\omega^2 x - 2d \omega y, d' = 7 \land \omega \geq 0] d' \geq 0 \]

increasingly damped oscillator
\begin{align*}
\omega \geq 0 \land d \geq 0 &\quad \rightarrow 2\omega^2 xy + 2y(-\omega^2 x - 2d \omega y) \leq 0 \\
\omega \geq 0 \land d \geq 0 &\quad \rightarrow [x' := y][y' := -\omega^2 x - 2d \omega y] 2\omega^2 xx' + 2yy' \leq 0 \\
\omega^2 x^2 + y^2 \leq c^2 &\quad \rightarrow [x' = y, y' = -\omega^2 x - 2d \omega y, d' = 7 \& \omega \geq 0 \land d \geq 0] \omega^2 x^2 + y^2 \leq c^2
\end{align*}

\begin{align*}
\omega \geq 0 &\quad \rightarrow 7 \geq 0 \\
\omega \geq 0 &\quad \rightarrow [d' := 7] d' \geq 0 \\
d \geq 0 &\quad \rightarrow [x' = y, y' = -\omega^2 x - 2d \omega y, d' = 7 \& \omega \geq 0] d' \geq 0
\end{align*}

increasingly damped oscillator
Could repeatedly diffcut in formulas to help the proof
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4. Summary
Theorem (Algebraic Completeness) (LICS’18, JACM’20)

\[ \text{dL calculus is a sound & complete axiomatization of algebraic invariants of polynomial differential equations. They are decidable by DI, DC, DG in dL.} \]

Theorem (Semialgebraic Completeness) (LICS’18, JACM’20)

\[ \text{dL calculus with RI is a sound & complete axiomatization of semialgebraic invariants of differential equations. They are decidable in dL.} \]
Darboux equalities are DG

\[ Q \rightarrow e' = ge \]
\[ e = 0 \rightarrow [x' = f(x) \& Q] e = 0 \]

(g ∈ \(\mathbb{R}[x]\))

Gaston Darboux 1878

Definable \(e'\) for Lie-derivative w.r.t. ODE
Darboux equalities are DG

\[ Q \rightarrow e' = ge \]
\[ e = 0 \rightarrow [x' = f(x) \& Q] e = 0 \]

(Gaston Darboux 1878)

\[ 2uu' + 2vv' = 2(u^2 + v^2)(u^2 + v^2 - 1) \]
\[ \Rightarrow [u' = -v - u + u^3 + uv^2] \]
\[ \Rightarrow v' = u - v + u^2 v + v^3 \]
\[ u^2 + v^2 - 1 = 0 \]
ODE Axiomatization: Derived Darboux Rules

Darboux equalities are DG

\[
Q \rightarrow e' = ge \\
\Rightarrow e = 0 \rightarrow [x' = f(x) \& Q]e = 0
\]

\[(g \in \mathbb{R}[x])\]

Proof Idea.

1. DG counterweight \(y' = -gy\) to reduce \(e = 0\) to \(ey = 0 \land y \neq 0\).
2. DG counter-counterweight \(z' = gz\) to reduce \(y \neq 0\) to \(yz = 1\).
3. \(ey = 0\) and \(yz = 1\) are now differential invariants by construction.

Derive

\[ [x' = f(x) \& Q](e)' = ge \rightarrow (e = 0 \rightarrow [x' = f(x) \& Q]e = 0) \]
Darboux inequalities are DG

\[
Q \rightarrow e' \geq ge \\
\text{if } e \geq 0 \rightarrow [x' = f(x) \& Q] e \geq 0
\]

(g \in \mathbb{R}[x])

Thomas Grönwall 1919
ODE Axiomatization: Derived Darboux Rules

Darboux inequalities are DG

\[
\begin{align*}
Q \Rightarrow e' & \geq ge \\
\frac{e \geq 0 \Rightarrow [x' = f(x) \& Q]}{e \geq 0 \Rightarrow [x' = f(x) \& Q]e \geq 0}
\end{align*}
\]

\((g \in \mathbb{R}[x])\)

Proof Idea.

