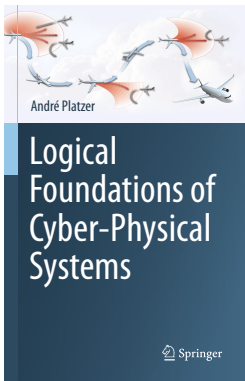
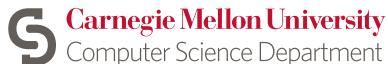


10: Differential Equations & Differential Invariants

Logical Foundations of Cyber-Physical Systems



André Platzer





- 1 Learning Objectives
- 2 A Gradual Introduction to Differential Invariants
 - Global Descriptive Power of Local Differential Equations
 - Intuition for Differential Invariants
 - Deriving Differential Equations
- 3 Differentials
 - Syntax
 - Semantics of Differential Symbols
 - Semantics of Differential Equations
 - Soundness
 - Example Proofs
- 4 Soundness Proof
- 5 Summary



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3 Differentials

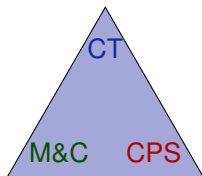
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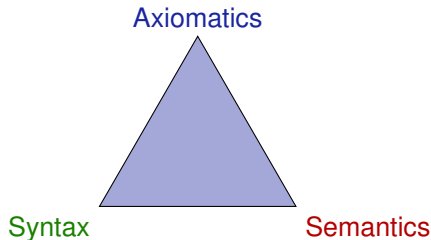


discrete vs. continuous analogies
rigorous reasoning about ODEs
induction for differential equations
differential facet of logical trinity



understanding continuous dynamics
relate discrete+continuous

semantics of ODEs
operational CPS effects



Syntax defines the notation

What problems are we allowed to write down?

Semantics what carries meaning.

What real or mathematical objects does the syntax stand for?

Axiomatics internalizes semantic relations into universal syntactic transformations.

How does the semantics of $e = \tilde{e}$ relate to the semantics of $e - \tilde{e} = 0$, syntactically? What about derivatives?



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ODE	Solution
$x' = 1, x(0) = x_0$	$x(t) = x_0 + t$
$x' = 5, x(0) = x_0$	$x(t) = x_0 + 5t$
$x' = x, x(0) = x_0$	$x(t) = x_0 e^t$
$x' = x^2, x(0) = x_0$	$x(t) = \frac{x_0}{1 - tx_0}$
$x' = \frac{1}{x}, x(0) = 1$	$x(t) = \sqrt{1 + 2t} \dots$
$y'(x) = -2xy, y(0) = 1$	$y(x) = e^{-x^2}$
$x'(t) = tx, x(0) = x_0$	$x(t) = x_0 e^{\frac{t^2}{2}}$
$x' = \sqrt{x}, x(0) = x_0$	$x(t) = \frac{t^2}{4} \pm t\sqrt{x_0} + x_0$
$x' = y, y' = -x, x(0) = 0, y(0) = 1$	$x(t) = \sin t, y(t) = \cos t$
$x' = 1 + x^2, x(0) = 0$	$x(t) = \tan t$
$x'(t) = \frac{2}{t^3} x(t)$	$x(t) = e^{-\frac{1}{t^2}}$ non-analytic
$x' = x^2 + x^4$???
$x'(t) = e^{t^2}$	non-elementary

Descriptive power of differential equations

- 1 Descriptive power: differential equations characterize continuous evolution only locally by the respective directions.
- 2 Simple differential equations describe complicated physical processes.
- 3 Complexity difference between local description and global behavior
- 4 Analyzing ODEs via their solutions undoes their descriptive power.
- 5 Let's exploit descriptive power of ODEs for proofs!

$$x'' = -x$$

$$x''(t) = e^{t^2}$$

$$x(t) = \sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - \dots$$

no elementary closed-form solution

You also prefer loop induction to unfolding all loop iterations, globally ...

