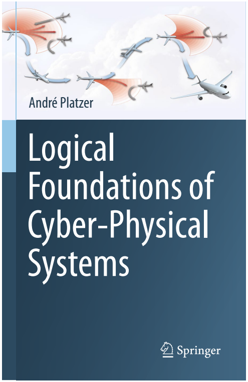
20: Virtual Substitution & Real Equations

Logical Foundations of Cyber-Physical Systems



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Carnegie Mellon University
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Outline

1. Learning Objectives
2. Framing the Miracle
3. Quantifier Elimination
   - Homomorphic Normalization for QE
   - Term Substitutions for Linear Equations
4. Square Root $\sqrt{\cdot}$ Virtual Substitution for Quadratics
   - Square Root Algebra
   - Virtual Substitutions of Square Roots
   - Example
5. Summary

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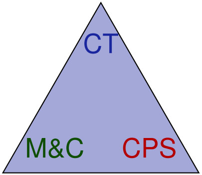
rigorous arithmetical reasoning
miracle of quantifier elimination
logical trinity for reals
switch between syntax & semantics at will
virtual substitution lemma
bridge gap between semantics and inexpressibles



analytic complexity
modeling tradeoffs

verifying CPS at scale

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Evaluating Real Arithmetic Formulas

$x^2 > 2 \land 2x < 3 \lor x^3 \leq x^2$

Evaluating Real Arithmetic Formulas

When $\omega(x) = 2$

$$\omega[\![x^2 > 2 \land 2x < 3 \lor x^3 \leq x^2]\!]$$

Evaluating Real Arithmetic Formulas

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$$\omega[\![x^2 > 2 \land 2x < 3 \lor x^3 \leq x^2]\!] = 2^2 > 2 \land 2 \cdot 2 < 3 \lor 2^3 \leq 2^2 = \mathit{false}$$

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When $\nu(x) = -1$

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Are the following formulas valid, i.e., true in all states?

$$x^2 > 2 \land 2x < 3 \lor x^3 \leq x^2$$
$$\forall x\, (x^2 > 2 \land 2x < 3 \lor x^3 \leq x^2)$$
$$\exists x\, (x^2 > 2 \land 2x < 3 \lor x^3 \leq x^2)$$

Evaluating Real Arithmetic Formulas

When $\omega(x) = 2$

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Evaluating Real Arithmetic Formulas

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Is validity of formulas decidable/semidecidable/undecidable/not semidecidable for:

1. Propositional logic [no variables]
- ✓ $\mathrm{FOL}[p, f, \ldots]$ uninterpreted
2. $\mathrm{FOL}_{\mathbb{N}}[+, \cdot, =]$
3. $\mathrm{FOL}_{\mathbb{R}}[+, \cdot, =, <]$
4. $\mathrm{FOL}_{\mathbb{Q}}[+, \cdot, =]$
5. $\mathrm{FOL}_{\mathbb{C}}[+, \cdot, =]$

Framing the Miracle: Quiz

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× FOL$_{\mathbb{N}}[+,\cdot,=]$ Peano arithmetic not semidecidable [Gödel'31]

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8. $\mathrm{FOL}_{\mathbb{N}}[+, =, 2|, 3|, \dots]$
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- ✓ $\text{FOL}_{\mathbb{R}}[+, =, \wedge, \exists]$ — decidable Gaussian elim. [179 CE]

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?	$\text{FOL}_{\mathbb{R}}[+, \cdot, \exp, =, <]$	unknown
×	$\text{FOL}_{\mathbb{R}}[+, \cdot, \sin, =, <]$ $\sin x = 0$	not semidecidable [Richardson'68]

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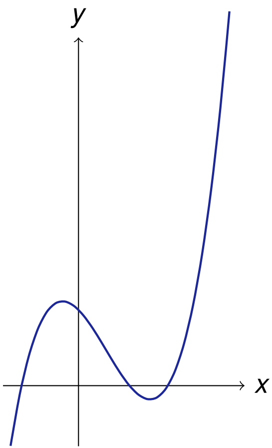
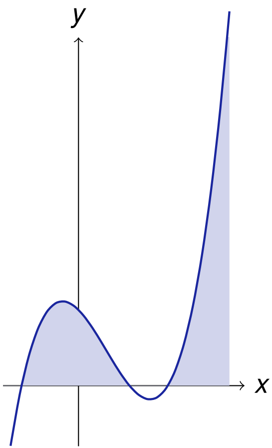


$$F \equiv \exists y\, (y \geq 0 \wedge 1 - x - 1.83x^2 + 1.66x^3 > y)$$



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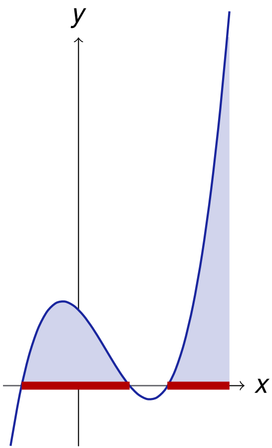
Quantifier Elimination ⇝ Projection



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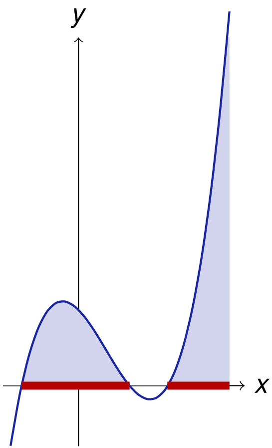

