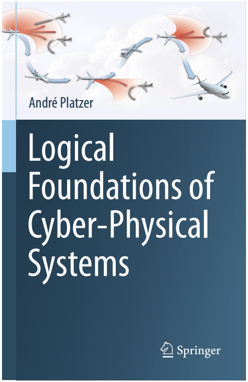
07: Control Loops & Invariants

Logical Foundations of Cyber-Physical Systems



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Department of Informatics

Computer Science Department
Carnegie Mellon University

Outline

1. Learning Objectives
2. Induction for Loops
   - Iteration Axiom
   - Induction Axiom
   - Induction Rule for Loops
   - Loop Invariants
   - Simple Example
   - Contextual Soundness Requirements
3. Operationalize Invariant Construction
   - Bouncing Ball
   - Rescuing Misplaced Constants
   - Safe Quantum
4. Summary

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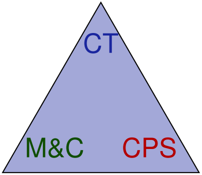
Learning Objectives

rigorous reasoning for repetitions
identifying and expressing invariants
global vs. local reasoning
relating iterations to invariants
finitely accessible infinities
operationalize invariant construction
splitting & generalizations



control loops
feedback mechanisms
dynamics of iteration

semantics of control loops
operational effects of control

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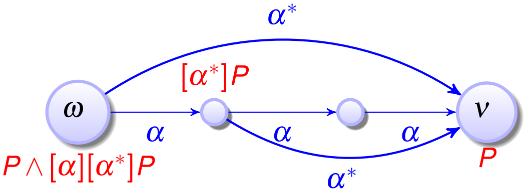
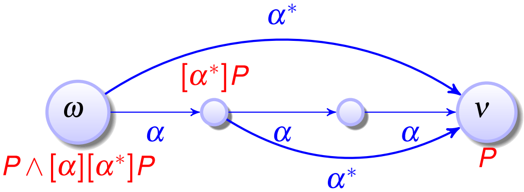
$[^*]\ [\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$



Iteration Axiom

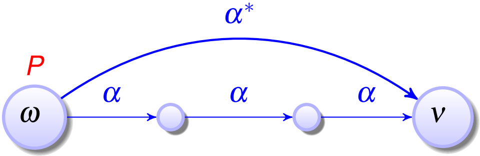
$[^*]$ $[\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$



Problem: Proof for $[\alpha^*]P$ needs proof of $[\alpha][\alpha^*]P$

Lemma ( )

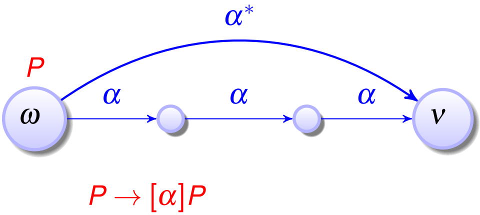
$$\text{I}\ [\alpha^*]P \leftrightarrow P \land$$



Induction Axiom

Lemma ( )

$$\mathrm{I}\ [\alpha^*]P \leftrightarrow P \land \qquad (P \rightarrow [\alpha]P)$$



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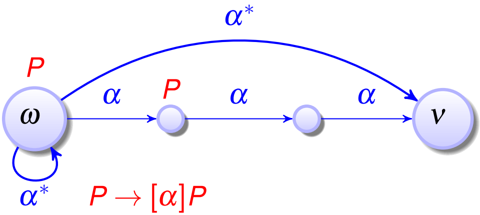
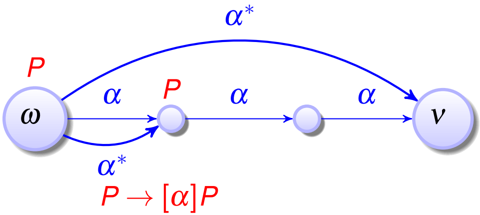


Induction Axiom

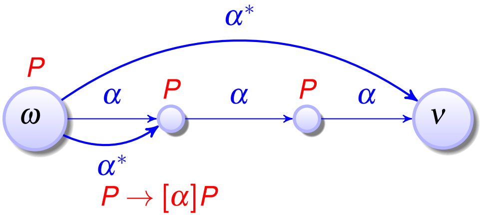
Lemma (I is sound)

$$\text{I}\ [\alpha^*]P \leftrightarrow P \land [\alpha^*](P \to [\alpha]P)$$



Lemma (I is sound)

$$\text{I} \quad [\alpha^*]P \leftrightarrow P \wedge [\alpha^*](P \rightarrow [\alpha]P)$$



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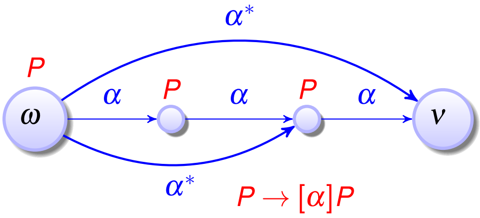

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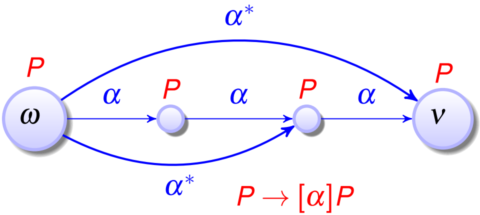

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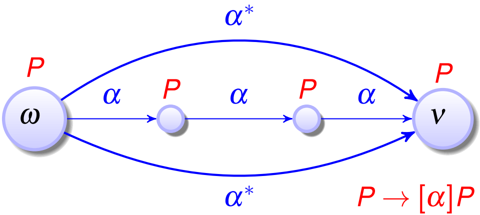

Induction Axiom

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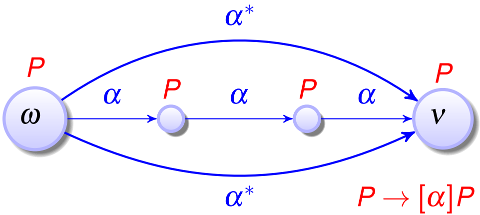

Problem: Inductive proof for $[\alpha^*]P$ needs proof of $[\alpha^*](P \rightarrow [\alpha]P)$

Induction Rule for Loops

Generalize induction step $[\alpha^*](P \rightarrow [\alpha]P)$ by Gödel

$$\text{G} \ \frac{P}{[\alpha]P}$$

Lemma (Loop induction rule ind is sound)

$$\text{ind} \ \frac{P \vdash [\alpha]P}{P \vdash [\alpha^*]P}$$

Induction Rule for Loops

Generalize induction step $[\alpha^*](P \to [\alpha]P)$ by Gödel

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Lemma (Loop induction rule ind is sound)

$$\textit{ind}\ \frac{P \vdash [\alpha]P}{P \vdash [\alpha^*]P}$$

Proof (Derived rule).

$$\text{I}\ \dfrac{\wedge\text{R}\ \dfrac{\text{id}\ \dfrac{*}{P \vdash P} \qquad \text{G}\ \dfrac{\to\text{R}\ \dfrac{P \vdash [\alpha]P}{\vdash P \to [\alpha]P}}{P \vdash [\alpha^*](P \to [\alpha]P)}}{P \vdash P \wedge [\alpha^*](P \to [\alpha]P)}}{P \vdash [\alpha^*]P}$$

□

Induction Rule for Loops

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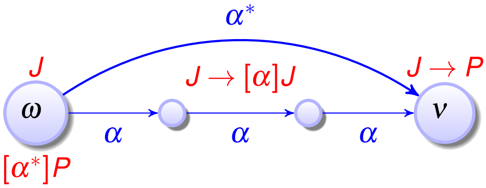
Problem: Use of G in ind may lose information: □
$[\alpha^*](P \to [\alpha]P)$ true in ω but $P \vdash [\alpha]P$ is not valid.

Generalize postcondition to strong loop invariant J by M[·] $\dfrac{P \to Q}{[\alpha]P \to [\alpha]Q}$

Lemma (Loop invariant rule loop is sound)

$$\text{loop}\ \frac{\Gamma \vdash J, \Delta \quad J \vdash [\alpha]J \quad J \vdash P}{\Gamma \vdash [\alpha^*]P, \Delta}$$



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Proof (Derived rule).

$$\text{cut}\ \dfrac{\to\text{R}\ \dfrac{\text{ind}\ \dfrac{J \vdash [\alpha]J}{J \vdash [\alpha^*]J}}{\Gamma \vdash J \to [\alpha^*]J, \Delta} \qquad \to\text{L}\ \dfrac{\Gamma \vdash J, \Delta \qquad \text{M}[\cdot]\ \dfrac{J \vdash P}{[\alpha^*]J \vdash [\alpha^*]P}}{\Gamma, J \to [\alpha^*]J \vdash [\alpha^*]P, \Delta}}{\Gamma \vdash [\alpha^*]P, \Delta}$$

□

Loop Invariants

Generalize postcondition to strong loop invariant J by $\quad \mathrm{M}[\cdot]\ \dfrac{P \to Q}{[\alpha]P \to [\alpha]Q}$

Lemma (Loop invariant rule loop is sound)

$$\text{loop}\ \frac{\Gamma \vdash J, \Delta \quad J \vdash [\alpha]J \quad J \vdash P}{\Gamma \vdash [\alpha^*]P, \Delta}$$

Proof (Derived rule).