Descriptive power of differential equations

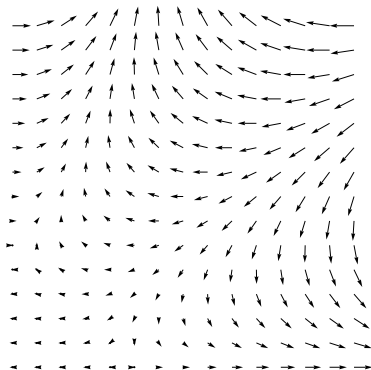
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$$x''(t) = e^{t^2} \quad \text{no elementary closed-form solution}$$

Differential Invariant

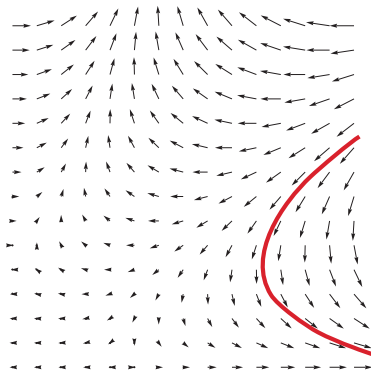
$$\frac{\Gamma \vdash F, \Delta \quad F \vdash ??? F \quad F \vdash P}{\Gamma \vdash [x' = f(x)]P, \Delta}$$



$$['] [x' = f(x)]P \leftrightarrow \forall t \geq 0 [x := y(t)]P \quad (y' = f(y), y(0) = x)$$

Differential Invariant

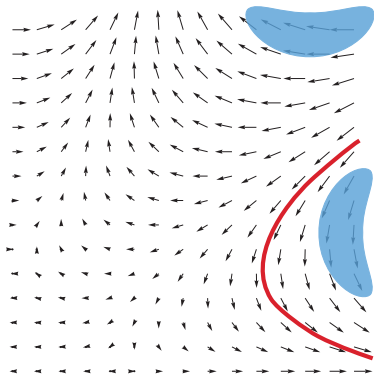
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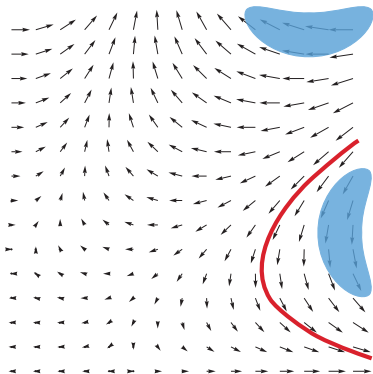
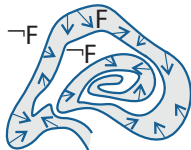


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Differential Invariant

$$\frac{\Gamma \vdash F, \Delta \quad F \vdash ??? F \quad F \vdash P}{\Gamma \vdash [x' = f(x)]P, \Delta}$$

Want: formula F remains true in the direction of the dynamics



$$['] [x' = f(x)]P \leftrightarrow \forall t \geq 0 [x := y(t)]P \quad (y' = f(y), y(0) = x)$$

Next step is undefined for ODEs. But don't need to know where exactly the system evolves to. Just that it remains somewhere in F .
 Show: only evolves into directions in which formula F stays true.

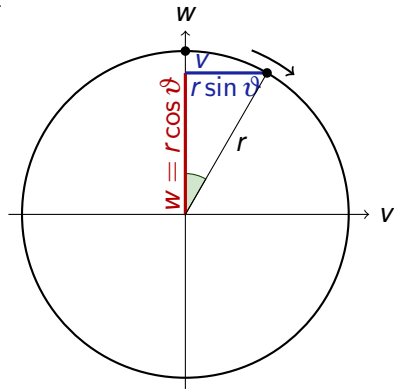


Guiding Example

$$v^2 + w^2 = r^2 \rightarrow [v' = w, w' = -v] v^2 + w^2 = r^2$$

Guiding Example: Rotational Dynamics

$$v^2 + w^2 = r^2 \rightarrow [v' = w, w' = -v] v^2 + w^2 = r^2$$



Guiding Example: Rotational Dynamics

$$v^2 + w^2 = r^2 \rightarrow [v' = w, w' = -v] v^2 + w^2 = r^2$$

$$\rightarrow^R \vdash v^2 + w^2 - r^2 = 0 \rightarrow [v' = w, w' = -v] v^2 + w^2 - r^2 = 0$$



