### 1: Operators of Differential Dynamic Logic (dL)

<table>
<thead>
<tr>
<th>dL</th>
<th>Operator</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e = \hat{e} )</td>
<td>equals</td>
<td>true iff values of terms ( e ) and ( \hat{e} ) are equal</td>
</tr>
<tr>
<td>( e \geq \hat{e} )</td>
<td>greater-or-equal</td>
<td>true iff value of ( e ) greater-or-equal to value of ( \hat{e} )</td>
</tr>
<tr>
<td>( p(e_1, \ldots, e_k) )</td>
<td>predicate</td>
<td>true iff ( p ) holds for the value of ( (e_1, \ldots, e_k) )</td>
</tr>
<tr>
<td>( \neg P )</td>
<td>negation / not</td>
<td>true if ( P ) is false</td>
</tr>
<tr>
<td>( P \land Q )</td>
<td>conjunction / and</td>
<td>true if both ( P ) and ( Q ) are true</td>
</tr>
<tr>
<td>( P \lor Q )</td>
<td>disjunction / or</td>
<td>true if ( P ) is true or if ( Q ) is true</td>
</tr>
<tr>
<td>( P \rightarrow Q )</td>
<td>implication / implies</td>
<td>true if ( P ) is false or ( Q ) is true</td>
</tr>
<tr>
<td>( P \leftrightarrow Q )</td>
<td>bi-implication / equivalent</td>
<td>true if ( P ) and ( Q ) are both true or both false</td>
</tr>
<tr>
<td>( \forall x \ P )</td>
<td>universal quantifier / for all</td>
<td>true if ( P ) is true for all real values of variable ( x )</td>
</tr>
<tr>
<td>( \exists x \ P )</td>
<td>existential quantifier / exists</td>
<td>true if ( P ) is true for some real value of variable ( x )</td>
</tr>
<tr>
<td>( [a]P )</td>
<td>([\cdot]) modality / box</td>
<td>true if ( P ) is true after all runs of HP ( a )</td>
</tr>
<tr>
<td>( \langle a \rangle P )</td>
<td>(\langle\cdot\rangle) modality / diamond</td>
<td>true if ( P ) is true after at least one run of HP ( a )</td>
</tr>
</tbody>
</table>

### 2: Statements and effects of Hybrid Programs (HPs)

<table>
<thead>
<tr>
<th>HP</th>
<th>Operation</th>
<th>Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x := e )</td>
<td>discrete assignment</td>
<td>assigns value of term ( e ) to variable ( x )</td>
</tr>
<tr>
<td>( x := \ast )</td>
<td>nondeterministic assignment</td>
<td>assigns any real value to variable ( x )</td>
</tr>
<tr>
<td>( x' = f(x) \land Q )</td>
<td>continuous evolution</td>
<td>evolve along differential equation ( x' = f(x) ) within evolution domain ( Q ) for any duration</td>
</tr>
<tr>
<td>( ?Q )</td>
<td>test</td>
<td>check truth of first-order formula ( Q ) at current state</td>
</tr>
<tr>
<td>( a; b )</td>
<td>sequential composition</td>
<td>HP ( b ) starts after HP ( a ) finishes</td>
</tr>
<tr>
<td>( a \cup b )</td>
<td>nondeterministic choice</td>
<td>choice between alternatives HP ( a ) or HP ( b )</td>
</tr>
<tr>
<td>( a^* )</td>
<td>nondeterministic repetition</td>
<td>repeats HP ( a ) ( n )-times for any ( n \in \mathbb{N} )</td>
</tr>
</tbody>
</table>

### 3: Semantics of dL formula \( P \) in interpretation \( I \) is the set of states \( [P] \subseteq S \) in which it is true