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□

Problem: Finding invariant J can be a challenge.

Misplaced $[\alpha^*]$ suggests that J needs to carry along info about α^* history.

A Simple Discrete Loop Example

$$\text{loop}\ \frac{\Gamma \vdash J, \Delta \quad J \vdash [\alpha]J \quad J \vdash P}{\Gamma \vdash [\alpha^*]P, \Delta}$$

$$\text{→R}\ \dfrac{\text{loop}\ \dfrac{x{\geq}8 \land 5{\geq}y \land y{\geq}0 \vdash J \quad J \vdash [x := x+y;\, y := x-2\cdot y]J \quad J \vdash x \geq 0}{x{\geq}8 \land 5{\geq}y \land y{\geq}0 \vdash [(x := x+y;\, y := x-2\cdot y)^*]\, x \geq 0}}{\vdash x{\geq}8 \land 5{\geq}y \land y{\geq}0 \rightarrow [(x := x+y;\, y := x-2\cdot y)^*]\, x \geq 0}$$

1. $J \equiv x \geq 0$

$$\text{loop}\ \frac{\Gamma \vdash J, \Delta \quad J \vdash [\alpha]J \quad J \vdash P}{\Gamma \vdash [\alpha^*]P, \Delta}$$

$$\to\text{R}\ \dfrac{\text{loop}\ \dfrac{x{\geq}8 \land 5{\geq}y \land y{\geq}0 \vdash J \quad J \vdash [x := x+y;\, y := x-2\cdot y]J \quad J \vdash x \geq 0}{x{\geq}8 \land 5{\geq}y \land y{\geq}0 \vdash [(x := x+y;\, y := x-2\cdot y)^*]\, x \geq 0}}{\vdash x{\geq}8 \land 5{\geq}y \land y{\geq}0 \to [(x := x+y;\, y := x-2\cdot y)^*]\, x \geq 0}$$

1. $J \equiv x \geq 0$ stronger: Lacks info about y

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1. $J \equiv x \geq 0$ stronger: Lacks info about y
2. $J \equiv x \geq 8 \land 5 \geq y \land y \geq 0$

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1. $J \equiv x \geq 0$ — stronger: Lacks info about y
2. $J \equiv x \geq 8 \land 5 \geq y \land y \geq 0$ — weaker: Changes immediately

A Simple Discrete Loop Example

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4. $J \equiv x \geq y \land y \geq 0$

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1. $J \equiv x \geq 0$ — stronger: Lacks info about y
2. $J \equiv x \geq 8 \land 5 \geq y \land y \geq 0$ — weaker: Changes immediately
3. $J \equiv x \geq 0 \land y \geq 0$ — no: y may become negative if $x < y$
4. $J \equiv x \geq y \land y \geq 0$ — correct loop invariant

Forgot to Add Sequent Context Γ, Δ to Premises

$$\frac{\Gamma \vdash J, \Delta \quad \Gamma??, J \vdash [\alpha]J, \Delta?? \quad \Gamma??, J \vdash P, \Delta??}{\Gamma \vdash [\alpha^*]P, \Delta}$$

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$$↯\ \frac{x = 0 \vdash x \leq 1 \quad x = 0, x \leq 1 \vdash [x := x + 1]x \leq 1 \quad x \leq 1 \vdash x \leq 1}{x = 0, x \leq 1 \vdash [(x := x + 1)^*]x \leq 1}$$

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Unsound! Be careful where your assumptions go,
or your CPS might go where it shouldn't.

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$$\frac{}{A \vdash [\big(\text{grav}; (?x{=}0; v{:=}{-}cv \cup ?x{\neq}0)\big)^*]B(x,v)}$$

$$A \equiv 0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0$$

$$B(x,v) \equiv 0 \le x \land x \le H$$

$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$$\text{loop}\;\frac{A \vdash j(x,v) \qquad \dfrac{}{j(x,v) \vdash [\text{grav}; (?x{=}0; v{:=}{-cv} \cup ?x{\neq}0)]j(x,v)} \qquad j(x,v) \vdash B(x,v)}{A \vdash [\big(\text{grav}; (?x{=}0; v{:=}{-cv} \cup ?x{\neq}0)\big)^*]B(x,v)}$$

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$$
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$$A \equiv 0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0$$

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$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$$
\dfrac{A \vdash \mathrm{j}(x,v) \qquad \dfrac{\dfrac{\mathrm{j}(x,v) \vdash [\mathrm{grav}]\mathrm{j}(x,v) \quad \text{[}\cup\text{]}\dfrac{\dfrac{}{\mathrm{j}(x,v) \vdash [?x{=}0; v{:=}{-}cv]\mathrm{j}(x,v) \land [?x{\neq}0]\mathrm{j}(x,v)}}{\mathrm{j}(x,v) \vdash [?x{=}0; v{:=}{-}cv \cup ?x{\neq}0]\mathrm{j}(x,v)}}{\mathrm{j}(x,v) \vdash [\mathrm{grav}][?x{=}0; v{:=}{-}cv \cup ?x{\neq}0]\mathrm{j}(x,v)}\,\text{MR}}{\dfrac{\mathrm{j}(x,v) \vdash [\mathrm{grav}; (?x{=}0; v{:=}{-}cv \cup ?x{\neq}0)]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \vdash [\mathrm{grav}; (?x{=}0; v{:=}{-}cv \cup ?x{\neq}0)]\mathrm{j}(x,v)}}\,\text{[;]} \qquad \mathrm{j}(x,v) \vdash B(x,v)}{A \vdash \left[\big(\mathrm{grav}; (?x{=}0; v{:=}{-}cv \cup ?x{\neq}0)\big)^*\right]B(x,v)}\,\text{loop}
$$

$$A \equiv 0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0$$

$$B(x,v) \equiv 0 \le x \land x \le H$$

$$\mathrm{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$$\text{loop}\;\dfrac{A \vdash j(x,v) \qquad \dfrac{\text{[;]}\;\dfrac{\text{MR}\;\dfrac{j(x,v) \vdash [\text{grav}]j(x,v) \qquad \text{[}\cup\text{]}\;\dfrac{\land\text{R}\;\dfrac{\dfrac{}{j(x,v) \vdash [?x{=}0;\, v{:=}{-}cv]j(x,v)} \qquad \dfrac{}{j(x,v) \vdash [?x{\neq}0]j(x,v)}}{j(x,v) \vdash [?x{=}0;\, v{:=}{-}cv]j(x,v) \land [?x{\neq}0]j(x,v)}}{j(x,v) \vdash [?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}}{j(x,v) \vdash [\text{grav}][?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}}{j(x,v) \vdash [\text{grav};(?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0)]j(x,v)}}{j(x,v) \vdash [\text{grav};(?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0)]j(x,v)} \qquad j(x,v) \vdash B(x,v)}{A \vdash \left[\left(\text{grav};(?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0)\right)^*\right]B(x,v)}$$

$$A \equiv 0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0$$

$$B(x,v) \equiv 0 \le x \land x \le H$$

$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$$
\text{loop}\;\dfrac{A \vdash j(x,v) \qquad \dfrac{\text{[;]}\;\dfrac{\text{MR}\;\dfrac{j(x,v) \vdash [grav]j(x,v) \qquad \text{[}\cup\text{]}\;\dfrac{\wedge R\;\dfrac{\text{[;]}\;\dfrac{\overline{j(x,v) \vdash [?x{=}0][v{:=}{-}cv]j(x,v)}}{j(x,v) \vdash [?x{=}0{\color{red};}\, v{:=}{-}cv]j(x,v)} \qquad \overline{j(x,v) \vdash [?x{\neq}0]j(x,v)}}{j(x,v) \vdash [?x{=}0;\, v{:=}{-}cv]j(x,v) \wedge [?x{\neq}0]j(x,v)}}{j(x,v) \vdash [?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}}{j(x,v) \vdash [grav][?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}}{j(x,v) \vdash [grav;(?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0)]j(x,v)}}{j(x,v) \vdash [grav;(?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0)]j(x,v)} \qquad j(x,v) \vdash B(x,v)}{A \vdash \left[\left(grav;(?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0)\right)^*\right]B(x,v)}
$$

$$
\begin{aligned}
A &\equiv 0 \le x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \ge c \ge 0\\
B(x,v) &\equiv 0 \le x \wedge x \le H\\
grav &\equiv \{x' = v, v' = -g \,\&\, x \ge 0\}
\end{aligned}
$$