$$F \equiv \exists y\,(y \geq 0 \land 1 - x - 1.83x^2 + 1.66x^3 > y)$$

QE

$$\mathrm{QE}(F) \equiv -0.75 < x \land x < 0.68 \lor x > 1.18$$

Quantifier Elimination $\leftrightsquigarrow$ Projection

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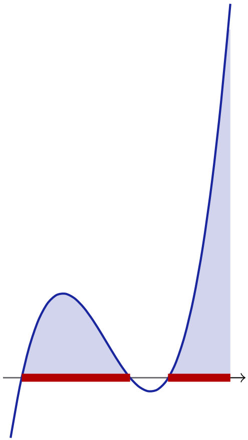
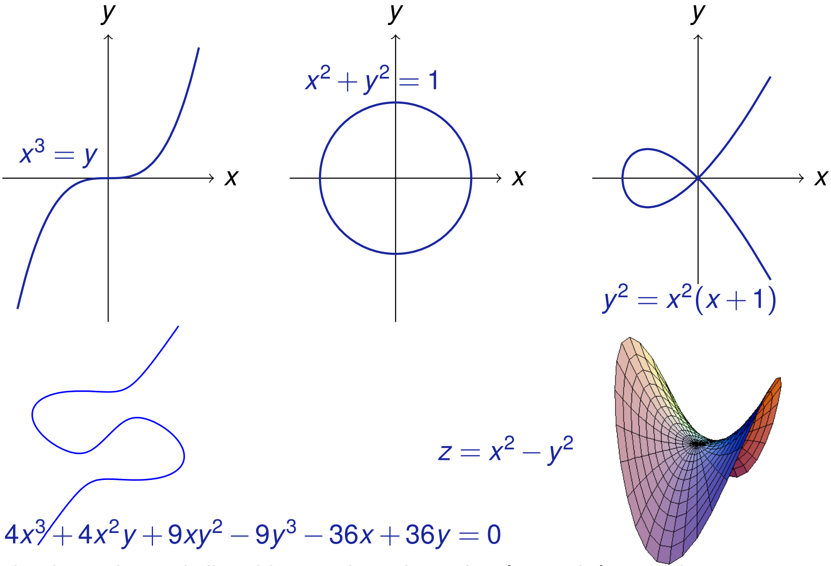


$$F \equiv \exists y\,(y \geq 0 \land 1 - x - 1.83x^2 + 1.66x^3 > y)$$

QE

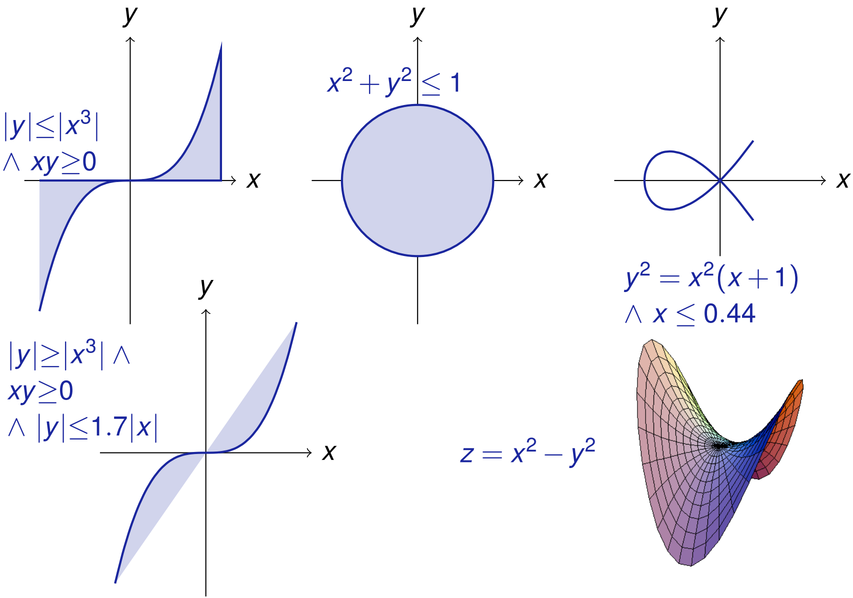
$$\mathrm{QE}(F) \equiv -0.75 < x \land x < 0.68 \lor x > 1.18$$

If all but one variable has fixed value: Finite union of intervals.
Univariate polynomials have finitely many roots. Signs change finitely often.



Algebraic variety: defined by conjunction of polynomial equations

Polynomial Inequalities $\leftrightsquigarrow$ Semialgebraic Sets



Theorem (Tarski'31)

First-order logic of real arithmetic is decidable since it admits quantifier elimination, i.e., for each formula P, compute quantifier-free formula $\mathrm{QE}(P)$ *that is equivalent, i.e.,* $P \leftrightarrow \mathrm{QE}(P)$ *is valid.*

Quantifier Elimination in Real Arithmetic

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Theorem (Complexity, Davenport&Heintz'88,Weispfenning'88)

(Time and space) complexity of QE for $\mathbb{R}$ is doubly exponential in the number n of quantifier (alternations).

$$2^{2^{O(n)}}$$

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$$2^{2^{O(n)}}$$

Answer even for one *free* variable and only linear polynomials

Quantifier Elimination Examples

$$\begin{aligned}
\mathrm{QE}(\exists x\,(2x^2 + c \le 5)) &\equiv \\
\mathrm{QE}(\forall c\,\exists x\,(2x^2 + c \le 5)) &\equiv
\end{aligned}$$

$$\begin{aligned}
\mathrm{QE}(\exists x\,(a = b + x^2)) &\equiv \\
\mathrm{QE}(\exists x\,(x^2 = 2)) &\equiv \\
\mathrm{QE}(\exists x\,(x^2 = 2 \land y = x)) &\equiv
\end{aligned}$$

Quantifier Elimination Examples

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\begin{aligned}
\mathrm{QE}(\exists x\,(2x^2+c\le 5)) &\equiv c\le 5\\
\mathrm{QE}(\forall c\,\exists x\,(2x^2+c\le 5)) &\equiv \mathrm{QE}(\forall c\,\mathrm{QE}(\exists x\,(2x^2+c\le 5)) \equiv \mathrm{QE}(\forall c\,(c\le 5))\\
&\equiv -100\le 5\land 5\le 5\land 100\le 5\\
\mathrm{QE}(\exists x\,(a=b+x^2)) &\equiv\\
\mathrm{QE}(\exists x\,(x^2=2)) &\equiv\\
\mathrm{QE}(\exists x\,(x^2=2\land y=x)) &\equiv
\end{aligned}
$$