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Syntax

$e ::= x \mid c \mid e + k \mid e - k \mid e \cdot k \mid e / k$

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Derivatives

$$(e + k)' =$$

$$(e - k)' =$$

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$$(e / k)' =$$

$$(c())' =$$

for constants/numbers $c()$

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... What do these primes mean? ...

Syntax $e ::= x \mid c \mid e + k \mid e - k \mid e \cdot k \mid e/k \mid (e)'$

internalize primes into dL syntax

Derivatives

$$(e + k)' = (e)' + (k)'$$

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Semantics

$$\omega[(e)'] =$$

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$$\omega[(e)'] = \frac{d\omega[e]}{dt}$$

what's the time derivative?

Semantics

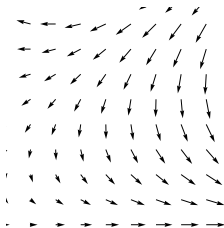
$$\omega[(e)'] = \frac{d\omega[e]}{dt}$$

what's the time derivative?

what's the time?

Semantics

$$\omega[(e)'] = \frac{d\omega[e]}{dt} \text{ nonsense!}$$

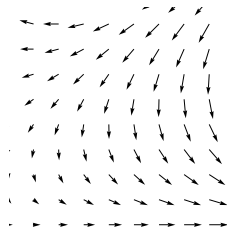


what's the time derivative?
depends on the differential equation?

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Semantics

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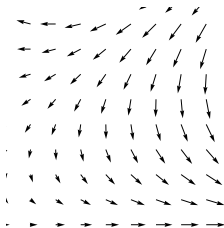
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Not compositional!

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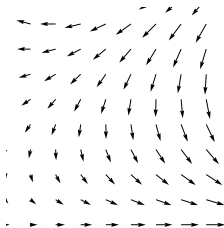


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well-defined in isolated state ω at all?

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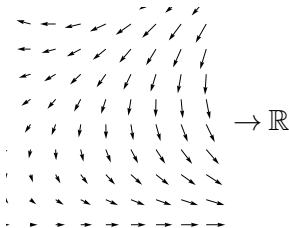
what's the time?

Not compositional!

No time-derivative without time!

Semantics

$$\omega[(e)'] = \sum_x \omega(x') \frac{\partial [e]}{\partial x}(\omega)$$



what's the time derivative?

depends on the differential equation?

well-defined in isolated state ω at all?

meaning is a function of x and x' .

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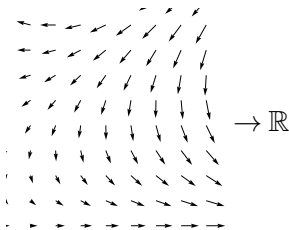
Differential form!

Semantics

$$\omega[(e)'] = \sum_x \omega(x') \frac{\partial [e]}{\partial x}(\omega)$$

Partial

$$\frac{\partial [e]}{\partial x}(\omega) = \lim_{\kappa \rightarrow \omega(x)} \frac{\omega_x^\kappa [e] - \omega [e]}{\kappa - \omega(x)}$$



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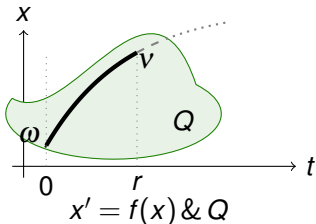
Differential form!