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$$
\text{loop}\;\dfrac{A \vdash \mathrm{j}(x,v) \qquad \text{[;]}\,\dfrac{\text{MR}\,\dfrac{\mathrm{j}(x,v) \vdash [grav]\mathrm{j}(x,v) \qquad [\cup]\,\dfrac{\wedge R\,\dfrac{\text{[;]}\,\dfrac{[?],\!\to\! R\,\dfrac{\overline{\mathrm{j}(x,v),\, x{=}0 \vdash [v{:=}{-cv}]\mathrm{j}(x,v)}}{\mathrm{j}(x,v) \vdash [{\color{red}?x{=}0}][v{:=}{-cv}]\mathrm{j}(x,v)}}{\mathrm{j}(x,v) \vdash [?x{=}0;\, v{:=}{-cv}]\mathrm{j}(x,v)} \qquad \overline{\mathrm{j}(x,v) \vdash [?x{\neq}0]\mathrm{j}(x,v)}}{\mathrm{j}(x,v) \vdash [?x{=}0;\, v{:=}{-cv}]\mathrm{j}(x,v) \wedge [?x{\neq}0]\mathrm{j}(x,v)}}{\mathrm{j}(x,v) \vdash [?x{=}0;\, v{:=}{-cv} \cup ?x{\neq}0]\mathrm{j}(x,v)}}{\mathrm{j}(x,v) \vdash [grav][?x{=}0;\, v{:=}{-cv} \cup ?x{\neq}0]\mathrm{j}(x,v)}}{\dfrac{\mathrm{j}(x,v) \vdash [grav;(?x{=}0;\, v{:=}{-cv} \cup ?x{\neq}0)]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \vdash [grav;(?x{=}0;\, v{:=}{-cv} \cup ?x{\neq}0)]\mathrm{j}(x,v)}} \qquad \mathrm{j}(x,v) \vdash B(x,v)}{A \vdash \big[\big(grav;(?x{=}0;\, v{:=}{-cv} \cup ?x{\neq}0)\big)^*\big]B(x,v)}
$$

$$A \equiv 0 \le x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \ge c \ge 0$$

$$B(x,v) \equiv 0 \le x \wedge x \le H$$

$$grav \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$$
\dfrac{
  A \vdash \text{j}(x,v)
  \quad
  \text{[;]}\dfrac{
    \text{MR}\dfrac{
      \text{j}(x,v) \vdash [\text{grav}]\text{j}(x,v)
      \quad
      [\cup]\dfrac{
        \wedge\text{R}\dfrac{
          \text{[;]}\dfrac{
            \text{[?]},\to\text{R}\dfrac{
              \text{[:=]}\dfrac{\text{j}(x,v), x{=}0 \vdash \text{j}(x,-cv)}{\text{j}(x,v), x{=}0 \vdash [v{:=}-cv]\text{j}(x,v)}
            }{\text{j}(x,v) \vdash [?x{=}0][v{:=}-cv]\text{j}(x,v)}
          }{\text{j}(x,v) \vdash [?x{=}0; v{:=}-cv]\text{j}(x,v)}
          \quad
          \dfrac{}{\text{j}(x,v) \vdash [?x{\neq}0]\text{j}(x,v)}
        }{\text{j}(x,v) \vdash [?x{=}0; v{:=}-cv]\text{j}(x,v) \wedge [?x{\neq}0]\text{j}(x,v)}
      }{\text{j}(x,v) \vdash [?x{=}0; v{:=}-cv \cup ?x{\neq}0]\text{j}(x,v)}
    }{\text{j}(x,v) \vdash [\text{grav}][?x{=}0; v{:=}-cv \cup ?x{\neq}0]\text{j}(x,v)}
  }{
    \dfrac{\text{j}(x,v) \vdash [\text{grav}; (?x{=}0; v{:=}-cv \cup ?x{\neq}0)]\text{j}(x,v)}{\text{j}(x,v) \vdash [\text{grav}; (?x{=}0; v{:=}-cv \cup ?x{\neq}0)]\text{j}(x,v)}
  }
  \quad
  \text{j}(x,v) \vdash B(x,v)
}{A \vdash [(\text{grav}; (?x{=}0; v{:=}-cv \cup ?x{\neq}0))^*]B(x,v)}\ \text{loop}
$$

$$A \equiv 0 \le x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \ge c \ge 0$$

$$B(x,v) \equiv 0 \le x \wedge x \le H$$

$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$$[:=]\ \frac{\mathrm{j}(x,v),\, x{=}0 \vdash \mathrm{j}(x,-cv)}{\mathrm{j}(x,v),\, x{=}0 \vdash [v:=-cv]\mathrm{j}(x,v)}$$

$$[?],\to\mathrm{R}\ \frac{\mathrm{j}(x,v),\, x{=}0 \vdash [v:=-cv]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \vdash [?x{=}0][v:=-cv]\mathrm{j}(x,v)}$$

$$[;]\ \frac{\mathrm{j}(x,v) \vdash [?x{=}0][v:=-cv]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \vdash [?x{=}0;\, v:=-cv]\mathrm{j}(x,v)}
\qquad
[?]\ \frac{\mathrm{j}(x,v),\, x{\neq}0 \vdash \mathrm{j}(x,v)}{\mathrm{j}(x,v) \vdash [?x{\neq}0]\mathrm{j}(x,v)}$$

$$\wedge\mathrm{R}\ \frac{\mathrm{j}(x,v) \vdash [?x{=}0;\, v:=-cv]\mathrm{j}(x,v) \qquad \mathrm{j}(x,v) \vdash [?x{\neq}0]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \vdash [?x{=}0;\, v:=-cv]\mathrm{j}(x,v) \wedge [?x{\neq}0]\mathrm{j}(x,v)}$$

$$[\cup]\ \frac{\mathrm{j}(x,v) \vdash [?x{=}0;\, v:=-cv]\mathrm{j}(x,v) \wedge [?x{\neq}0]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \vdash [?x{=}0;\, v:=-cv \cup ?x{\neq}0]\mathrm{j}(x,v)}$$

$$\mathrm{MR}\ \frac{\mathrm{j}(x,v) \vdash [\mathrm{grav}]\mathrm{j}(x,v) \qquad \mathrm{j}(x,v) \vdash [?x{=}0;\, v:=-cv \cup ?x{\neq}0]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \vdash [\mathrm{grav}][?x{=}0;\, v:=-cv \cup ?x{\neq}0]\mathrm{j}(x,v)}$$

$$[;]\ \frac{\mathrm{j}(x,v) \vdash [\mathrm{grav}][?x{=}0;\, v:=-cv \cup ?x{\neq}0]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \vdash [\mathrm{grav};(?x{=}0;\, v:=-cv \cup ?x{\neq}0)]\mathrm{j}(x,v)}$$

$$\mathrm{loop}\ \frac{A \vdash \mathrm{j}(x,v) \qquad \mathrm{j}(x,v) \vdash [\mathrm{grav};(?x{=}0;\, v:=-cv \cup ?x{\neq}0)]\mathrm{j}(x,v) \qquad \mathrm{j}(x,v) \vdash B(x,v)}{A \vdash \left[\left(\mathrm{grav};(?x{=}0;\, v:=-cv \cup ?x{\neq}0)\right)^*\right]B(x,v)}$$

$$A \equiv 0 \le x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \ge c \ge 0$$

$$B(x,v) \equiv 0 \le x \wedge x \le H$$

$$\mathrm{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$$
\text{loop}\;\dfrac{A \vdash j(x,v) \qquad \text{[;]}\;\dfrac{\text{MR}\;\dfrac{j(x,v) \vdash [grav]j(x,v) \qquad [\cup]\;\dfrac{\wedge R\;\dfrac{[;]\;\dfrac{[?],\to R\;\dfrac{[:=]\;\dfrac{j(x,v), x{=}0 \vdash j(x,-cv)}{j(x,v), x{=}0 \vdash [v:=-cv]j(x,v)}}{j(x,v) \vdash [?x{=}0][v:=-cv]j(x,v)}}{j(x,v) \vdash [?x{=}0;\, v:=-cv]j(x,v)} \qquad [?]\;\dfrac{j(x,v), x{\neq}0 \vdash j(x,v)}{j(x,v) \vdash [?x{\neq}0]j(x,v)}}{j(x,v) \vdash [?x{=}0;\, v:=-cv]j(x,v) \wedge [?x{\neq}0]j(x,v)}}{j(x,v) \vdash [?x{=}0;\, v:=-cv \cup ?x{\neq}0]j(x,v)}}{j(x,v) \vdash [grav][?x{=}0;\, v:=-cv \cup ?x{\neq}0]j(x,v)}}{j(x,v) \vdash [grav;(?x{=}0;\, v:=-cv \cup ?x{\neq}0)]j(x,v)} \qquad j(x,v) \vdash B(x,v)}{A \vdash [\big(grav;(?x{=}0;\, v:=-cv \cup ?x{\neq}0)\big)^*]B(x,v)}
$$

$$A \equiv 0 \le x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \ge c \ge 0$$

$$B(x,v) \equiv 0 \le x \wedge x \le H$$

$$grav \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$A \vdash \mathrm{j}(x,v)$