Quantifier Elimination Examples

$$
\begin{aligned}
\mathrm{QE}(\exists x\,(2x^2+c\le 5)) &\equiv c\le 5\\
\mathrm{QE}(\forall c\,\exists x\,(2x^2+c\le 5)) &\equiv \mathrm{QE}(\forall c\,\mathrm{QE}(\exists x\,(2x^2+c\le 5)) \equiv \mathrm{QE}(\forall c\,(c\le 5))\\
&\equiv -100\le 5\land 5\le 5\land 100\le 5\equiv \mathit{false}\\
\mathrm{QE}(\exists x\,(a=b+x^2)) &\equiv\\
\mathrm{QE}(\exists x\,(x^2=2)) &\equiv\\
\mathrm{QE}(\exists x\,(x^2=2\land y=x)) &\equiv
\end{aligned}
$$

Quantifier Elimination Examples

$$
\begin{aligned}
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&\equiv -100\le 5\land 5\le 5\land 100\le 5 \equiv \mathit{false}\\
\mathrm{QE}(\exists x\,(a=b+x^2)) &\equiv a\ge b\\
\mathrm{QE}(\exists x\,(x^2=2)) &\equiv\\
\mathrm{QE}(\exists x\,(x^2=2\land y=x)) &\equiv
\end{aligned}
$$

Quantifier Elimination Examples

$$
\begin{aligned}
\mathrm{QE}(\exists x\,(2x^2+c\le 5)) &\equiv c\le 5\\
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&\equiv -100\le 5\land 5\le 5\land 100\le 5\equiv \mathit{false}\\
\mathrm{QE}(\exists x\,(a=b+x^2)) &\equiv a\ge b\\
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Quantifier Elimination Examples

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&\equiv -100\le 5\land 5\le 5\land 100\le 5\equiv \mathit{false}\\
\mathrm{QE}(\exists x\,(a=b+x^2)) &\equiv a\ge b\\
\mathrm{QE}(\exists x\,(x^2=2)) &\equiv \mathit{true}\\
\mathrm{QE}(\exists x\,(x^2=2\land y=x)) &\equiv y=\pm\sqrt{2}
\end{aligned}
$$

Quantifier Elimination Examples

$$
\begin{aligned}
\mathrm{QE}(\exists x\,(2x^2+c\le 5)) &\equiv c \le 5\\
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&\equiv -100\le 5\land 5\le 5\land 100\le 5 \equiv \mathit{false}\\
\mathrm{QE}(\exists x\,(a=b+x^2)) &\equiv a\ge b\\
\mathrm{QE}(\exists x\,(x^2=2)) &\equiv \mathit{true}\\
\mathrm{QE}(\exists x\,(x^2=2\land y=x)) &\equiv y=\pm\sqrt{2}\equiv y^2=2
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{QE}(A \wedge B) &\equiv \\
\mathrm{QE}(A \vee B) &\equiv \\
\mathrm{QE}(\neg A) &\equiv \\
\mathrm{QE}(\forall x\, A) &\equiv \\
\mathrm{QE}(\exists x\, A) &\equiv && A \text{ has quantifiers}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{QE}(A \land B) &\equiv \mathrm{QE}(A) \land \mathrm{QE}(B) \\
\mathrm{QE}(A \lor B) &\equiv \mathrm{QE}(A) \lor \mathrm{QE}(B) \\
\mathrm{QE}(\neg A) &\equiv \neg\, \mathrm{QE}(A) \\
\mathrm{QE}(\forall x\, A) &\equiv \mathrm{QE}(\neg \exists x\, \neg A) \\
\mathrm{QE}(\exists x\, A) &\equiv \mathrm{QE}(\exists x\, \mathrm{QE}(A)) && A \text{ has quantifiers}
\end{aligned}
$$

Framework: Logical Normalization for QE

$$
\begin{aligned}
\mathrm{QE}(A \land B) &\equiv \mathrm{QE}(A) \land \mathrm{QE}(B)\\
\mathrm{QE}(A \lor B) &\equiv \mathrm{QE}(A) \lor \mathrm{QE}(B)\\
\mathrm{QE}(\neg A) &\equiv \neg\,\mathrm{QE}(A)\\
\mathrm{QE}(\forall x\, A) &\equiv \mathrm{QE}(\neg \exists x\, \neg A)\\
\mathrm{QE}(\exists x\, A) &\equiv \mathrm{QE}(\exists x\, \mathrm{QE}(A)) && A \text{ has quantifiers}\\
\mathrm{QE}(\exists x\, (A \lor B)) &\equiv\\
\mathrm{QE}(\exists x\, \neg(A \land B)) &\equiv\\
\mathrm{QE}(\exists x\, \neg(A \lor B)) &\equiv\\
\mathrm{QE}(\exists x\, \neg\neg A) &\equiv
\end{aligned}
$$

Framework: Logical Normalization for QE

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\mathrm{QE}(A \land B) &\equiv \mathrm{QE}(A) \land \mathrm{QE}(B)\\
\mathrm{QE}(A \lor B) &\equiv \mathrm{QE}(A) \lor \mathrm{QE}(B)\\
\mathrm{QE}(\neg A) &\equiv \neg\,\mathrm{QE}(A)\\
\mathrm{QE}(\forall x\, A) &\equiv \mathrm{QE}(\neg \exists x\, \neg A)\\
\mathrm{QE}(\exists x\, A) &\equiv \mathrm{QE}(\exists x\, \mathrm{QE}(A)) && A \text{ has quantifiers}\\
\mathrm{QE}(\exists x\, (A \lor B)) &\equiv \mathrm{QE}(\exists x\, A) \lor \mathrm{QE}(\exists x\, B)\\
\mathrm{QE}(\exists x\, \neg(A \land B)) &\equiv \mathrm{QE}(\exists x\, (\neg A \lor \neg B))\\
\mathrm{QE}(\exists x\, \neg(A \lor B)) &\equiv \mathrm{QE}(\exists x\, (\neg A \land \neg B))\\
\mathrm{QE}(\exists x\, \neg\neg A) &\equiv \mathrm{QE}(\exists x\, A)
\end{aligned}
$$