Definition (Hybrid program semantics)

$(\llbracket \cdot \rrbracket : \text{HP} \rightarrow \wp(\mathcal{S} \times \mathcal{S}))$

$\llbracket x' = f(x) \& Q \rrbracket = \{(\varphi(0)|_{\{x'\}^c}, \varphi(r)) : \varphi(z) \models x' = f(x) \wedge Q \text{ for all } 0 \leq z \leq r$
 for a solution $\varphi : [0, r] \rightarrow \mathcal{S}$ of any duration $r \in \mathbb{R}\}$

where $\varphi(z)(x') \stackrel{\text{def}}{=} \frac{d\varphi(t)(x)}{dt}(z)$

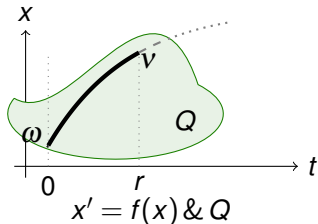


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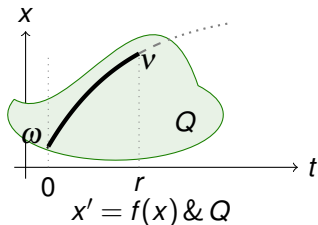
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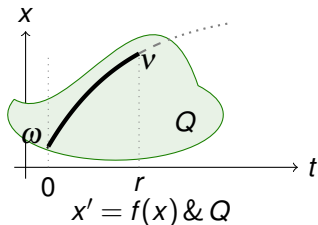
$\llbracket x' = f(x) \& Q \rrbracket = \{(\omega, \nu) : \varphi(z) \models x' = f(x) \wedge Q \text{ for all } 0 \leq z \leq r$
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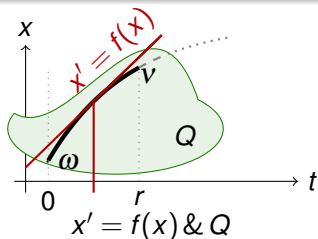


Initial value of x' in ω is irrelevant since defined by ODE.
 Final value of x' is carried over to the final state v .

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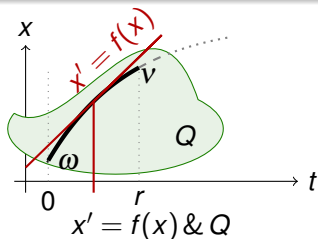


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Lemma (Differential lemma) (Differential value vs. Time-derivative)

If $\varphi \models x' = f(x) \wedge Q$ for duration $r > 0$, then for all $0 \leq z \leq r$, $FV(e) \subseteq \{x\}$:

$$\text{Syntactic ' } \rightarrow \varphi(z) \llbracket (e)' \rrbracket = \frac{d\varphi(t) \llbracket e \rrbracket}{dt}(z) \leftarrow \text{Analytic '}$$

Lemma (Differential assignment) (Effect on Differentials)

If $\varphi \models x' = f(x) \wedge Q$ then $\varphi \models P \leftrightarrow [x' := f(x)]P$

Lemma (Derivations) (Equations of Differentials)

$$(e + k)' = (e)' + (k)'$$

$$(e \cdot k)' = (e)' \cdot k + e \cdot (k)'$$

$$(c())' = 0$$

for constants/numbers $c()$

$$(x)' = x'$$

for variables $x \in \mathcal{V}$

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DE $[x' = f(x) \& Q]P \leftrightarrow [x' = f(x) \& Q][x' := f(x)]P$

Axiomatics

DI $([x' = f(x)]e = 0 \leftrightarrow e = 0) \leftarrow [x' = f(x)](e)' = 0$

Differential Substitution Lemmas

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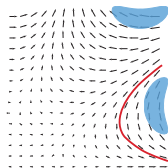
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rate of change of e along ODE is 0

Differential Invariant

$$\text{dl } \frac{\vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x)]e = 0}$$

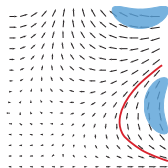


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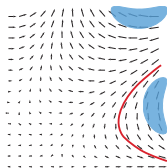


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Proof (dl is a derived rule).