- \( [e \geq \hat{e}] = \{ \omega \in S : \omega[e] \geq \omega[\hat{e}] \} \)
- \( [p(e_1, \ldots, e_k)] = \{ \omega \in S : (\omega[e_1], \ldots, \omega[e_k]) \in I(p) \} \) for predicate symbol \( p \)
- \( [\neg P] = [P]^c = S \setminus [P] \)
- \( [P \land Q] = [P] \cap [Q] \)
- \( [\exists x \ P] = \{ \omega \in S : \exists \nu \in [P] \text{ for some state } \nu \text{ with } \nu = \omega \text{ except for the real value of } x \} \)
- \( [\forall x \ P] = \{ \omega \in S : \forall \nu \in [P] \text{ for all states } \nu \text{ with } \nu = \omega \text{ except for the real value of } x \} \)
- \( [\langle a \rangle P] = [a] \circ [P] = \{ \omega : \nu \in [P] \text{ for some state } \nu \text{ such that } (\omega, \nu) \in [a] \} \)
- \( [\lbrack a \rbrack P] = \lbrack\lbrack a \rbrack P \rbrack = \{ \omega : \nu \in [P] \text{ for all states } \nu \text{ such that } (\omega, \nu) \in [a] \} \)

### 4: Semantics of HP \( a \) in interpretation \( I \) is relation \( \lbrack a \rbrack \subseteq S \times S \) between initial and final states

- \( [x := e] = \{ (\omega, \nu) : \nu = \omega \text{ except } \nu[x] = \omega[e] \} \)
- \( [?Q] = \{ (\omega, \nu) : \nu \in [Q] \} \)
- \( [x' = f(x) \land Q] = \{ (\omega, \nu) : \varphi \vdash x' = f(x) \land Q \text{ for some solution } \varphi : [0, r] \to S \text{ with } \varphi(r) = \nu, \varphi(0) = \omega \} \)
- \( [a \cup b] = [a] \cup [b] \)
- \( [a; b] = [a] \circ [b] = \{ (\omega, \nu) : (\omega, \mu) \in [a], (\mu, \nu) \in [b] \} \)
- \( [a^*] = \bigcup_{n \in \mathbb{N}} [a^n] \) with \( a^{n+1} \equiv (a^n); a \) and \( a^0 \equiv (?true) \)

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5: Axiomatization

\( (a) P \leftrightarrow \neg [a] \neg P \)

\( [e] \ [x := e] p(x) \leftrightarrow p(e) \)

\( [\forall] \ [x := \forall] p(x) \leftrightarrow \forall x p(x) \)

\( [?Q] p \leftrightarrow (Q \rightarrow p) \)

\( [a \cup b] P \leftrightarrow [a] P \land [b] P \)

\( [a; b] P \leftrightarrow [a][b] P \)

\( [a^*] P \leftrightarrow P \land [a][a^*] P \)

\( K \ [a] (P \rightarrow Q) \rightarrow ([a] P \rightarrow [a] Q) \)

\( I \ [a^*] P \leftrightarrow P \land [a^*] (P \rightarrow [a] P) \)

\( V \ p \rightarrow [a] p \quad (\text{FV}(p) \cap \text{BV}(a) = \emptyset) \)

\( \frac{P \rightarrow Q}{[a] P \rightarrow [a] Q} \) (M)

\( \frac{P}{[a] P} \) (G)

\( \frac{\forall x p(x)}{\forall x p(x)} \) (∀)

\( \frac{p \rightarrow q}{q} \) (MP)

\( \frac{f(\bar{x}) = g(\bar{x})}{c(f(\bar{x})) = c(g(\bar{x}))} \) (CT)

\( \frac{f(\bar{x}) = g(\bar{x})}{p(f(\bar{x})) \leftrightarrow p(g(\bar{x}))} \) (CQ)

\( \frac{P \rightarrow Q}{C(P) \leftrightarrow C(Q)} \) (CE)

6: Differential equation axioms and differential axioms

\( \text{DW} \ [x' = f(x) \land Q] P \leftrightarrow [x' = f(x) \land Q](Q \rightarrow P) \)