1. DG counterweight \(y' = -gy\) to reduce \(e \geq 0\) to \(ey \geq 0 \land y > 0\).
2. DG counter-counterweight \(z' = \frac{g}{2}z\) to reduce \(y > 0\) to \(yz^2 = 1\).
3. \(yz^2 = 1\) and (after DC with \(y > 0\)) \(ey \geq 0\) are differential invariants by construction as \((ey)' = e' y - gye \geq 0\) from premise since \(y > 0\).

Derive

\([x' = f(x) \& Q](e)' \geq ge \Rightarrow (e \geq 0 \Rightarrow [x' = f(x) \& Q]e \geq 0)\)
Darboux inequalities are DG

\[
Q \rightarrow e' \geq ge
\]

\[
e \geq 0 \rightarrow [x' = f(x) \& Q] e \geq 0
\]

\[
(g \in \mathbb{R}[x])
\]

\[
(1-u^2-v^2)' \geq -\frac{1}{2}(u^2+v^2)(1-u^2-v^2)
\]

\[
\rightarrow \left\{ u' = -v + \frac{u}{4}(1-u^2-v^2) \\
 v' = u + \frac{v}{4}(1-u^2-v^2) \right\} \quad 1-u^2-v^2 > 0
\]
Darboux inequalities are DG

\[ Q \rightarrow e' \geq ge \]
\[ e \succcurlyeq 0 \rightarrow [x' = f(x) \& Q] e \succcurlyeq 0 \]

\( (g \in \mathbb{R}[x]) \)

\[ y' = -gy \]
\[ e'y > 0 \]
\[ e' \geq ge \]

\[ (1 - u^2 - v^2)' \geq -\frac{1}{2}(u^2 + v^2)(1 - u^2 - v^2) \]
\[ \ldots \rightarrow \begin{bmatrix} u' = -v + \frac{u}{4}(1 - u^2 - v^2) \\ v' = u + \frac{v}{4}(1 - u^2 - v^2) \\ y' = \frac{1}{2}(u^2 + v^2)y \end{bmatrix} \]
\[ 1 - u^2 - v^2 > 0 \]

\[ (1 - u^2 - v^2)y > 0 \]
Darboux inequalities are DG

\[ Q \to e' \geq ge \]

\[ e \succcurlyeq 0 \to [x' = f(x) \& Q] e \succcurlyeq 0 \]

\[ y' = -gy \]

\[ yz^2 = 1 \]

\[ z' = g/2 \cdot z \]

\[ (1-u^2-v^2)' \geq -\frac{1}{2}(u^2+v^2)(1-u^2-v^2) \]

\[ \ldots \to [u' = -v + \frac{u}{4}(1-u^2-v^2), v' = u + \frac{v}{4}(1-u^2-v^2), y' = \frac{1}{2}(u^2+v^2)y, z' = -\frac{1}{4}(u^2+v^2)z \]

\[ 1-u^2-v^2 > 0 \]

\[ (1-u^2-v^2)y > 0 \]

\[ yz^2 = 1 \]
Darboux Inequalities are Differential Ghosts: Details

\[ Q \rightarrow (-gy)z^2 + y(2z(g/2z)) = 0 \]
\[ yz^2 = 1 \rightarrow [x' = f(x), y' = -gy, z' = g/2z & Q] \]
\[ y > 0 \rightarrow \exists z [x' = f(x), y' = -gy, z' = g/2z & Q] \]
\[ y > 0 \rightarrow [x' = f(x), y' = -gy & Q] \]

\[ Q \rightarrow e' \geq ge \]
\[ R \rightarrow e' \geq ge, y > 0 \rightarrow e' y - gy e \geq 0 \]

\[ e \not\geq 0, y > 0 \rightarrow [x' = f(x), y' = -gy & Q] \]
\[ e : y \geq 0 \rightarrow [x' = f(x), y' = -gy & Q] \]
\[ e \geq 0, y > 0 \rightarrow [x' = f(x), y' = -gy & Q] \]
\[ e \geq 0 \rightarrow \exists y [x' = f(x), y' = -gy & Q] \]
\[ e \geq 0 \rightarrow [x' = f(x) & Q] \]