$$\text{DI} \frac{}{e = 0 \vdash [x' = f(x)]e = 0}$$

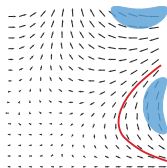


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Proof (dl is a derived rule).

$$\begin{array}{l} \text{DE} \frac{\vdash [x' = f(x)](e)' = 0}{e = 0 \vdash [x' = f(x)]e = 0} \\ \text{DI} \frac{\vdash [x' = f(x)](e)' = 0}{e = 0 \vdash [x' = f(x)]e = 0} \end{array}$$

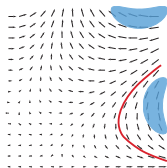


Differential Invariant

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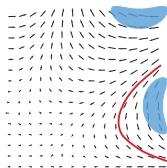
□

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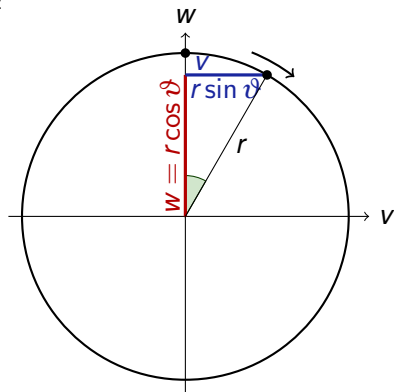
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$$\text{G} \frac{P}{[\alpha]P} \quad \square$$

Guiding Example: Rotational Dynamics

$$v^2 + w^2 = r^2 \rightarrow [v' = w, w' = -v] v^2 + w^2 = r^2$$



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$$v^2 + w^2 = r^2 \rightarrow [v' = w, w' = -v]v^2 + w^2 = r^2 \quad \text{dl} \quad \frac{Q \vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x) \& Q]e = 0}$$

$$\begin{array}{c} \text{[:=]} \\ \hline \vdash [v' := w][w' := -v]2vv' + 2ww' - 2rr' = 0 \\ \hline \text{dl} \\ \frac{v^2 + w^2 - r^2 = 0 \vdash [v' = w, w' = -v]v^2 + w^2 - r^2 = 0}{\rightarrow \text{R}} \\ \hline \vdash v^2 + w^2 - r^2 = 0 \rightarrow [v' = w, w' = -v]v^2 + w^2 - r^2 = 0 \end{array}$$



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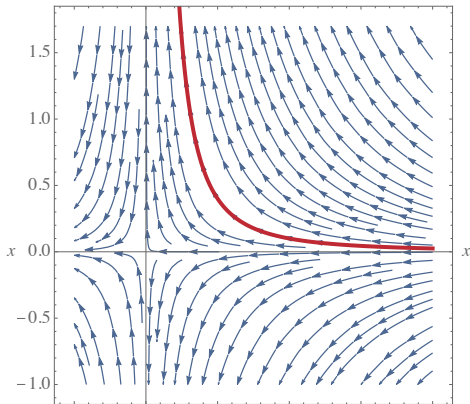
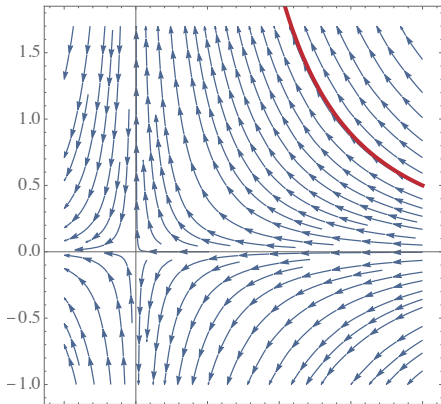
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Simple proof without solving ODE, just by differentiating



Example Proof

$$\rightarrow \mathbb{R} \quad \vdash x^2 y - 2 = 0 \rightarrow [x' = -x^2, y' = 2xy] x^2 y - 2 = 0$$