$\mathrm{j}(x,v) \vdash [\mathrm{grav}](\mathrm{j}(x,v))$

$\mathrm{j}(x,v), x{=}0 \vdash \mathrm{j}(x,(-cv))$

$\mathrm{j}(x,v), x{\neq}0 \vdash \mathrm{j}(x,v)$

$\mathrm{j}(x,v) \vdash B(x,v)$

$$A \equiv 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \land x \leq H$$

$$\mathrm{grav} \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

$0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0 \vdash j(x,v)$

$j(x,v) \vdash [\{x'=v, v'=-g \,\&\, x\ge 0\}](j(x,v))$

$j(x,v), x=0 \vdash j(x,(-cv))$

$j(x,v), x\neq 0 \vdash j(x,v)$

$j(x,v) \vdash 0 \le x \land x \le H$

$$A \equiv 0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0$$

$$B(x,v) \equiv 0 \le x \land x \le H$$

$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$$0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0 \vdash j(x,v)$$
$$j(x,v) \vdash [\{x'=v, v'=-g \,\&\, x\ge 0\}](j(x,v))$$
$$j(x,v), x=0 \vdash j(x,(-cv))$$
$$j(x,v), x\neq 0 \vdash j(x,v)$$
$$j(x,v) \vdash 0 \le x \land x \le H$$

2. $j(x,v) \equiv 0 \le x \land x \le H$

$$A \equiv 0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0$$
$$B(x,v) \equiv 0 \le x \land x \le H$$
$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \vdash j(x,v)$

$j(x,v) \vdash [\{x'=v, v'=-g \,\&\, x\geq 0\}](j(x,v))$

$j(x,v), x=0 \vdash j(x,(-cv))$

$j(x,v), x\neq 0 \vdash j(x,v)$

$j(x,v) \vdash 0 \leq x \wedge x \leq H$

2. $j(x,v) \equiv 0 \leq x \wedge x \leq H$ — weak: fails ODE if $v \gg 0$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$

$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

$0 \le x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \ge c \ge 0 \vdash \mathrm{j}(x,v)$

$\mathrm{j}(x,v) \vdash [\{x'=v, v'=-g \,\&\, x \ge 0\}](\mathrm{j}(x,v))$

$\mathrm{j}(x,v), x=0 \vdash \mathrm{j}(x,(-cv))$

$\mathrm{j}(x,v), x \ne 0 \vdash \mathrm{j}(x,v)$

$\mathrm{j}(x,v) \vdash 0 \le x \wedge x \le H$

1. $\mathrm{j}(x,v) \equiv x \ge 0$
2. $\mathrm{j}(x,v) \equiv 0 \le x \wedge x \le H$ weak: fails ODE if $v \gg 0$

$$A \equiv 0 \le x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \ge c \ge 0$$
$$B(x,v) \equiv 0 \le x \wedge x \le H$$
$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0 \vdash \mathrm{j}(x,v)$

$\mathrm{j}(x,v) \vdash [\{x'=v, v'=-g \,\&\, x \ge 0\}](\mathrm{j}(x,v))$

$\mathrm{j}(x,v), x=0 \vdash \mathrm{j}(x,(-cv))$

$\mathrm{j}(x,v), x \ne 0 \vdash \mathrm{j}(x,v)$

$\mathrm{j}(x,v) \vdash 0 \le x \land x \le H$

1. $\mathrm{j}(x,v) \equiv x \ge 0$ — weaker: fails postcondition if $x > H$
2. $\mathrm{j}(x,v) \equiv 0 \le x \land x \le H$ — weak: fails ODE if $v \gg 0$

$$A \equiv 0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0$$

$$B(x,v) \equiv 0 \le x \land x \le H$$

$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$0 \le x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \ge c \ge 0 \vdash j(x,v)$

$j(x,v) \vdash [\{x'=v, v'=-g \,\&\, x \ge 0\}](j(x,v))$

$j(x,v), x=0 \vdash j(x,(-cv))$

$j(x,v), x \ne 0 \vdash j(x,v)$

$j(x,v) \vdash 0 \le x \wedge x \le H$

1. $j(x,v) \equiv x \ge 0$ — weaker: fails postcondition if $x > H$
2. $j(x,v) \equiv 0 \le x \wedge x \le H$ — weak: fails ODE if $v \gg 0$
3. $j(x,v) \equiv x = 0 \wedge v = 0$

$$A \equiv 0 \le x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \ge c \ge 0$$

$$B(x,v) \equiv 0 \le x \wedge x \le H$$

$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0 \vdash \mathrm{j}(x,v)$

$\mathrm{j}(x,v) \vdash [\{x'=v, v'=-g \,\&\, x\geq 0\}](\mathrm{j}(x,v))$

$\mathrm{j}(x,v), x=0 \vdash \mathrm{j}(x,(-cv))$

$\mathrm{j}(x,v), x\neq 0 \vdash \mathrm{j}(x,v)$

$\mathrm{j}(x,v) \vdash 0 \leq x \land x \leq H$

1. $\mathrm{j}(x,v) \equiv x \geq 0$ — weaker: fails postcondition if $x > H$
2. $\mathrm{j}(x,v) \equiv 0 \leq x \land x \leq H$ — weak: fails ODE if $v \gg 0$
3. $\mathrm{j}(x,v) \equiv x = 0 \land v = 0$ — strong: fails initial condition if $x > 0$

$$A \equiv 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \land x \leq H$$

$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

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$$0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0 \vdash \mathrm{j}(x,v)$$

$$\mathrm{j}(x,v) \vdash [\{x'=v, v'=-g \,\&\, x\ge 0\}](\mathrm{j}(x,v))$$

$$\mathrm{j}(x,v), x=0 \vdash \mathrm{j}(x,(-cv))$$

$$\mathrm{j}(x,v), x\ne 0 \vdash \mathrm{j}(x,v)$$

$$\mathrm{j}(x,v) \vdash 0 \le x \land x \le H$$

1. $\mathrm{j}(x,v) \equiv x \ge 0$ — weaker: fails postcondition if $x > H$
2. $\mathrm{j}(x,v) \equiv 0 \le x \land x \le H$ — weak: fails ODE if $v \gg 0$
3. $\mathrm{j}(x,v) \equiv x = 0 \land v = 0$ — strong: fails initial condition if $x > 0$
4. $\mathrm{j}(x,v) \equiv x = 0 \lor x = H \land v = 0$

$$A \equiv 0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0$$

$$B(x,v) \equiv 0 \le x \land x \le H$$

$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0 \vdash j(x,v)$

$j(x,v) \vdash [\{x'=v, v'=-g \,\&\, x\ge 0\}](j(x,v))$

$j(x,v), x=0 \vdash j(x,(-cv))$

$j(x,v), x\neq 0 \vdash j(x,v)$

$j(x,v) \vdash 0 \le x \land x \le H$

1. $j(x,v) \equiv x \ge 0$ — weaker: fails postcondition if $x > H$
2. $j(x,v) \equiv 0 \le x \land x \le H$ — weak: fails ODE if $v \gg 0$
3. $j(x,v) \equiv x = 0 \land v = 0$ — strong: fails initial condition if $x > 0$
4. $j(x,v) \equiv x = 0 \lor x = H \land v = 0$ — no space for intermediate states

$$A \equiv 0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0$$

$$B(x,v) \equiv 0 \le x \land x \le H$$

$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0 \vdash \mathrm{j}(x,v)$

$\mathrm{j}(x,v) \vdash [\{x'=v, v'=-g \,\&\, x \ge 0\}](\mathrm{j}(x,v))$

$\mathrm{j}(x,v), x=0 \vdash \mathrm{j}(x,(-cv))$

$\mathrm{j}(x,v), x \ne 0 \vdash \mathrm{j}(x,v)$

$\mathrm{j}(x,v) \vdash 0 \le x \land x \le H$

1. $\mathrm{j}(x,v) \equiv x \ge 0$ — weaker: fails postcondition if $x > H$
2. $\mathrm{j}(x,v) \equiv 0 \le x \land x \le H$ — weak: fails ODE if $v \gg 0$
3. $\mathrm{j}(x,v) \equiv x = 0 \land v = 0$ — strong: fails initial condition if $x > 0$
4. $\mathrm{j}(x,v) \equiv x = 0 \lor x = H \land v = 0$ — no space for intermediate states
5. $\mathrm{j}(x,v) \equiv 2gx = 2gH - v^2 \land x \ge 0$

$$A \equiv 0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0$$

$$B(x,v) \equiv 0 \le x \land x \le H$$

$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

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$$0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0 \vdash \mathrm{j}(x,v)$$
$$\mathrm{j}(x,v) \vdash [\{x'=v, v'=-g \,\&\, x\ge 0\}](\mathrm{j}(x,v))$$
$$\mathrm{j}(x,v), x=0 \vdash \mathrm{j}(x,(-cv))$$
$$\mathrm{j}(x,v), x\ne 0 \vdash \mathrm{j}(x,v)$$
$$\mathrm{j}(x,v) \vdash 0 \le x \land x \le H$$