P.S. \( z' = g/2z \) superfluous for open inequalities \( e > 0 \) and \( e \neq 0 \).
Vectorial Darboux are DG

\[ Q \rightarrow \mathbf{e}' = Ge \]

\[ e = 0 \rightarrow [x' = f(x) \& Q] e = 0 \]

\( G \in \mathbb{R}[x]^{n \times n} \)

Definable \( \mathbf{e}' \) for component-wise Lie-derivative w.r.t. ODE
ODE Axiomatization: Derived Darboux Rules

Vectorial Darboux are DG

\[
Q \rightarrow e' = Ge \\
\rightarrow e = 0 \rightarrow [x' = f(x) & Q]e = 0
\]

\( (G \in \mathbb{R}[x]^{n \times n}) \)

Proof Idea.

1. DG counterweight \( y' = -G^T y \) to change \( e = 0 \) to \( e \cdot y = 0 \).
2. But: \( e \cdot y = 0 \nRightarrow e = 0 \) even if \( y \neq 0 \).
3. Redo: time-varying independent DG matrix \( Y' = -YG \) with \( Ye = 0 \).
4. \( Ye = 0 \nRightarrow e = 0 \) if \( \det Y \neq 0 \).
5. DC \( \det Y \neq 0 \) proves by dbx with Liouville: \( \det(Y)' = -\text{tr}(G)\det(Y) \)
6. Continuous change of basis \( Y^{-1} \) balancing out motion of \( e \): constant!
7. Continuous change to new evolving variables is sound by DG.

Derive

\[
[x' = f(x) & Q](e)' = Ge \rightarrow (e = 0 \rightarrow [x' = f(x) & Q]e = 0)
\]
Time is defined so that motion looks simple ≈ Poincaré
Vectorial Darboux are DG

\[
Q \rightarrow e' = Ge \\
\Rightarrow e = 0 \rightarrow [x' = f(x) \& Q]e = 0
\]

\[
(G \in \mathbb{R}[x]^{n \times n})
\]

Proof Idea.

\[
\begin{align*}
& R: (e)' = Ge \rightarrow -2e \cdot (e)' \geq g(-\|e\|^2) \\
& (\forall)': (e)' = Ge \rightarrow (\neg \|e\|^2)' \geq g(-\|e\|^2) \\
& M: [x' = f(x) \& Q](e)' = Ge \rightarrow [x' = f(x) \& Q](-\|e\|^2)' \geq g(-\|e\|^2) \\
& DBX: [x' = f(x) \& Q](e)' = Ge, -\|e\|^2 \geq 0 \rightarrow [x' = f(x) \& Q](-\|e\|^2) \geq 0 \\
& M: [x' = f(x) \& Q](e)' = Ge, e = 0 \rightarrow [x' = f(x) \& Q]e = 0
\end{align*}
\]

where \( g \overset{\text{def}}{=} 1 + \sum_{i=1}^{n} \sum_{j=1}^{n} G_{ij}^2 \)

1 + squared Frobenius

Derive

\[
[x' = f(x) \& Q](e)' = Ge \rightarrow (e = 0 \rightarrow [x' = f(x) \& Q]e = 0)
\]
ODE Axiomatization: Derived Invariant Rules

Vectorial Darboux are DG

\[ Q \rightarrow \mathbf{e}' = G\mathbf{e} \]
\[ \mathbf{e} = 0 \rightarrow [x' = f(x) \& Q] \mathbf{e} = 0 \]
ODE Axiomatization: Derived Invariant Rules

Vectorial Darboux are DG

\[ Q \rightarrow e' = Ge \]
\[ e = 0 \rightarrow [x' = f(x) \& Q]e = 0 \]

Differential Radical Invariants are DG

\[ \Gamma, Q \rightarrow \bigwedge_{i=0}^{N-1} e^{(i)} = 0 \quad Q \rightarrow e^{(N)} = \sum_{i=0}^{N-1} g_i e^{(i)} \]
\[ \Gamma \rightarrow [x' = f(x) \& Q]e = 0 \]
Vectorial Darboux are DG

\[ Q \rightarrow e' = Ge \]

\[ e = 0 \rightarrow [x' = f(x) \& Q]e = 0 \]