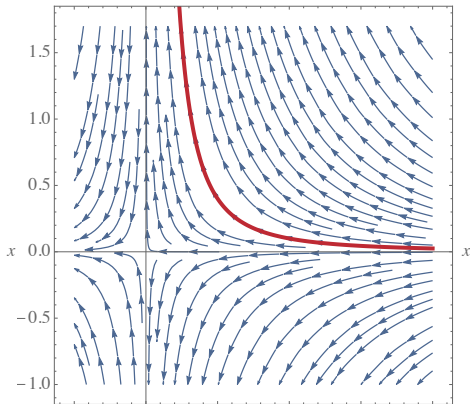
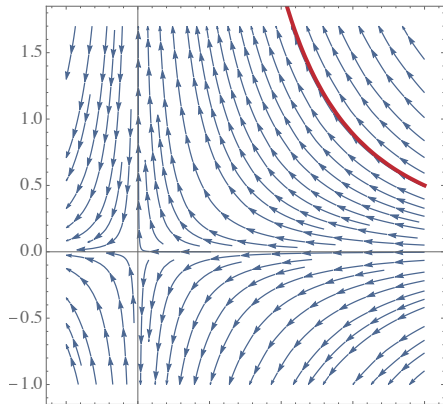




Example Proof

$$\frac{d}{dt} \overline{x^2 y - 2 = 0} \vdash [x' = -x^2, y' = 2xy] x^2 y - 2 = 0$$

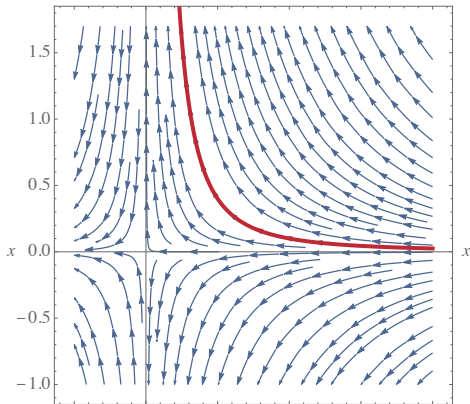
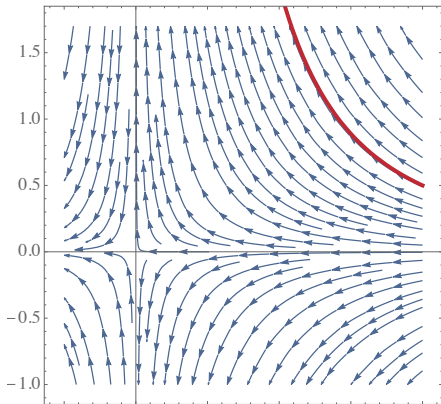
$$\rightarrow_{\mathbb{R}} \vdash x^2 y - 2 = 0 \rightarrow [x' = -x^2, y' = 2xy] x^2 y - 2 = 0$$





Example Proof

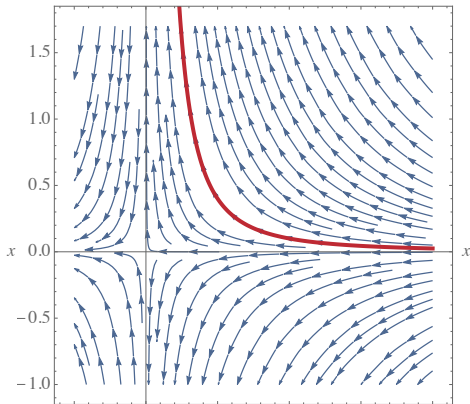
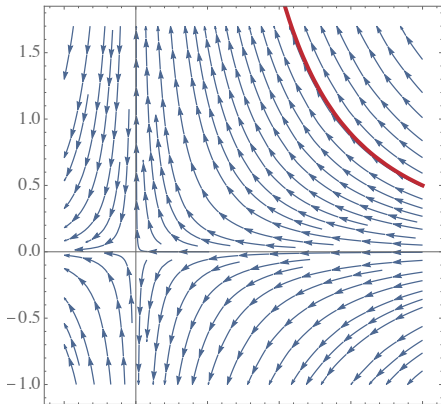
$$\begin{array}{c} \text{[:=]} \\ \hline \vdash [x' := -x^2][y' := 2xy] 2xx'y + x^2y' - 0 = 0 \\ \hline \text{dl} \\ x^2y - 2 = 0 \vdash [x' = -x^2, y' = 2xy] x^2y - 2 = 0 \\ \hline \text{\(\rightarrow\)}_R \\ y \vdash x^2y - 2 = 0 \rightarrow [x' = -x^2, y' = 2xy] x^2y - 2 = 0 \end{array}$$