1. $\mathrm{j}(x,v) \equiv x \ge 0$ — weaker: fails postcondition if $x > H$
2. $\mathrm{j}(x,v) \equiv 0 \le x \land x \le H$ — weak: fails ODE if $v \gg 0$
3. $\mathrm{j}(x,v) \equiv x = 0 \land v = 0$ — strong: fails initial condition if $x > 0$
4. $\mathrm{j}(x,v) \equiv x = 0 \lor x = H \land v = 0$ — no space for intermediate states
5. $\mathrm{j}(x,v) \equiv 2gx=2gH-v^2 \land x\ge 0$ — works: implicitly links v and x

$$A \equiv 0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0$$
$$B(x,v) \equiv 0 \le x \land x \le H$$
$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$$0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0 \vdash 2gx{=}2gH{-}v^2 \land x{\geq}0$$
$$2gx{=}2gH{-}v^2 \land x{\geq}0 \vdash [\{x'{=}v, v'{=}{-}g \,\&\, x{\geq}0\}](2gx{=}2gH{-}v^2 \land x{\geq}0)$$
$$2gx{=}2gH{-}v^2 \land x{\geq}0, x{=}0 \vdash 2gx{=}2gH{-}({-}cv)^2 \land x{\geq}0$$
$$2gx{=}2gH{-}v^2 \land x{\geq}0, x{\neq}0 \vdash 2gx{=}2gH{-}v^2 \land x{\geq}0$$
$$2gx{=}2gH{-}v^2 \land x{\geq}0 \vdash 0 \leq x \land x \leq H$$

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$x(t) = H - \frac{g}{2}t^2$ $v(t) = -gt$

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$$x(t) = H - \frac{g}{2}t^2 \rightsquigarrow 2gx(t) = 2gH - g^2t^2 \qquad v(t)^2 = g^2t^2 \leftsquigarrow v(t) = -gt$$

$$[']\quad \frac{}{j(x,v) \vdash [x'=v, v'=-g \,\&\, x\geq 0]j(x,v)}$$

$$\dfrac{\dfrac{}{\text{j}(x,v) \vdash \forall t{\geq}0\,[x{:=}H-\frac{g}{2}t^2;\, v{:=}-gt]\big(x{\geq}0 \rightarrow \text{j}(x,v)\big)}\;[;]}{\text{j}(x,v) \vdash [x'{=}v, v'{=}-g\,\&\, x{\geq}0]\text{j}(x,v)}\;[']$$

$$\dfrac{\dfrac{\dfrac{}{j(x,v) \vdash \forall t{\geq}0\,[x:=H-\frac{g}{2}t^2][v:=-gt]\big(x{\geq}0 \rightarrow j(x,v)\big)}[:=]}{j(x,v) \vdash \forall t{\geq}0\,[x:=H-\frac{g}{2}t^2;\, v:=-gt]\big(x{\geq}0 \rightarrow j(x,v)\big)}[;]}{j(x,v) \vdash [x'=v, v'=-g\,\&\,x{\geq}0]j(x,v)}['\,]$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{j(x,v) \vdash \forall t{\geq}0\, [x:=H-\frac{g}{2}t^2](x\geq 0 \rightarrow j(x,-gt))}[:=]}{j(x,v) \vdash \forall t{\geq}0\, [x:=H-\frac{g}{2}t^2][v:=-gt](x\geq 0 \rightarrow j(x,v))}[:=]}{j(x,v) \vdash \forall t{\geq}0\, [x:=H-\frac{g}{2}t^2;\, v:=-gt](x\geq 0 \rightarrow j(x,v))}[;]}{j(x,v) \vdash [x'=v, v'=-g \,\&\, x\geq 0]j(x,v)}[']
$$

$$\forall R \quad \dfrac{}{j(x,v) \vdash \forall t{\geq}0\, \big(H-\frac{g}{2}t^2 \geq 0 \to j(H-\frac{g}{2}t^2, -gt)\big)}$$

$$[:=] \quad \dfrac{}{j(x,v) \vdash \forall t{\geq}0\, [x:=H-\frac{g}{2}t^2]\big(x\geq 0 \to j(x,-gt)\big)}$$

$$[:=] \quad \dfrac{}{j(x,v) \vdash \forall t{\geq}0\, [x:=H-\frac{g}{2}t^2][v:=-gt]\big(x\geq 0 \to j(x,v)\big)}$$

$$[;] \quad \dfrac{}{j(x,v) \vdash \forall t{\geq}0\, [x:=H-\frac{g}{2}t^2; v:=-gt]\big(x\geq 0 \to j(x,v)\big)}$$

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$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{j(x,v) \vdash t\geq 0 \rightarrow H-\frac{g}{2}t^2\geq 0 \rightarrow j(H-\frac{g}{2}t^2,-gt)}\rightarrow R}{j(x,v) \vdash \forall t\geq 0\, (H-\frac{g}{2}t^2\geq 0 \rightarrow j(H-\frac{g}{2}t^2,-gt))}\forall R}{j(x,v) \vdash \forall t\geq 0\, [x:=H-\frac{g}{2}t^2](x\geq 0 \rightarrow j(x,-gt))}[:=]}{j(x,v) \vdash \forall t\geq 0\, [x:=H-\frac{g}{2}t^2][v:=-gt](x\geq 0 \rightarrow j(x,v))}[:=]}{j(x,v) \vdash \forall t\geq 0\, [x:=H-\frac{g}{2}t^2; v:=-gt](x\geq 0 \rightarrow j(x,v))}[;]}{j(x,v) \vdash [x'=v, v'=-g \,\&\, x\geq 0]j(x,v)}[']
$$

$$
\begin{array}{ll}
 & j(x,v), t\geq 0, H-\frac{g}{2}t^2\geq 0 \vdash j(H-\frac{g}{2}t^2,-gt) \\
\hline
\rightarrow\text{R} & j(x,v) \vdash t\geq 0 \rightarrow H-\frac{g}{2}t^2\geq 0 \rightarrow j(H-\frac{g}{2}t^2,-gt) \\
\hline
\forall\text{R} & j(x,v) \vdash \forall t\geq 0\,\big(H-\frac{g}{2}t^2\geq 0 \rightarrow j(H-\frac{g}{2}t^2,-gt)\big) \\
\hline
[:=] & j(x,v) \vdash \forall t\geq 0\,[x:=H-\frac{g}{2}t^2]\big(x\geq 0 \rightarrow j(x,-gt)\big) \\
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\hline
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\end{array}
$$

$$j(x,v) \equiv 2gx = 2gH - v^2 \land x \geq 0$$

$$\dfrac{}{2gx=2gH-v^2 \land x\geq 0, H-\frac{g}{2}t^2\geq 0 \vdash 2g(H-\frac{g}{2}t^2)=2gH-(gt)^2 \land (H-\frac{g}{2}t^2)\geq 0}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{j(x,v), t\geq 0, H-\frac{g}{2}t^2\geq 0 \vdash j(H-\frac{g}{2}t^2, -gt)}{j(x,v) \vdash t\geq 0 \to H-\frac{g}{2}t^2\geq 0 \to j(H-\frac{g}{2}t^2,-gt)}\ \to\!R}{j(x,v) \vdash \forall t\geq 0\, (H-\frac{g}{2}t^2\geq 0 \to j(H-\frac{g}{2}t^2,-gt))}\ \forall R}{j(x,v) \vdash \forall t\geq 0\, [x:=H-\frac{g}{2}t^2](x\geq 0 \to j(x,-gt))}\ [:=]}{j(x,v) \vdash \forall t\geq 0\, [x:=H-\frac{g}{2}t^2][v:=-gt](x\geq 0 \to j(x,v))}\ [:=]}{j(x,v) \vdash \forall t\geq 0\, [x:=H-\frac{g}{2}t^2; v:=-gt](x\geq 0 \to j(x,v))}\ [;]}{j(x,v) \vdash [x'=v, v'=-g \,\&\, x\geq 0]j(x,v)}\ [']$$

$$\dfrac{\dfrac{}{2gx=2gH-v^2 \vdash 2g(H-\frac{g}{2}t^2)=2gH-(gt)^2} \quad \dfrac{}{H-\frac{g}{2}t^2\geq 0 \vdash H-\frac{g}{2}t^2\geq 0}}{2gx=2gH-v^2\wedge x\geq 0, H-\frac{g}{2}t^2\geq 0 \vdash 2g(H-\frac{g}{2}t^2)=2gH-(gt)^2 \wedge (H-\frac{g}{2}t^2)\geq 0}\ \wedge\mathrm{R}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{j(x,v), t\geq 0, H-\frac{g}{2}t^2\geq 0 \vdash j(H-\frac{g}{2}t^2, -gt)}{j(x,v) \vdash t\geq 0 \rightarrow H-\frac{g}{2}t^2\geq 0 \rightarrow j(H-\frac{g}{2}t^2, -gt)}\ \rightarrow\mathrm{R}}{j(x,v) \vdash \forall t\geq 0\, (H-\frac{g}{2}t^2\geq 0 \rightarrow j(H-\frac{g}{2}t^2, -gt))}\ \forall\mathrm{R}}{j(x,v) \vdash \forall t\geq 0\, [x:=H-\frac{g}{2}t^2](x\geq 0 \rightarrow j(x,-gt))}\ [:=]}{j(x,v) \vdash \forall t\geq 0\, [x:=H-\frac{g}{2}t^2][v:=-gt](x\geq 0 \rightarrow j(x,v))}\ [:=]}{j(x,v) \vdash \forall t\geq 0\, [x:=H-\frac{g}{2}t^2; v:=-gt](x\geq 0 \rightarrow j(x,v))}\ [;]}{j(x,v) \vdash [x'=v, v'=-g \,\&\, x\geq 0]j(x,v)}\ [']$$