Differential Radical Invariants are DG

\[ \Gamma, Q \rightarrow \bigwedge_{i=0}^{N-1} e^{(i)} = 0 \quad Q \rightarrow e^{(N)} = \sum_{i=0}^{N-1} g_i e^{(i)} \]

\[ \Gamma \rightarrow [x' = f(x) \& Q]e = 0 \]

Proof Idea.

by vdbx with 
\[ G = \begin{pmatrix} 
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \cdots & 0 & 1 \\
g_0 & g_1 & \cdots & g_{N-2} & g_{N-1} 
\end{pmatrix}, \quad e = \begin{pmatrix} 
e \\
e^{(1)} \\
\vdots \\
e^{(N-1)} 
\end{pmatrix} \]
ODE Axiomatization: Derived Invariant Rules

Vectorial Darboux are DG

\[ Q \rightarrow e' = Ge \]
\[ e = 0 \rightarrow [x' = f(x) \& Q]e = 0 \]

Differential Radical Invariants are DG

\[ \Gamma, Q \rightarrow \bigwedge_{i=0}^{N-1} e^{(i)} = 0 \quad Q \rightarrow e^{(N)} = \sum_{i=0}^{N-1} g_i e^{(i)} \]
\[ \Gamma \rightarrow [x' = f(x) \& Q]e = 0 \]

Semialgebraic Invariants are derived

\[ Q \rightarrow e'^* \succeq 0 \]
\[ Q \rightarrow e^{(N)} = \sum_{i=0}^{N-1} g_i e^{(i)} \]
\[ \Gamma \rightarrow [x' = f(x) \& Q]e \succeq 0 \]

\[ e'^* = 0 \]

\[ N \text{ exists} \]
ODE Axiomatization: Derived Invariant Rules

Vectorial Darboux are DG

\[
\begin{align*}
Q \rightarrow e' &= Ge \\
e &= 0 \rightarrow [x' = f(x) \& Q]e = 0
\end{align*}
\]

Differential Radical Invariants are DG

\[
\begin{align*}
\Gamma, Q \rightarrow \bigwedge_{i=0}^{N-1} e^{(i)} &= 0 \\
Q \rightarrow e^{(N)} &= \sum_{i=0}^{N-1} g_i e^{(i)} \\
\Gamma \rightarrow [x' = f(x) \& Q]e &= 0
\end{align*}
\]

Semialgebraic Invariants are derived

\[
\begin{align*}
Q \rightarrow e'^* \succeq 0 \\
Q \rightarrow e^{(N)} &= \sum_{i=0}^{N-1} g_i e^{(i)} \\
\Gamma \rightarrow [x' = f(x) \& Q]e \succeq 0
\end{align*}
\]

\[
e'^* \geq 0 \equiv e \geq 0 \land (e = 0 \rightarrow (e')'^* \geq 0) \\
e'^* > 0 \equiv e > 0 \lor (e = 0 \land (e')'^* > 0)
\]
Proofs with higher Lie derivatives

Proofs use continuously changing basis to keep invariants at constant local coordinates

Local coordinates: \((\frac{x}{x}, \frac{y}{y})\)

Sound and complete ODE invariance proofs
Extended Axiomatization

Unique Solutions
\[
\langle x' = f(x) \& Q_1 \land Q_2 \rangle P \leftrightarrow \langle x' = f(x) \& Q_1 \rangle P \land \langle x' = f(x) \& Q_2 \rangle P
\]

Continuous Existence
\[
\langle x' = f(x) \& e > 0 \rangle x \neq y \leftrightarrow e > 0
\]

Real Induction
\[
[x' = f(x)] P \leftrightarrow \forall y [x' = f(x) \& P \lor x = y] \\
(x = y \rightarrow P \land \langle x' = f(x) \& P \rangle x \neq y)
\]
Extended Axiomatization

Unique Solutions

\[ \langle x' = f(x) & Q_1 \land Q_2 \rangle P \]
\[ \iff \langle x' = f(x) \rangle P \land \langle x' = f(x) & Q_2 \rangle P \]

Continuous Existence

\[ \langle x' = f(x) & e > 0 \rangle x \neq y \iff e > 0 \]

Real Induction

\[ [x' = f(x)] P \iff \forall y [x' = f(x) & P \lor x = y] \]
\[ (x = y \to P \land \langle x' = f(x) & P \rangle x \neq y) \]