Example Proof

$$\begin{array}{c} \mathbb{R} \\ \hline \vdash 2x(-x^2)y + x^2(2xy) = 0 \\ \hline [:=] \\ \vdash [x' := -x^2][y' := 2xy] 2xx'y + x^2y' - 0 = 0 \\ \hline \text{dI} \\ x^2y - 2 = 0 \vdash [x' = -x^2, y' = 2xy] x^2y - 2 = 0 \\ \hline \rightarrow \mathbb{R} \\ y \\ \vdash x^2y - 2 = 0 \rightarrow [x' = -x^2, y' = 2xy] x^2y - 2 = 0 \end{array}$$





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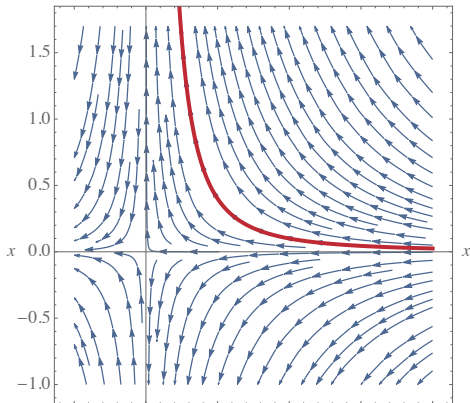
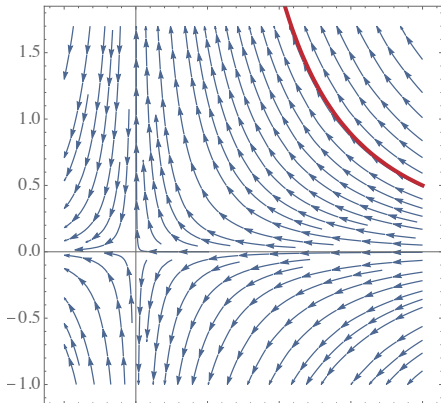
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$$\mathbb{R} \quad \frac{}{\vdash 2x(-x^2)y + x^2(2xy) = 0}$$

$$[:=] \quad \frac{}{\vdash [x' := -x^2][y' := 2xy] 2xx'y + x^2y' - 0 = 0}$$

$$dI \quad \frac{x^2y - 2 = 0}{\vdash [x' = -x^2, y' = 2xy] x^2y - 2 = 0}$$

$$\rightarrow R_y \quad \frac{}{\vdash x^2y - 2 = 0 \rightarrow [x' = -x^2, y' = 2xy] x^2y - 2 = 0}$$





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Lemma (Differential lemma) (Differential value vs. Time-derivative)

If $\varphi \models x' = f(x) \wedge Q$ for duration $r > 0$, then for all $0 \leq z \leq r$, $FV(e) \subseteq \{x\}$:

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$$\omega \llbracket (e)' \rrbracket = \sum_x \omega(x') \frac{\partial \llbracket e \rrbracket}{\partial x}(\omega)$$

Definition (Hybrid program semantics) ($\llbracket \cdot \rrbracket : \text{HP} \rightarrow \wp(\mathcal{S} \times \mathcal{S})$)

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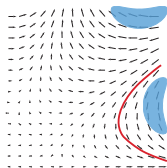
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6

Appendix

- Differential Equations vs. Loops
- Differential Invariant Terms and Invariant Functions