Framework: Logical Normalization for QE

$$
\begin{aligned}
\mathrm{QE}(A \wedge B) &\equiv \mathrm{QE}(A) \wedge \mathrm{QE}(B) \\
\mathrm{QE}(A \vee B) &\equiv \mathrm{QE}(A) \vee \mathrm{QE}(B) \\
\mathrm{QE}(\neg A) &\equiv \neg\, \mathrm{QE}(A) \\
\mathrm{QE}(\forall x\, A) &\equiv \mathrm{QE}(\neg \exists x\, \neg A) \\
\mathrm{QE}(\exists x\, A) &\equiv \mathrm{QE}(\exists x\, \mathrm{QE}(A)) && A \text{ has quantifiers} \\
\mathrm{QE}(\exists x\, (A \vee B)) &\equiv \mathrm{QE}(\exists x\, A) \vee \mathrm{QE}(\exists x\, B) \\
\mathrm{QE}(\exists x\, \neg(A \wedge B)) &\equiv \mathrm{QE}(\exists x\, (\neg A \vee \neg B)) \\
\mathrm{QE}(\exists x\, \neg(A \vee B)) &\equiv \mathrm{QE}(\exists x\, (\neg A \wedge \neg B)) \\
\mathrm{QE}(\exists x\, \neg\neg A) &\equiv \mathrm{QE}(\exists x\, A) \\
\mathrm{QE}(\exists x\, (A \wedge (B \vee C))) &\equiv \\
\mathrm{QE}(\exists x\, ((A \vee B) \wedge C)) &\equiv
\end{aligned}
$$

Framework: Logical Normalization for QE

$$
\begin{aligned}
\mathrm{QE}(A \land B) &\equiv \mathrm{QE}(A) \land \mathrm{QE}(B) \\
\mathrm{QE}(A \lor B) &\equiv \mathrm{QE}(A) \lor \mathrm{QE}(B) \\
\mathrm{QE}(\neg A) &\equiv \neg\, \mathrm{QE}(A) \\
\mathrm{QE}(\forall x\, A) &\equiv \mathrm{QE}(\neg \exists x\, \neg A) \\
\mathrm{QE}(\exists x\, A) &\equiv \mathrm{QE}(\exists x\, \mathrm{QE}(A)) && \text{A has quantifiers} \\
\mathrm{QE}(\exists x\, (A \lor B)) &\equiv \mathrm{QE}(\exists x\, A) \lor \mathrm{QE}(\exists x\, B) \\
\mathrm{QE}(\exists x\, \neg(A \land B)) &\equiv \mathrm{QE}(\exists x\, (\neg A \lor \neg B)) \\
\mathrm{QE}(\exists x\, \neg(A \lor B)) &\equiv \mathrm{QE}(\exists x\, (\neg A \land \neg B)) \\
\mathrm{QE}(\exists x\, \neg\neg A) &\equiv \mathrm{QE}(\exists x\, A) \\
\mathrm{QE}(\exists x\, (A \land (B \lor C))) &\equiv \mathrm{QE}(\exists x\, ((A \land B) \lor (A \land C))) && \text{expensive} \\
\mathrm{QE}(\exists x\, ((A \lor B) \land C)) &\equiv \mathrm{QE}(\exists x\, ((A \land C) \lor (B \land C))) && \text{expensive}
\end{aligned}
$$

Framework: Logical Normalization for QE

Normal Form $\text{QE}(\exists x\,(A_1 \land \ldots \land A_k))$ with atomic A_i

$$
\begin{aligned}
\text{QE}(A \land B) &\equiv \text{QE}(A) \land \text{QE}(B) \\
\text{QE}(A \lor B) &\equiv \text{QE}(A) \lor \text{QE}(B) \\
\text{QE}(\neg A) &\equiv \neg\,\text{QE}(A) \\
\text{QE}(\forall x\,A) &\equiv \text{QE}(\neg\exists x\,\neg A) \\
\text{QE}(\exists x\,A) &\equiv \text{QE}(\exists x\,\text{QE}(A)) && A \text{ has quantifiers} \\
\text{QE}(\exists x\,(A \lor B)) &\equiv \text{QE}(\exists x\,A) \lor \text{QE}(\exists x\,B) \\
\text{QE}(\exists x\,\neg(A \land B)) &\equiv \text{QE}(\exists x\,(\neg A \lor \neg B)) \\
\text{QE}(\exists x\,\neg(A \lor B)) &\equiv \text{QE}(\exists x\,(\neg A \land \neg B)) \\
\text{QE}(\exists x\,\neg\neg A) &\equiv \text{QE}(\exists x\,A) \\
\text{QE}(\exists x\,(A \land (B \lor C))) &\equiv \text{QE}(\exists x\,((A \land B) \lor (A \land C))) && \text{expensive} \\
\text{QE}(\exists x\,((A \lor B) \land C)) &\equiv \text{QE}(\exists x\,((A \land C) \lor (B \land C))) && \text{expensive}
\end{aligned}
$$

Normal Form $\mathrm{QE}(\exists x\,(p_1 \sim_i 0 \land \ldots \land p_k \sim_k 0))$ and $\sim_i \in \{>,=,\geq,\neq\}$

$$
\begin{aligned}
p = q &\equiv p - q = 0\\
p \geq q &\equiv p - q \geq 0\\
p > q &\equiv p - q > 0\\
p \neq q &\equiv p - q \neq 0\\
p \leq q &\equiv q - p \geq 0\\
p < q &\equiv q - p > 0\\
\neg(p \geq q) &\equiv p < q\\
\neg(p > q) &\equiv p \leq q\\
\neg(p = q) &\equiv p \neq q\\
\neg(p \neq q) &\equiv p = q
\end{aligned}
$$