Differential Adjoint

\[ \langle x' = f(x) & Q(x) \rangle x = y \iff \langle y' = -f(y) & Q(y) \rangle y = x \]
ODE Axiomatization: Derived Invariance Rule

\[ x_0' = 1 \]

Real Induction Rule

\[
P \rightarrow \langle x' = f(x) \& P \rangle \quad \neg P \rightarrow \langle x' = -f(x) \& \neg P \rangle
\]

\[ P \rightarrow [x' = f(x)]P \]

\[ \langle x' = f(x) \& P \rangle \equiv \langle y := x \rangle \langle x' = f(x) \& P \vee x = y \rangle x \neq y \]

Local progress to \( P \)

Real Induction

\[ [x' = f(x)]P \leftrightarrow \forall y [x' = f(x) \& P \vee x = y] \]

\[ (x = y \rightarrow P \wedge \langle x' = f(x) \& P \rangle x \neq y) \]

Differential Adjoint

\[ \langle x' = f(x) \& Q(x) \rangle x = y \leftrightarrow \langle y' = -f(y) \& Q(y) \rangle y = x \]
ODE Axiomatization: Derived Local Progress

Local Progress Step

\[ e > 0 \lor e = 0 \land \langle x' = f(x) \land e' \geq 0 \rangle \circ \]
\[ \rightarrow \langle x' = f(x) \land e \geq 0 \rangle \circ \]

Local Progress ≥

\[ e'^* \geq 0 \rightarrow \langle x' = f(x) \land e \geq 0 \rangle \circ \]

Local Progress >

\[ e'^* > 0 \rightarrow \langle x' = f(x) \land e > 0 \rangle \circ \]

Local Progress Semialgebraic

\[ \langle x' = f(x) \land P \rangle \circ \leftrightarrow P'^* \]
ODE Axiomatization: Derived Local Progress

Local Progress Step

\[ e > 0 \lor e = 0 \land \langle x' = f(x) \land e' \geq 0 \rangle \circ \]
\[ \rightarrow \langle x' = f(x) \land e \geq 0 \rangle \circ \]

Local Progress $\geq$

\[ e'^* \geq 0 \rightarrow \langle x' = f(x) \land e \geq 0 \rangle \circ \]

Local Progress $>$

\[ e'^* > 0 \rightarrow \langle x' = f(x) \land e > 0 \rangle \circ \]

Local Progress Semialgebraic

\[ \langle x' = f(x) \land P \rangle \circ \leftrightarrow P'^* \]

\[ (P \land Q)'^* \equiv P'^* \land Q'^* \]
\[ (P \lor Q)'^* \equiv P'^* \lor Q'^* \]
Theorem (Algebraic Completeness) (LICS’18, JACM’20)

dL calculus is a sound & complete axiomatization of algebraic invariants of polynomial differential equations. They are decidable with a derived axiom:

\[(DRI) \quad [x' = f(x) & Q] e = 0 \iff (Q \rightarrow e'^* = 0) \quad (Q \text{ open})\]

Theorem (Semialgebraic Completeness) (LICS’18, JACM’20)

dL calculus with RI is a sound & complete axiomatization of semialgebraic invariants of differential equations. They are decidable with derived axiom:

\[(SAI) \quad \forall x \left( P \rightarrow [x' = f(x)] P \right) \leftrightarrow \forall x \left( P \rightarrow P'^* \right) \land \forall x \left( \neg P \rightarrow (\neg P)'^*\right)\]

Definable \( e'^* \) is short for all/significant Lie derivative w.r.t. ODE

Definable \( e'^*\neg \) is w.r.t. backwards ODE \( x' = -f(x) \). Also for \( P \).
### Theorem (Analytic Completeness) (LICS’18,JACM’20)

**dL calculus is a sound & complete axiomatization of analytic invariants of analytic differential equations.**

\[
\text{(DRI)} \quad [x' = f(x) & Q]e = 0 \iff (Q \rightarrow e' = 0) \quad (Q \text{ open})
\]