Lemma (Differential equations are their own loop)

$$\llbracket (x' = f(x))^* \rrbracket = \llbracket x' = f(x) \rrbracket$$

loop α^*

repeat any number $n \in \mathbb{N}$ of times

can repeat 0 times

effect depends on previous loop iteration

local generator is loop body α

full global execution trace

unwinding proof by iteration $[*]$

inductive proof with loop invariant

ODE $x' = f(x)$

evolve for any duration $r \in \mathbb{R}$

can evolve for duration 0

effect depends on the past solution

local generator $x' = f(x)$

global solution $\varphi : [0, r] \rightarrow \mathcal{S}$

proof by global solution with $[']$

proof with differential invariant



$$\rightarrow R \quad \frac{}{\vdash x^2 + y^2 = 0 \rightarrow [x' = 4y^3, y' = -4x^3] x^2 + y^2 = 0}$$



$$\text{cut,MR} \frac{x^2 + y^2 = 0 \vdash [x' = 4y^3, y' = -4x^3] x^2 + y^2 = 0}{\rightarrow R \vdash x^2 + y^2 = 0 \rightarrow [x' = 4y^3, y' = -4x^3] x^2 + y^2 = 0}$$



$$\begin{array}{c} \text{dl} \\ \hline x^4 + y^4 = 0 \vdash [x' = 4y^3, y' = -4x^3] x^4 + y^4 = 0 \\ \hline \text{cut,MR} \\ x^2 + y^2 = 0 \vdash [x' = 4y^3, y' = -4x^3] x^2 + y^2 = 0 \\ \hline \rightarrow R \\ \vdash x^2 + y^2 = 0 \rightarrow [x' = 4y^3, y' = -4x^3] x^2 + y^2 = 0 \end{array}$$



$$\begin{array}{c}
 \text{[:=]} \quad \frac{}{\vdash [x':=4y^3][y':=-4x^3](4x^3x' + 4y^3y') = 0} \\
 \text{dl} \quad \frac{x^4 + y^4 = 0 \vdash [x' = 4y^3, y' = -4x^3]x^4 + y^4 = 0}{x^2 + y^2 = 0 \vdash [x' = 4y^3, y' = -4x^3]x^2 + y^2 = 0} \\
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 \rightarrow R \quad \frac{}{\vdash x^2 + y^2 = 0 \rightarrow [x' = 4y^3, y' = -4x^3]x^2 + y^2 = 0}
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$$\begin{array}{c}
 \mathbb{R} \\
 \hline
 \vdash 4x^3(4y^3) + 4y^3(-4x^3) = 0 \\
 \hline
 [:=] \\
 \vdash [x' := 4y^3][y' := -4x^3](4x^3 x' + 4y^3 y') = 0 \\
 \hline
 \text{dl} \\
 x^4 + y^4 = 0 \vdash [x' = 4y^3, y' = -4x^3] x^4 + y^4 = 0 \\
 \hline
 \text{cut, MR} \\
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 dl \quad \frac{x^4 + y^4 = 0 \vdash [x' = 4y^3, y' = -4x^3]x^4 + y^4 = 0}{x^2 + y^2 = 0 \vdash [x' = 4y^3, y' = -4x^3]x^2 + y^2 = 0} \\
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Theorem (Sophus Lie)

$$DI_c \frac{\vdash [x' := f(x)](e)' = 0}{\vdash \forall c (e = c \rightarrow [x' = f(x)]e = c)}$$

premise and conclusion are equivalent on mathematical domains, i.e., connected open sets.



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premise and conclusion are equivalent on mathematical domains, i.e., connected open sets.

Clou: $(e - c)' = (e)'$ independent of additive constants

Stronger Induction Hypotheses

- 1 As usual in math and in proofs with loops:
- 2 Inductive proofs may need stronger induction hypotheses to succeed.
- 3 Differentially inductive proofs may need a stronger differential inductive structure to succeed.
- 4 Even if $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 0\} = \{(x, y) \in \mathbb{R}^2 : x^4 + y^4 = 0\}$ have the same solutions, they have different differential structure.



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