\( \text{DC} \ ([x' = f(x) \land Q] P \leftrightarrow [x' = f(x) \land Q \land C] P) \leftrightarrow [x' = f(x) \land Q] C \)

\( \text{DE} \ ([x' = f(x) \land Q] P \leftrightarrow [x' = f(x) \land Q][x' := f(x)] P) \)

\( \text{DI} \ ([x' = f(x) \land Q] P \leftrightarrow [?Q] P \leftrightarrow (Q \rightarrow [x' = f(x) \land Q])(P)') \)

\( \text{DG} \ [x' = f(x) \land Q] P \leftrightarrow \exists y [x' = f(x), y' = a(x)y + b(x) \land Q] P \)

\( \text{DS} \ [x' = c \land q(x)] p(x) \leftrightarrow \forall t \geq 0 \ ((\forall 0 \leq s \leq t q(x + cs)) \rightarrow [x := x + ct] p(x)) \)

\( c' \ (c)' = 0 \)

\( x' (x)' = x' \)

\( (e + d)' = (e)' + (d)' \)

\( (e \cdot d)' = (e)' \cdot d + e \cdot (d)' \)

\( [y := g(x)][y' := 1]((f(g(x)))' = (f(y))' \cdot (g(x))') \)

7: First-order axioms

\( \forall i \ (\forall x p(x)) \rightarrow p(c) \)

\( \forall \rightarrow \forall x (P \rightarrow Q) \rightarrow (\forall x P \rightarrow \forall x Q) \)

\( \forall y p \rightarrow \forall x p \quad (x \notin \text{FV}(p)) \)
### 8: dL Sequent calculus proof rules

<table>
<thead>
<tr>
<th>Rule</th>
<th>Premises</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>loop</td>
<td>$\Gamma, J \vdash J \cdot J \cdot [a]J \cdot J \vdash P$</td>
<td>$\Gamma \vdash [a]^*P, \Delta$</td>
</tr>
<tr>
<td>MR</td>
<td>$\Gamma \vdash [a]P, \Delta$</td>
<td>$\Gamma \vdash [a]Q, \Delta$ $Q \vdash P$</td>
</tr>
<tr>
<td>ML</td>
<td>$\Gamma, [a]Q \vdash \Delta$</td>
<td>$\Gamma, [a]P \vdash \Delta$ $P \vdash Q$</td>
</tr>
<tr>
<td>G</td>
<td>$\Gamma \vdash [a]P, \Delta$</td>
<td>$\Gamma \vdash [a]P, \Delta$ $P \vdash Q$</td>
</tr>
<tr>
<td>GVR</td>
<td>$\Gamma_{const} \vdash p, \Delta_{const}$</td>
<td>$\Gamma \vdash [a]p, \Delta$</td>
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</table>

### 9: Differential equation sequent calculus proof rules

<table>
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<th>Rule</th>
<th>Premises</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>dW</td>
<td>$Q \vdash x' = f(x) &amp; Q \vdash P, \Delta$</td>
<td>$\Gamma \vdash <a href="p(x)">x' = f(x)</a>'$</td>
</tr>
<tr>
<td>dI</td>
<td>$\Gamma, q(x) \vdash p(x), \Delta$ $q(x) \vdash <a href="p(x)">x' = f(x)</a>'$</td>
<td>$\Gamma \vdash x' = f(x) &amp; q(x)p(x), \Delta$</td>
</tr>
<tr>
<td>dC</td>
<td>$\Gamma \vdash x' = f(x) &amp; Q, C, \Delta$</td>
<td>$\Gamma \vdash [x' = f(x) &amp; Q &amp; C]P, \Delta$</td>
</tr>
<tr>
<td>dG</td>
<td>$\Gamma \vdash \exists y [x' = f(x), y' = a(x)y + b(x) &amp; Q], \Delta$</td>
<td>$\Gamma \vdash [x' = f(x) &amp; Q]P, \Delta$</td>
</tr>
</tbody>
</table>