### Theorem (Semianalytic Completeness) (LICS’18,JACM’20)

**dL calculus with RI is a sound & complete axiomatization of semianalytic invariants of differential equations.**

\[
\text{(SAI)} \quad \forall x \left( P \rightarrow [x' = f(x)] P \right) \iff \forall x \left( P \rightarrow P' \right) \wedge \forall x \left( \neg P \rightarrow (\neg P)' \right)
\]

(S) **Smooth** function interpretations \( h : \mathbb{R}^k \rightarrow \mathbb{R} \)

(P) **Partial derivatives** of \( h(y_1, \ldots, y_k) \) have syntactic term representation \( \frac{\partial h}{\partial y_i} \)

(R) **Computable differential radicals**: compute \( N, g_i \) for \( e^{(N)} = \sum_{i=0}^{N-1} g_i e^{(i)} \)
Completeness for Noetherian Functions

Definition (Noetherian Function)

\( h : H \subseteq \mathbb{R}^k \rightarrow \mathbb{R} \) is Noetherian function iff \( h(y) = p(y, h_1(y), \ldots, h_r(y)) \) for a polynomial \( p \) and Noetherian chain \( h_1, \ldots, h_r : H \subseteq \mathbb{R}^k \rightarrow \mathbb{R} \), i.e., real analytic

\[
\frac{\partial h_j}{\partial y_i}(y) = q_{ij}(y, h_1(y), \ldots, h_r(y))
\]

for some polynomial \( q_{ij} \in \mathbb{R}[y, z] \)

Example: \( \frac{\partial \sin}{\partial y}(y) = \cos(y) \) and \( \frac{\partial \cos}{\partial y}(y) = -\sin(y) \) and \( \frac{\partial \exp}{\partial y}(y) = \exp(y) \)

Theorem

Noetherian functions satisfy SPR conditions.

⇒ Completeness for logic + differential equations with Noetherian functions.

(S) Smooth function interpretations \( h : \mathbb{R}^k \rightarrow \mathbb{R} \)

(P) Partial derivatives of \( h(y_1, \ldots, y_k) \) have syntactic term representation \( \frac{\partial h}{\partial y_i} \)

(R) Computable differential radicals: compute \( N, g_i \) for \( e^{(N)} = \sum_{i=0}^{N-1} g_i e^{(i)} \)
1. Differential Dynamic Logic
   - Syntax
   - Axiomatization
   - Relative Completeness / ODE

2. Proofs for Differential Equations
   - Differential Invariants / Cuts / Ghosts

3. Completeness for Differential Equation Invariants
   - Darboux are Differential Ghosts
   - Derived Differential Radical Invariants
   - Real Induction
   - Derived Local Progress
   - Completeness for Invariants
   - Completeness for Noetherian Functions

4. Summary
Logical Foundation for Differential Equation Invariants

**differential dynamic logic**

\[ dL = DL + HP \]

1. Poincaré: qualitative ODE
2. Complete axiomatization
3. Algebraic ODE invariants
4. Semialgebraic ODE invariants
5. Algebraic hybrid systems
6. Local ODE progress
7. Decide by dL proof/disproof
8. Uniform substitution axioms
9. Analytic extensions: Noetherian

**Properties**

1. MVT
2. Prefix
3. Picard-Lind
4. \( \mathbb{R} \)-complete
5. Existence
6. Uniqueness

1. Differential invariants
2. Differential cuts
3. Differential ghosts
4. Real induction
5. Continuous existence
6. Unique solutions

Impressive power of differential ghosts

LICS’18, JACM’20
Differential Equation Invariance Axiomatization vs. Derived Rules
I Part: Elementary Cyber-Physical Systems
1. Differential Equations & Domains
2. Choice & Control
3. Safety & Contracts
4. Dynamical Systems & Dynamic Axioms
5. Truth & Proof
6. Control Loops & Invariants
7. Events & Responses
8. Reactions & Delays

II Part: Differential Equations Analysis
9. Differential Equations & Differential Invariants
10. Differential Equations & Proofs
11. Ghosts & Differential Ghosts
12. Differential Invariants & Proof Theory

III Part: Adversarial Cyber-Physical Systems
13-16. Hybrid Systems & Hybrid Games

IV Part: Comprehensive CPS Correctness