Virtual Substitution

$$\exists x\, F \leftrightarrow \bigvee_{t \in T} A_t \land F_x^t$$

where terms T substituted (virtually) into F depend on F
where A_t are quantifier-free additional compatibility conditions

Scalability requires simplifier for intermediate results

Quantifier Elimination by Virtual Substitution

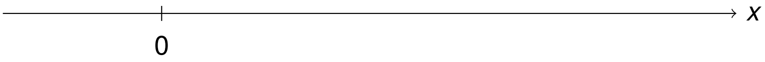
Virtual Substitution

Quantifier $\exists x\, F \leftrightarrow \bigvee_{t \in T} A_t \wedge F_x^t$ Quantifier-free

where terms T substituted (virtually) into F depend on F
where A_t are quantifier-free additional compatibility conditions

Scalability requires simplifier for intermediate results

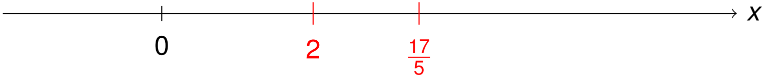
Naïve Virtual Substitution by Example



Can we get rid of the quantifier without changing the semantics?

$$\exists x \left(x > 2 \land x < \frac{17}{5} \right)$$

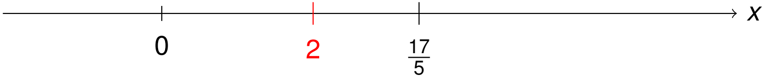
Naïve Virtual Substitution by Example



Can we get rid of the quantifier without changing the semantics?

$$\exists x (x > 2 \land x < \frac{17}{5})$$

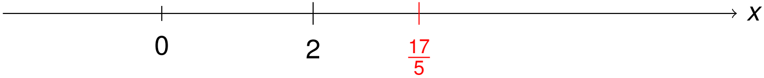
Naïve Virtual Substitution by Example



Can we get rid of the quantifier without changing the semantics?

$$
\begin{aligned}
& \exists x\,(x > 2 \land x < \tfrac{17}{5}) \\
\equiv \quad & (2 > 2 \land 2 < \tfrac{17}{5}) \qquad \text{boundary case “}x = 2\text{”}
\end{aligned}
$$

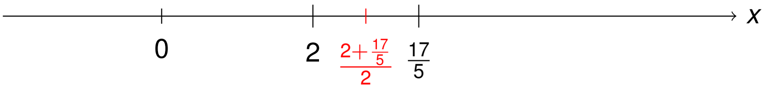
Naïve Virtual Substitution by Example



Can we get rid of the quantifier without changing the semantics?

$$
\begin{array}{lll}
 & \exists x\,(x > 2 \land x < \frac{17}{5}) & \\
\equiv & (2 > 2 \land 2 < \frac{17}{5}) & \text{boundary case “}x = 2\text{”} \\
\lor & (\frac{17}{5} > 2 \land \frac{17}{5} < \frac{17}{5}) & \text{boundary case “}x = \frac{17}{5}\text{”}
\end{array}
$$

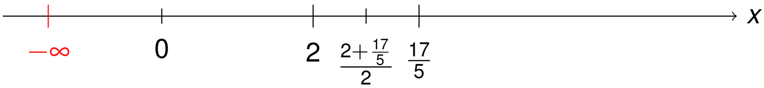
Naïve Virtual Substitution by Example



Can we get rid of the quantifier without changing the semantics?

$$
\begin{array}{lll}
 & \exists x (x > 2 \land x < \frac{17}{5}) & \\
\equiv & (2 > 2 \land 2 < \frac{17}{5}) & \text{boundary case "} x = 2 \text{"} \\
\lor & (\frac{17}{5} > 2 \land \frac{17}{5} < \frac{17}{5}) & \text{boundary case "} x = \frac{17}{5} \text{"} \\
\lor & (\frac{2+\frac{17}{5}}{2} > 2 \land \frac{2+\frac{17}{5}}{2} < \frac{17}{5}) & \text{intermediate case "} x = \frac{2+\frac{17}{5}}{2} \text{"}
\end{array}
$$

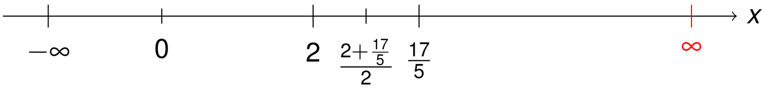
Naïve Virtual Substitution by Example



Can we get rid of the quantifier without changing the semantics?

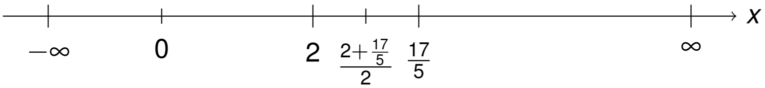
$$
\begin{array}{lll}
 & \exists x (x > 2 \land x < \frac{17}{5}) & \\
\equiv & (2 > 2 \land 2 < \frac{17}{5}) & \text{boundary case “}x = 2\text{”} \\
\lor & (\frac{17}{5} > 2 \land \frac{17}{5} < \frac{17}{5}) & \text{boundary case “}x = \frac{17}{5}\text{”} \\
\lor & (\frac{2+\frac{17}{5}}{2} > 2 \land \frac{2+\frac{17}{5}}{2} < \frac{17}{5}) & \text{intermediate case “}x = \frac{2+\frac{17}{5}}{2}\text{”} \\
\lor & (-\infty > 2 \land -\infty < \frac{17}{5}) & \text{extremal case “}x = -\infty\text{”}
\end{array}
$$

Naïve Virtual Substitution by Example



Can we get rid of the quantifier without changing the semantics?