### 10: Propositional sequent calculus proof rules

<table>
<thead>
<tr>
<th>Rule</th>
<th>Premises</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>¬R</td>
<td>$\Gamma, P \vdash \Delta$</td>
<td>$\Gamma \vdash \neg P, \Delta$</td>
</tr>
<tr>
<td>¬L</td>
<td>$\Gamma \vdash \neg P, \Delta$</td>
<td>$\Gamma \vdash P, \Delta$ $\Delta \vdash P &amp; Q, \Delta$</td>
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<tr>
<td>⊢L</td>
<td>$\Gamma \vdash C, \Delta$ $\Gamma, C \vdash \Delta$</td>
<td>$\Gamma \vdash \Delta$</td>
</tr>
<tr>
<td>⊢L</td>
<td>$\Gamma \vdash \neg P, \Delta$</td>
<td>$\Gamma \vdash P, \Delta$ $\Delta \vdash P &amp; Q, \Delta$</td>
</tr>
<tr>
<td>⊢L</td>
<td>$\Gamma \vdash \exists x \forall p(x), \Delta$</td>
<td>$\Gamma \vdash \exists x \forall p(x), \Delta$</td>
</tr>
<tr>
<td>⊢R</td>
<td>$\Gamma \vdash \forall x p(x), \Delta$</td>
<td>$\Gamma \vdash \exists x p(x), \Delta$</td>
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</table>

### 11: Quantifier sequent calculus proof rules

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<th>Premises</th>
<th>Conclusion</th>
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</thead>
<tbody>
<tr>
<td>∀R</td>
<td>$\Gamma \vdash p(y), \Delta$</td>
<td>$\Gamma \vdash \forall x p(x), \Delta$</td>
</tr>
<tr>
<td>∀L</td>
<td>$\Gamma, p(e) \vdash \Delta$</td>
<td>$\Gamma, \forall x p(x) \vdash \Delta$</td>
</tr>
<tr>
<td>∃R</td>
<td>$\Gamma \vdash p(x), \Delta$</td>
<td>$\Gamma \vdash \exists x p(x), \Delta$</td>
</tr>
<tr>
<td>∃L</td>
<td>$\Gamma \vdash \exists x p(x), \Delta$</td>
<td>$\Gamma \vdash \exists x p(x), \Delta$</td>
</tr>
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</table>

### 12: Derived rules

<table>
<thead>
<tr>
<th>Rule</th>
<th>Premises</th>
<th>Conclusion</th>
</tr>
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<tbody>
<tr>
<td>WR</td>
<td>$\vdash P$</td>
<td>$\Gamma \vdash P, \Delta$</td>
</tr>
<tr>
<td>WL</td>
<td>$P \vdash$</td>
<td>$\Gamma \vdash P &amp; Q, \Delta$</td>
</tr>
<tr>
<td>WLR</td>
<td>$\Gamma, P \vdash Q, \Delta$</td>
<td>$\Gamma \vdash P &amp; Q, \Delta$</td>
</tr>
</tbody>
</table>

### Derived rules (continued)

<table>
<thead>
<tr>
<th>Rule</th>
<th>Premises</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>↔cR</td>
<td>$\Gamma \vdash Q \leftrightarrow P, \Delta$</td>
<td>$\Gamma \vdash P \leftrightarrow Q, \Delta$</td>
</tr>
<tr>
<td>↔cL</td>
<td>$\Gamma \vdash P \leftrightarrow Q, \Delta$</td>
<td>$\Gamma \vdash P \leftrightarrow Q, \Delta$</td>
</tr>
<tr>
<td>→2cL</td>
<td>$\Gamma \vdash P \leftrightarrow Q, \Delta$</td>
<td>$\Gamma \vdash P \leftrightarrow Q, \Delta$</td>
</tr>
</tbody>
</table>
13: Derived axioms
\[\lceil\land\ [\alpha](P \land Q) \leftrightarrow [\alpha]P \land [\alpha]Q\]