$$\dfrac{\dfrac{\overset{*}{\overline{2gx=2gH-v^2 \vdash 2g(H-\frac{g}{2}t^2)=2gH-(gt)^2}}\ \mathbb{R} \qquad \overline{H-\frac{g}{2}t^2\geq 0 \vdash H-\frac{g}{2}t^2\geq 0}}{2gx=2gH-v^2 \land x\geq 0, H-\frac{g}{2}t^2\geq 0 \vdash 2g(H-\frac{g}{2}t^2)=2gH-(gt)^2 \land (H-\frac{g}{2}t^2)\geq 0}\ \land\mathrm{R}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{j(x,v), t\geq 0, H-\frac{g}{2}t^2\geq 0 \vdash j(H-\frac{g}{2}t^2,-gt)}{j(x,v) \vdash t\geq 0 \rightarrow H-\frac{g}{2}t^2\geq 0 \rightarrow j(H-\frac{g}{2}t^2,-gt)}\ \rightarrow\mathrm{R}}{j(x,v) \vdash \forall t\geq 0\,(H-\frac{g}{2}t^2\geq 0 \rightarrow j(H-\frac{g}{2}t^2,-gt))}\ \forall\mathrm{R}}{j(x,v) \vdash \forall t\geq 0\,[x:=H-\frac{g}{2}t^2](x\geq 0 \rightarrow j(x,-gt))}\ [:=]}{j(x,v) \vdash \forall t\geq 0\,[x:=H-\frac{g}{2}t^2][v:=-gt](x\geq 0 \rightarrow j(x,v))}\ [:=]}{j(x,v) \vdash \forall t\geq 0\,[x:=H-\frac{g}{2}t^2; v:=-gt](x\geq 0 \rightarrow j(x,v))}\ [;]}{j(x,v) \vdash [x'=v, v'=-g\,\&\,x\geq 0]j(x,v)}\ [']$$

$$
\dfrac{
\dfrac{\dfrac{*}{2gx=2gH-v^2 \vdash 2g(H-\frac{g}{2}t^2)=2gH-(gt)^2}\mathbb{R}
\quad
\dfrac{*}{H-\frac{g}{2}t^2\geq 0 \vdash H-\frac{g}{2}t^2\geq 0}\text{id}}
{2gx=2gH-v^2\land x\geq 0, H-\frac{g}{2}t^2\geq 0 \vdash 2g(H-\frac{g}{2}t^2)=2gH-(gt)^2\land(H-\frac{g}{2}t^2)\geq 0}\land\text{R}
}{}
$$

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{j(x,v), t\geq 0, H-\frac{g}{2}t^2\geq 0 \vdash j(H-\frac{g}{2}t^2,-gt)}{j(x,v) \vdash t\geq 0 \to H-\frac{g}{2}t^2\geq 0 \to j(H-\frac{g}{2}t^2,-gt)}\to\text{R}}{j(x,v) \vdash \forall t\geq 0\,(H-\frac{g}{2}t^2\geq 0 \to j(H-\frac{g}{2}t^2,-gt))}\forall\text{R}}{j(x,v) \vdash \forall t\geq 0\,[x:=H-\frac{g}{2}t^2](x\geq 0 \to j(x,-gt))}[:=]}{j(x,v) \vdash \forall t\geq 0\,[x:=H-\frac{g}{2}t^2][v:=-gt](x\geq 0 \to j(x,v))}[:=]}{j(x,v) \vdash \forall t\geq 0\,[x:=H-\frac{g}{2}t^2; v:=-gt](x\geq 0 \to j(x,v))}[;]}{j(x,v) \vdash [x'=v, v'=-g\,\&\, x\geq 0]j(x,v)}[']
$$

$$
\cfrac{\mathbb{R}\;\cfrac{*}{2gx{=}2gH{-}v^2 \vdash 2g(H{-}\frac{g}{2}t^2){=}2gH{-}(gt)^2} \qquad \mathrm{id}\;\cfrac{*}{H{-}\frac{g}{2}t^2{\geq}0 \vdash H{-}\frac{g}{2}t^2{\geq}0}}{2gx{=}2gH{-}v^2{\wedge}x{\geq}0, H{-}\frac{g}{2}t^2{\geq}0 \vdash 2g(H{-}\frac{g}{2}t^2){=}2gH{-}(gt)^2{\wedge}(H{-}\frac{g}{2}t^2){\geq}0}\;\wedge R
$$

$$
\begin{array}{ll}
 & \mathrm{j}(x,v), t{\geq}0, H{-}\frac{g}{2}t^2{\geq}0 \vdash \mathrm{j}(H{-}\frac{g}{2}t^2, -gt) \\
\rightarrow R & \\ \hline
 & \mathrm{j}(x,v) \vdash t{\geq}0 \rightarrow H{-}\frac{g}{2}t^2{\geq}0 \rightarrow \mathrm{j}(H{-}\frac{g}{2}t^2, -gt) \\
\forall R & \\ \hline
 & \mathrm{j}(x,v) \vdash \forall t{\geq}0\,(H{-}\frac{g}{2}t^2{\geq}0 \rightarrow \mathrm{j}(H{-}\frac{g}{2}t^2, -gt)) \\
[:=] & \\ \hline
 & \mathrm{j}(x,v) \vdash \forall t{\geq}0\,[x{:=}H{-}\frac{g}{2}t^2](x{\geq}0 \rightarrow \mathrm{j}(x, -gt)) \\
[:=] & \\ \hline
 & \mathrm{j}(x,v) \vdash \forall t{\geq}0\,[x{:=}H{-}\frac{g}{2}t^2][v{:=}{-}gt](x{\geq}0 \rightarrow \mathrm{j}(x,v)) \\
[;] & \\ \hline
 & \mathrm{j}(x,v) \vdash \forall t{\geq}0\,[x{:=}H{-}\frac{g}{2}t^2; v{:=}{-}gt](x{\geq}0 \rightarrow \mathrm{j}(x,v)) \\
['] & \\ \hline
 & \mathrm{j}(x,v) \vdash [x'{=}v, v'{=}{-}g\,\&\,x{\geq}0]\mathrm{j}(x,v)
\end{array}
$$

- Is Quantum done with his safety proof?

Proving Quantum the Acrophobic Bouncing Ball

$$\mathbb{R}\ \frac{*}{2gx=2gH-v^2 \vdash 2g(H-\frac{g}{2}t^2)=2gH-(gt)^2} \qquad \text{id}\ \frac{*}{H-\frac{g}{2}t^2\geq 0 \vdash H-\frac{g}{2}t^2\geq 0}$$

$$\wedge\text{R}\ \overline{2gx=2gH-v^2\wedge x\geq 0, H-\frac{g}{2}t^2\geq 0 \vdash 2g(H-\frac{g}{2}t^2)=2gH-(gt)^2\wedge(H-\frac{g}{2}t^2)\geq 0}$$

$$\text{j}(x,v), t\geq 0, H-\frac{g}{2}t^2\geq 0 \vdash \text{j}(H-\frac{g}{2}t^2,-gt)$$

$$\to\text{R}\ \overline{\text{j}(x,v) \vdash t\geq 0 \to H-\frac{g}{2}t^2\geq 0 \to \text{j}(H-\frac{g}{2}t^2,-gt)}$$

$$\forall\text{R}\ \overline{\text{j}(x,v) \vdash \forall t\geq 0\,(H-\frac{g}{2}t^2\geq 0 \to \text{j}(H-\frac{g}{2}t^2,-gt))}$$

$$[:=]\ \overline{\text{j}(x,v) \vdash \forall t\geq 0\,[x:=H-\frac{g}{2}t^2](x\geq 0 \to \text{j}(x,-gt))}$$

$$[:=]\ \overline{\text{j}(x,v) \vdash \forall t\geq 0\,[x:=H-\frac{g}{2}t^2][v:=-gt](x\geq 0 \to \text{j}(x,v))}$$

$$[;]\ \overline{\text{j}(x,v) \vdash \forall t\geq 0\,[x:=H-\frac{g}{2}t^2; v:=-gt](x\geq 0 \to \text{j}(x,v))}$$

$$[']\ \overline{\text{j}(x,v) \vdash [x'=v, v'=-g\,\&\,x\geq 0]\text{j}(x,v)}$$

- Is Quantum done with his safety proof?
- Oh no! The solutions we sneaked into $[']$ only solve the ODE/IVP if $x = H, v = 0$ which assumption $\text{j}(x,v)$ can't guarantee!