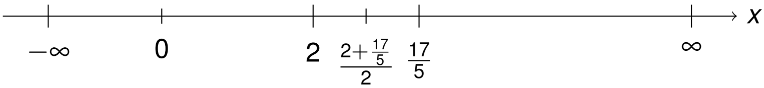
$$
\begin{array}{cll}
 & \exists x\,(x > 2 \land x < \frac{17}{5}) & \\
\equiv & (2 > 2 \land 2 < \frac{17}{5}) & \text{boundary case “}x = 2\text{”} \\
\lor & (\frac{17}{5} > 2 \land \frac{17}{5} < \frac{17}{5}) & \text{boundary case “}x = \frac{17}{5}\text{”} \\
\lor & (\frac{2+\frac{17}{5}}{2} > 2 \land \frac{2+\frac{17}{5}}{2} < \frac{17}{5}) & \text{intermediate case “}x = \frac{2+\frac{17}{5}}{2}\text{”} \\
\lor & (-\infty > 2 \land -\infty < \frac{17}{5}) & \text{extremal case “}x = -\infty\text{”} \\
\lor & (\infty > 2 \land \infty < \frac{17}{5}) & \text{extremal case “}x = \infty\text{”}
\end{array}
$$



Can we get rid of the quantifier without changing the semantics?

$$
\begin{array}{rll}
 & \exists x\,(x > 2 \land x < \frac{17}{5}) & \\
\equiv & (2 > 2 \land 2 < \frac{17}{5}) & \text{boundary case “}x = 2\text{”} \\
\lor & (\frac{17}{5} > 2 \land \frac{17}{5} < \frac{17}{5}) & \text{boundary case “}x = \frac{17}{5}\text{”} \\
\lor & (\frac{2+\frac{17}{5}}{2} > 2 \land \frac{2+\frac{17}{5}}{2} < \frac{17}{5}) & \text{intermediate case “}x = \frac{2+\frac{17}{5}}{2}\text{”} \\
\lor & (-\infty > 2 \land -\infty < \frac{17}{5}) & \text{extremal case “}x = -\infty\text{”} \\
\lor & (\infty > 2 \land \infty < \frac{17}{5}) & \text{extremal case “}x = \infty\text{”} \\
\equiv & \mathit{true} & \text{evaluate}
\end{array}
$$

Naïve Virtual Substitution by Example



Can we get rid of the quantifier without changing the semantics?

$$
\begin{array}{cll}
 & \exists x\,(x > 2 \land x < \frac{17}{5}) & \\
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\lor & (\frac{17}{5} > 2 \land \frac{17}{5} < \frac{17}{5}) & \text{boundary case “}x = \frac{17}{5}\text{”} \\
\lor & (\frac{2+\frac{17}{5}}{2} > 2 \land \frac{2+\frac{17}{5}}{2} < \frac{17}{5}) & \text{intermediate case “}x = \frac{2+\frac{17}{5}}{2}\text{”} \\
\lor & (-\infty > 2 \land -\infty < \frac{17}{5}) & \text{extremal case “}x = -\infty\text{”} \\
\lor & (\infty > 2 \land \infty < \frac{17}{5}) & \text{extremal case “}x = \infty\text{”} \\
\equiv & \mathit{true} & \text{evaluate}
\end{array}
$$

- ∞ is not in $\mathrm{FOL}_{\mathbb{R}}$
- Interior points aren't always terms in $\mathrm{FOL}_{\mathbb{R}}$ if nonlinear
- Substituting them into formulas requires attention

Linear Virtual Substitution

Theorem (Virtual Substitution: Linear Equation)

$$\exists x\,(bx + c = 0 \land F) \;\leftrightarrow$$

Linear Virtual Substitution

Theorem (Virtual Substitution: Linear Equation)

$$\exists x\,(bx + c = 0 \land F) \;\leftrightarrow\; F_x^{-c/b}$$

Linear solution

Theorem (Virtual Substitution: Linear Equation)

$$\exists x \, (bx + c = 0 \land F) \; \leftrightarrow \; b \neq 0 \land F_x^{-c/b}$$

Don't divide by 0

Linear Virtual Substitution

Theorem (Virtual Substitution: Linear Equation)

$$b \neq 0 \rightarrow \left(\exists x\,(bx + c = 0 \land F) \;\leftrightarrow\; b \neq 0 \land F_x^{-c/b}\right)$$

Only actually linear solution if $b \neq 0$

Linear Virtual Substitution

Theorem (Virtual Substitution: Linear Equation $x \notin b, c$)

$$b \neq 0 \rightarrow \left(\exists x\,(bx + c = 0 \land F) \leftrightarrow b \neq 0 \land F_x^{-c/b}\right) \quad \textit{if } x \notin b, c$$

Only linear if no x in b, c

Linear Virtual Substitution

Theorem (Virtual Substitution: Linear Equation $x \notin b, c$)

$$b \neq 0 \rightarrow \big(\exists x\,(bx + c = 0 \wedge F) \leftrightarrow b \neq 0 \wedge F_x^{-c/b}\big) \quad \textit{if } x \notin b, c$$

Conditional equivalence, so conditions may need to be checked or case-split

Linear Virtual Substitution

Theorem (Virtual Substitution: Linear Equation $x \notin b, c$)

$$b \neq 0 \rightarrow \big(\exists x\,(bx + c = 0 \land F) \leftrightarrow b \neq 0 \land F_x^{-c/b}\ \big) \quad \text{if } x \notin b, c$$

Lemma (Uniform substitution of linear equations)

The linear equation axiom is sound (b, c are arity 0 function symbols):

$$\exists lin \quad b \neq 0 \rightarrow \big(\exists x\,(b \cdot x + c = 0 \land q(x)) \leftrightarrow q(-c/b)\big)$$

$$\exists x\,(\underbrace{(y^2+4)}_{b} \cdot x + \underbrace{(yz-1)}_{c} = 0 \land x^3 + x \geq 0) \leftrightarrow \left(-\frac{yz-1}{y^2+4}\right)^3 + \left(-\frac{yz-1}{y^2+4}\right) \geq 0$$

Outline

1. Learning Objectives
2. Framing the Miracle
3. Quantifier Elimination
   - Homomorphic Normalization for QE
   - Term Substitutions for Linear Equations
4. Square Root $\sqrt{\cdot}$ Virtual Substitution for Quadratics
   - Square Root Algebra
   - Virtual Substitutions of Square Roots
   - Example
5. Summary