Proving Quantum the Acrophobic Bouncing Ball

$$\dfrac{\dfrac{\mathbb{R}\;\dfrac{*}{2gx{=}2gH{-}v^2 \vdash 2g(H{-}\frac{g}{2}t^2){=}2gH{-}(gt)^2} \qquad \text{id}\;\dfrac{*}{H{-}\frac{g}{2}t^2{\geq}0 \vdash H{-}\frac{g}{2}t^2{\geq}0}}{2gx{=}2gH{-}v^2{\wedge}x{\geq}0, H{-}\frac{g}{2}t^2{\geq}0 \vdash 2g(H{-}\frac{g}{2}t^2){=}2gH{-}(gt)^2{\wedge}(H{-}\frac{g}{2}t^2){\geq}0}\;\wedge\text{R}}{}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{j(x,v), t{\geq}0, H{-}\frac{g}{2}t^2{\geq}0 \vdash j(H{-}\frac{g}{2}t^2, -gt)}{j(x,v) \vdash t{\geq}0 \rightarrow H{-}\frac{g}{2}t^2{\geq}0 \rightarrow j(H{-}\frac{g}{2}t^2, -gt)}\;\rightarrow\text{R}}{j(x,v) \vdash \forall t{\geq}0\,(H{-}\frac{g}{2}t^2{\geq}0 \rightarrow j(H{-}\frac{g}{2}t^2, -gt))}\;\forall\text{R}}{j(x,v) \vdash \forall t{\geq}0\,[x{:=}H{-}\frac{g}{2}t^2](x{\geq}0 \rightarrow j(x, -gt))}\;[:=]}{j(x,v) \vdash \forall t{\geq}0\,[x{:=}H{-}\frac{g}{2}t^2][v{:=}{-}gt](x{\geq}0 \rightarrow j(x,v))}\;[:=]}{j(x,v) \vdash \forall t{\geq}0\,[x{:=}H{-}\frac{g}{2}t^2; v{:=}{-}gt](x{\geq}0 \rightarrow j(x,v))}\;[;]}{j(x,v) \vdash [x'{=}v, v'{=}{-}g\,\&\,x{\geq}0]j(x,v)}\;[']$$

- Is Quantum done with his safety proof?
- Oh no! The solutions we sneaked into $[']$ only solve the ODE/IVP if $x = H, v = 0$ which assumption $j(x,v)$ can't guarantee!
- Never use solutions without proof! ▸ Todo redo proof with true solution

$$\text{loop}\ \frac{}{A \vdash [\alpha^*]B(x,v)}$$

1. $j(x,v) \equiv 2gx = 2gH - v^2 \land x \geq 0$
2. $p \equiv c = 1 \land g > 0$

Clumsy Quantum Misplaced the Constants

$$\text{loop}\ \frac{}{A \vdash [\alpha^*]B(x,v)}$$

1. $j(x,v) \equiv 2gx{=}2gH{-}v^2 \wedge x{\geq}0$
2. $p \equiv c{=}1 \wedge g{>}0$
3. $J \equiv j(x,v) \wedge p$ as loop invariant

Clumsy Quantum Misplaced the Constants

$$\text{loop}\dfrac{\mathbb{R}\dfrac{*}{A \vdash \mathrm{j}(x,v) \land p} \quad []\land\dfrac{}{\mathrm{j}(x,v) \land p \vdash [\alpha](\mathrm{j}(x,v) \land p)} \quad \mathbb{R}\dfrac{}{\mathrm{j}(x,v) \land p \vdash B(x,v)}}{A \vdash [\alpha^*]B(x,v)}$$

1. $\mathrm{j}(x,v) \equiv 2gx = 2gH - v^2 \land x \geq 0$
2. $p \equiv c = 1 \land g > 0$
3. $J \equiv \mathrm{j}(x,v) \land p$ as loop invariant

Clumsy Quantum Misplaced the Constants

$[]\wedge\ [\alpha](P\wedge Q)\leftrightarrow[\alpha]P\wedge[\alpha]Q$

$$\dfrac{\dfrac{*}{A\vdash \mathrm{j}(x,v)\wedge p}\mathbb{R} \quad \dfrac{\dfrac{\dfrac{\text{above}}{\mathrm{j}(x,v)\wedge p\vdash[\alpha]\mathrm{j}(x,v)} \quad \dfrac{}{\mathrm{j}(x,v)\wedge p\vdash[\alpha]p}\mathrm{V}}{\mathrm{j}(x,v)\wedge p\vdash[\alpha]\mathrm{j}(x,v)\wedge[\alpha]p}\wedge\mathrm{R}}{\mathrm{j}(x,v)\wedge p\vdash[\alpha](\mathrm{j}(x,v)\wedge p)}[]\wedge \quad \dfrac{}{\mathrm{j}(x,v)\wedge p\vdash B(x,v)}\mathbb{R}}{A\vdash[\alpha^*]B(x,v)}\text{loop}$$

1. $\mathrm{j}(x,v)\equiv 2gx=2gH-v^2\wedge x\geq 0$
2. $p\equiv c=1\wedge g>0$
3. $J\equiv \mathrm{j}(x,v)\wedge p$ as loop invariant

$[]\wedge\ [\alpha](P \wedge Q) \leftrightarrow [\alpha]P \wedge [\alpha]Q$ $\qquad$ V $p \rightarrow [\alpha]p \quad (FV(p) \cap BV(\alpha) = \emptyset)$

$$\dfrac{\dfrac{*}{A \vdash j(x,v) \wedge p}\mathbb{R} \quad \dfrac{\dfrac{\dfrac{\text{above}}{j(x,v) \wedge p \vdash [\alpha]j(x,v)} \quad \dfrac{*}{j(x,v) \wedge p \vdash [\alpha]p}\text{V}}{j(x,v) \wedge p \vdash [\alpha]j(x,v) \wedge [\alpha]p}\wedge R}{j(x,v) \wedge p \vdash [\alpha](j(x,v) \wedge p)}[]\wedge \quad \dfrac{}{j(x,v) \wedge p \vdash B(x,v)}\mathbb{R}}{A \vdash [\alpha^*]B(x,v)}\text{loop}$$

1. $j(x,v) \equiv 2gx = 2gH - v^2 \wedge x \geq 0$
2. $p \equiv c = 1 \wedge g > 0$
3. $J \equiv j(x,v) \wedge p$ as loop invariant

$[]\wedge\ [\alpha](P \wedge Q) \leftrightarrow [\alpha]P \wedge [\alpha]Q$ $\qquad$ $V\ p \rightarrow [\alpha]p \quad (FV(p) \cap BV(\alpha) = \emptyset)$

$$\dfrac{\dfrac{*}{A \vdash \mathrm{j}(x,v) \wedge p}\mathbb{R} \quad \dfrac{\dfrac{\dfrac{\text{above}}{\mathrm{j}(x,v) \wedge p \vdash [\alpha]\mathrm{j}(x,v)} \quad \dfrac{*}{\mathrm{j}(x,v) \wedge p \vdash [\alpha]p}V}{\mathrm{j}(x,v) \wedge p \vdash [\alpha]\mathrm{j}(x,v) \wedge [\alpha]p}\wedge R}{\mathrm{j}(x,v) \wedge p \vdash [\alpha](\mathrm{j}(x,v) \wedge p)}[]\wedge \quad \dfrac{*}{\mathrm{j}(x,v) \wedge p \vdash B(x,v)}\mathbb{R}}{A \vdash [\alpha^*]B(x,v)}\text{loop}$$

1. $\mathrm{j}(x,v) \equiv 2gx = 2gH - v^2 \wedge x \geq 0$
2. $p \equiv c = 1 \wedge g > 0$
3. $J \equiv \mathrm{j}(x,v) \wedge p$ as loop invariant

Note: constants $c = 1 \wedge g > 0$ that never change are usually elided from J

Proposition (Quantum can bounce around safely)

$$0 \le x \land x = H \land v = 0 \land g > 0 \land 1 = c \rightarrow$$
$$[(\{x' = v, v' = -g \,\&\, x \ge 0\}; (?x = 0; v := -cv \cup ?x \ne 0))^*](0 \le x \land x \le H)$$

$$\textbf{requires}(0 \le x \land x = H \land v = 0)$$
$$\textbf{requires}(g > 0 \land 1 = c)$$
$$\textbf{ensures}(0 \le x \land x \le H)$$
$$\{\{x' = v, v' = -g \,\&\, x \ge 0\};$$
$$(?x = 0; v := -cv \cup ?x \ne 0))\}^* \textbf{@invariant}(2gx = 2gH - v^2 \land x \ge 0)$$

Invariant Contracts

Invariants play a crucial rôle in CPS design. Capture them if you can.
Use **@invariant**() contracts in your hybrid programs.

Outline

1. Learning Objectives
2. Induction for Loops
   - Iteration Axiom
   - Induction Axiom
   - Induction Rule for Loops
   - Loop Invariants
   - Simple Example
   - Contextual Soundness Requirements
3. Operationalize Invariant Construction
   - Bouncing Ball
   - Rescuing Misplaced Constants
   - Safe Quantum
4. Summary

Invariants

The lion's share of understanding comes from understanding what does change (variants/progress measures) and what doesn't change (invariants).