Theorem (Virtual Substitution: Quadratic Equation)

$$\exists x\,(ax^2 + bx + c = 0 \land F) \leftrightarrow$$

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation)

$$\exists x\,(ax^2 + bx + c = 0 \land F) \leftrightarrow$$

$$F_x^{(-b+\sqrt{b^2-4ac})/(2a)}$$

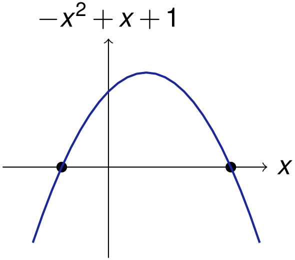
Quadratic solution

Theorem (Virtual Substitution: Quadratic Equation)

$$\exists x\,(ax^2+bx+c=0 \land F) \leftrightarrow$$

$$\left(F_x^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_x^{(-b-\sqrt{b^2-4ac})/(2a)}\right)$$

Or negative square root solution



Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation)

$$\exists x\,(ax^2 + bx + c = 0 \land F) \leftrightarrow$$

$$a \neq 0 \land \qquad \left(F_x^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_x^{(-b-\sqrt{b^2-4ac})/(2a)}\right)$$

Don't divide by 0

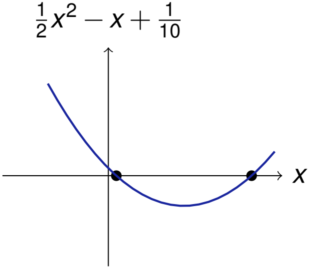
Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation)

$$\exists x\, (ax^2 + bx + c = 0 \land F) \leftrightarrow$$

$$a \neq 0 \land b^2 - 4ac \geq 0 \land \left(F_x^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_x^{(-b-\sqrt{b^2-4ac})/(2a)}\right)$$

Real solution if $\sqrt{\cdot}$ exists by discriminant



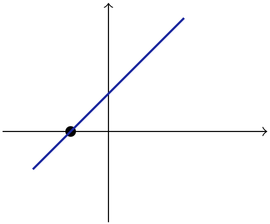
Theorem (Virtual Substitution: Quadratic Equation)

$$\exists x\,(ax^2 + bx + c = 0 \land F) \leftrightarrow$$

$$a = 0 \land b \neq 0 \land F_x^{-c/b}$$

$$\lor\, a \neq 0 \land b^2 - 4ac \geq 0 \land \left(F_x^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_x^{(-b-\sqrt{b^2-4ac})/(2a)}\right)$$

Instead linear solution if $a = 0$ (may case-split)



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Only equivalent if not all 0 which gives trivial equation (else use F)

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

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Only linear or quadratic if no x in a, b, c

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\end{aligned}
$$

1. Quantifier-free equivalent

Quadratic Virtual Substitution

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2. Just not a formula . . .

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Virtual Substitution into Polynomial

Virtually substitute $(a+b\sqrt{c})/d$ into a polynomial p:

$$p_{\bar{x}}^{(a+b\sqrt{c})/d} \stackrel{\text{def}}{=}$$

Convention: On this slide c' is not a derivative but just another name ...

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Square Root Algebra

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$\sqrt{c}$-algebra

Algebra of terms $(a+b\sqrt{c})/d$ with polynomials $a,b,c,d \in \mathbb{Q}[x_1,..,x_n]$:

$$((a+b\sqrt{c})/d)+((a'+b'\sqrt{c})/d') =$$
$$((a+b\sqrt{c})/d)\cdot((a'+b'\sqrt{c})/d') =$$

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where $c \geq 0, d, d' \neq 0$

$$((a+b\sqrt{c})/d) + ((a'+b'\sqrt{c})/d') = ((ad'+da') + (bd'+db')\sqrt{c})/(dd')$$
$$((a+b\sqrt{c})/d) \cdot ((a'+b'\sqrt{c})/d') = ((aa'+bb'c) + (ab'+ba')\sqrt{c})/(dd')$$

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Virtual $\sqrt{\cdot}$ Substitution

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$\sqrt{c}$-comparisons $\quad d \neq 0 \wedge c \geq 0$

$$(a+0\sqrt{c})/d = 0 \equiv$$

$$(a+0\sqrt{c})/d \leq 0 \equiv$$

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$$(a+b\sqrt{c})/d = 0 \equiv ab \leq 0 \wedge a^2 - b^2c = 0$$

$$(a+b\sqrt{c})/d \leq 0 \equiv ad \leq 0 \wedge a^2 - b^2c \geq 0 \vee bd \leq 0 \wedge a^2 - b^2c \leq 0$$

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Theorem (Virtual Substitution: Quadratic Equation $x \notin a,b,c$)

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\end{aligned}
$$

Lemma (Virtual Substitution Lemma for $\sqrt{\cdot}$)

$$F_x^{(a+b\sqrt{c})/d} \equiv F_{\bar{x}}^{(a+b\sqrt{c})/d}$$

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$FOL_{\mathbb{R}}$

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Extended logic $\quad F_x^{(a+b\sqrt{c})/d} \equiv F_{\bar{x}}^{(a+b\sqrt{c})/d} \quad$ $FOL_{\mathbb{R}}$

$$\omega_x^r \in [\![F]\!] \textit{ iff } \omega \in [\![F_{\bar{x}}^{(a+b\sqrt{c})/d}]\!] \textit{ where } r = (\omega[\![a]\!] + \omega[\![b]\!]\sqrt{\omega[\![c]\!]})/(\omega[\![d]\!]) \in \mathbb{R}$$

Example: Quadratic Curiosity

$$a \neq 0 \rightarrow (\exists x\,(ax^2+bx+c=0 \land ax^2+bx+c \leq 0) \leftrightarrow b^2-4ac \geq 0 \land true)$$

$$
\begin{aligned}
&(ax^2+bx+c)_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)}\\
=&a((-b+\sqrt{b^2-4ac})/(2a))^2+b((-b+\sqrt{b^2-4ac})/(2a))+c\\
=&a((b^2+b^2-4ac+(-b-b)\sqrt{b^2-4ac})/(4a^2))+(-b^2+b\sqrt{b^2-4ac})/(2a)+c\\
=&(ab^2+ab^2-4a^2c+(-ab-ab)\sqrt{b^2-4ac})/(4a^2)+(-b^2+2ac+b\sqrt{b^2-4ac})/(2a)\\
=&((ab^2+ab^2-4a^2c)2a+(-b^2+2ac)4a^2+((-ab-ab)2a+b4a^2)\sqrt{b^2-4ac})/(8a^3)\\
=&(\cancel{2a^2b^2}+\cancel{2a^2b^2}-\cancel{8a^3c}-\cancel{4a^2b^2}+\cancel{8a^3c}+(-\cancel{2a^2b}-\cancel{2a^2b}+\cancel{4a^2b})\sqrt{b^2-4ac})/(8a^3)\\
=&(0+0\sqrt{b^2-4ac})/(8a^3)=(0+0\sqrt{..})/1=0
\end{aligned}
$$

$$(ax^2+bx+c=0)_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \equiv ((0+0\sqrt{..})/1=0) \equiv (0\cdot 1=0) \equiv true$$

$$(ax^2+bx+c\leq 0)_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \equiv (\underbrace{(0+0\sqrt{..})/1}_{0} \leq 0) \equiv (0\cdot 1 \leq 0) \equiv true$$

Example: Nonnegative Roots of Quadratic Polynomials

$$a \neq 0 \to \big(\exists x\,(ax^2+bx+c=0 \land x \geq 0)$$
$$\leftrightarrow b^2-4ac \geq 0 \land \big(2ba \leq 0 \land 4ac \geq 0 \lor -2a \leq 0 \land 4ac \leq 0$$
$$\lor 2ba \leq 0 \land 4ac \geq 0 \lor 2a \leq 0 \land 4ac \leq 0)\big)$$

$$-(-b+\sqrt{b^2-4ac})/(2a) = ((-1+0\sqrt{b^2-4ac})/1)\cdot((-b+\sqrt{b^2-4ac})/(2a)$$
$$= (b-\sqrt{b^2-4ac})/(2a)$$

$$(-x \leq 0)_{\bar{x}}^{(b-\sqrt{b^2-4ac})/(2a)}$$
$$\equiv b2a \leq 0 \land \cancel{b^2}-(-1)^2(\cancel{b^2}-4ac) \geq 0 \lor -1\cdot 2a \leq 0 \land \cancel{b^2}-(-1)^2(\cancel{b^2}-4ac) \leq 0$$
$$\equiv 2ba \leq 0 \land 4ac \geq 0 \lor -2a \leq 0 \land 4ac \leq 0$$

$$(-x \leq 0)_{\bar{x}}^{(b+\sqrt{b^2-4ac})/(2a)}$$
$$\equiv b2a \leq 0 \land \cancel{b^2}-1^2(\cancel{b^2}-4ac) \geq 0 \lor 1\cdot 2a \leq 0 \land \cancel{b^2}-1^2(\cancel{b^2}-4ac) \leq 0$$
$$\equiv 2ba \leq 0 \land 4ac \geq 0 \lor 2a \leq 0 \land 4ac \leq 0$$

Outline

1. Learning Objectives
2. Framing the Miracle
3. Quantifier Elimination
   - Homomorphic Normalization for QE
   - Term Substitutions for Linear Equations
4. Square Root $\sqrt{\cdot}$ Virtual Substitution for Quadratics
   - Square Root Algebra
   - Virtual Substitutions of Square Roots
   - Example
5. Summary

$\sqrt{\cdot}$ Square Root Algebra

Virtual Substitution of $(a+b\sqrt{c})/d$ into Comparisons

$$(p \sim 0)_{\bar{x}}^{(a+b\sqrt{c})/d} \equiv (p_{\bar{x}}^{(a+b\sqrt{c})/d} \sim 0) \quad \text{accordingly for } \land, \lor, \ldots$$

$\sqrt{c}$-algebra $\qquad d \neq 0 \land c \geq 0$

$$((a+b\sqrt{c})/d) + ((a'+b'\sqrt{c})/d') = ((ad'+da') + (bd'+db')\sqrt{c})/(dd')$$

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$$(a+b\sqrt{c})/d < 0 \equiv ad < 0 \land a^2 - b^2c > 0$$
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Logical Foundations of Cyber-Physical Systems.
Springer, Switzerland, 2018.
URL: http://www.springer.com/978-3-319-63587-3, doi:10.1007/978-3-319-63588-0.

Volker Weispfenning.
Quantifier elimination for real algebra — the quadratic case and beyond.
Appl. Algebra Eng. Commun. Comput., 8(2):85–101, 1997.
doi:10.1007/s002000050055.

André Platzer.
Logical Analysis of Hybrid Systems: Proving Theorems for Complex Dynamics.
Springer, Heidelberg, 2010.
doi:10.1007/978-3-642-14509-4.

Jacek Bochnak, Michel Coste, and Marie-Francoise Roy.
Real Algebraic Geometry, volume 36 of *Ergeb. Math. Grenzgeb.*
Springer, Berlin, 1998.
doi:10.1007/978-3-662-03718-8.

Saugata Basu, Richard Pollack, and Marie-Françoise Roy.
Algorithms in Real Algebraic Geometry.
Springer, Berlin, 2nd edition, 2006.
doi:10.1007/3-540-33099-2.

Alfred Tarski.
A Decision Method for Elementary Algebra and Geometry.
University of California Press, Berkeley, 2nd edition, 1951.
doi:10.1007/978-3-7091-9459-1_3.

George E. Collins.
Quantifier elimination for real closed fields by cylindrical algebraic decomposition.
In H. Barkhage, editor, *Automata Theory and Formal Languages*, volume 33 of *LNCS*, pages 134–183, Berlin, 1975. Springer.
doi:10.1007/3-540-07407-4_17.