Invariants are a fundamental force of CS

Variants are another fundamental force of CS

Summary: Loops, Generalizations, Splittings

I $[\alpha^*]P \leftrightarrow P \land [\alpha^*](P \rightarrow [\alpha]P)$

G $\dfrac{P}{[\alpha]P}$

M[·] $\dfrac{P \rightarrow Q}{[\alpha]P \rightarrow [\alpha]Q}$

loop $\dfrac{\Gamma \vdash J, \Delta \quad J \vdash [\alpha]J \quad J \vdash P}{\Gamma \vdash [\alpha^*]P, \Delta}$

MR $\dfrac{\Gamma \vdash [\alpha]Q, \Delta \quad Q \vdash P}{\Gamma \vdash [\alpha]P, \Delta}$

[]∧ $[\alpha](P \land Q) \leftrightarrow [\alpha]P \land [\alpha]Q$

V $p \rightarrow [\alpha]p \quad (FV(p) \cap BV(\alpha) = \emptyset)$

Outline

5. Appendix
   - Iteration Axiom
   - Iterations & Splitting the Box
   - Iteration & Generalizations

Iteration Axiom

compositional semantics $\Rightarrow$ compositional rules!

Loops of Proofs: Iterations

$[^*]\ [\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$

$$\frac{}{A \vdash [\alpha^*]B}$$

$$[^*]\quad [\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$$

$$[^*]\ \dfrac{\dfrac{}{A \vdash B \land [\alpha][\alpha^*]B}}{A \vdash [\alpha^*]B}$$

Loops of Proofs: Iterations

$[*]\;\; [\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$

$$\dfrac{\dfrac{\overline{A \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)}}{A \vdash B \land [\alpha]\color{red}{[\alpha^*]B}}\;[*]}{A \vdash [\alpha^*]B}\;[*]$$

$[*]\ [\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash B \land [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)}}{A \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)}\,[*]}{A \vdash B \land [\alpha][\alpha^*]B}\,[*]}{A \vdash [\alpha^*]B}\,[*]$$

Loops of Proofs: Iterations & Splitting the Box

$$[*]\quad [\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$$

$$[]\land\quad [\alpha](P \land Q) \leftrightarrow [\alpha]P \land [\alpha]Q$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \vdash B \land [\alpha]B \land [\alpha][\alpha](B \land [\alpha][\alpha^*]B)}}{A \vdash B \land [\alpha](B \land [\alpha](B \land [\alpha][\alpha^*]B))}\;[]\land}{A \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)}\;[*]}{A \vdash B \land [\alpha][\alpha^*]B}\;[*]}{A \vdash [\alpha^*]B}\;[*]$$

Loops of Proofs: Iterations & Splitting the Box

$$[^*]\quad [\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$$

$$[]\land\quad [\alpha](P \land Q) \leftrightarrow [\alpha]P \land [\alpha]Q$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash B \land [\alpha]B \land [\alpha]([\alpha]B \land [\alpha][\alpha][\alpha^*]B)}}{A \vdash B \land [\alpha]B \land [\alpha]{\color{red}[\alpha](B \land [\alpha][\alpha^*]B)}}\,[]\land}{A \vdash B \land [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)}\,[]\land}{A \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)}\,[^*]}{A \vdash B \land [\alpha][\alpha^*]B}\,[^*]}{A \vdash [\alpha^*]B}\,[^*]
$$

Loops of Proofs: Iterations & Splitting the Box

$$[^*]\quad [\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$$

$$[]\wedge\quad [\alpha](P \wedge Q) \leftrightarrow [\alpha]P \wedge [\alpha]Q$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash B \wedge [\alpha]B \wedge [\alpha][\alpha]B \wedge [\alpha][\alpha][\alpha][\alpha^*]B}}{A \vdash B \wedge [\alpha]B \wedge [\alpha]([\alpha]B \wedge [\alpha][\alpha][\alpha^*]B)}[]\wedge}{A \vdash B \wedge [\alpha]B \wedge [\alpha][\alpha](B \wedge [\alpha][\alpha^*]B)}[]\wedge}{A \vdash B \wedge [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}[]\wedge}{A \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}[^*]}{A \vdash B \wedge [\alpha][\alpha^*]B}[^*]}{A \vdash [\alpha^*]B}[^*]$$

$$[^*]\quad [\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$$

$$[]\land\quad [\alpha](P \land Q) \leftrightarrow [\alpha]P \land [\alpha]Q$$

$$A \vdash B \quad A \vdash [\alpha]B \quad A \vdash [\alpha][\alpha]B \quad A \vdash [\alpha][\alpha][\alpha][\alpha^*]B$$

$$\land\text{R}\ \overline{\quad A \vdash B \land [\alpha]B \land [\alpha][\alpha]B \land [\alpha][\alpha][\alpha][\alpha^*]B \quad}$$

$$[]\land\ \overline{\quad A \vdash B \land [\alpha]B \land [\alpha]([\alpha]B \land [\alpha][\alpha][\alpha^*]B) \quad}$$

$$[]\land\ \overline{\quad A \vdash B \land [\alpha]B \land [\alpha][\alpha](B \land [\alpha][\alpha^*]B) \quad}$$

$$[]\land\ \overline{\quad A \vdash B \land [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big) \quad}$$

$$[^*]\ \overline{\quad A \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B) \quad}$$

$$[^*]\ \overline{\quad A \vdash B \land [\alpha][\alpha^*]B \quad}$$

$$[^*]\ \overline{\quad A \vdash [\alpha^*]B \quad}$$

Loops of Proofs: Iterations & Splitting the Box

$$[^*]\quad [\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$$

$$[]\land\quad [\alpha](P \land Q) \leftrightarrow [\alpha]P \land [\alpha]Q$$

$$
\begin{array}{ll}
 & A \vdash B \qquad A \vdash [\alpha]B \qquad A \vdash [\alpha][\alpha]B \qquad A \vdash [\alpha][\alpha][\alpha][\alpha^*]B \\
\land R & \overline{A \vdash B \land [\alpha]B \land [\alpha][\alpha]B \land [\alpha][\alpha][\alpha][\alpha^*]B} \\
[]\land & \overline{A \vdash B \land [\alpha]B \land [\alpha]([\alpha]B \land [\alpha][\alpha][\alpha^*]B)} \\
[]\land & \overline{A \vdash B \land [\alpha]B \land [\alpha][\alpha](B \land [\alpha][\alpha^*]B)} \\
[]\land & \overline{A \vdash B \land [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)} \\
[^*] & \overline{A \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)} \\
[^*] & \overline{A \vdash B \land [\alpha][\alpha^*]B} \\
[^*] & \overline{A \vdash [\alpha^*]B}
\end{array}
$$

1. Simple approach . . . if we don't mind unrolling until the end of time
2. Useful for finding counterexamples

Loops of Proofs: Iterations & Generalizations

[*] $[\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash B \land [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)}}{A \vdash B \land [\alpha](B \land [\alpha]{\color{red}[\alpha^*]B})}\,[*]}{A \vdash B \land [\alpha][\alpha^*]B}\,[*]}{A \vdash [\alpha^*]B}\,[*]$$

$$[^*]\ [\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$$

$$\text{MR}\ \frac{\Gamma \vdash [\alpha]Q, \Delta \quad Q \vdash P}{\Gamma \vdash [\alpha]P, \Delta}$$

$$\wedge\text{R}\ \cfrac{A \vdash B \qquad \cfrac{}{A \vdash [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)}}{\cfrac{A \vdash \color{red}{B \land [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)}}{[^*]\ \cfrac{A \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)}{[^*]\ \cfrac{A \vdash B \land [\alpha][\alpha^*]B}{[^*]\ A \vdash [\alpha^*]B}}}}$$

Loops of Proofs: Iterations & Generalizations

$$[^*]\quad [\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$$

$$\text{MR}\ \frac{\Gamma \vdash [\alpha]Q, \Delta \qquad Q \vdash P}{\Gamma \vdash [\alpha]P, \Delta}$$

$$
[^*]\dfrac{[^*]\dfrac{[^*]\dfrac{\land\text{R}\dfrac{A \vdash B \qquad \text{MR}\dfrac{A \vdash [\alpha]J_1 \qquad \dfrac{}{J_1 \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)}}{A \vdash [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)}}{A \vdash B \land [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)}}{A \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)}}{A \vdash B \land [\alpha][\alpha^*]B}}{A \vdash [\alpha^*]B}
$$

Loops of Proofs: Iterations & Generalizations

$$[*] \quad [\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$$

$$\text{MR} \quad \dfrac{\Gamma \vdash [\alpha]Q, \Delta \qquad Q \vdash P}{\Gamma \vdash [\alpha]P, \Delta}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash B \qquad \dfrac{A \vdash [\alpha]J_1 \qquad \dfrac{J_1 \vdash B \qquad \dfrac{}{J_1 \vdash [\alpha](B \land [\alpha][\alpha^*]B)}}{J_1 \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)}\land\text{R}}{A \vdash [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)}\text{MR}}{A \vdash B \land [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)}\land\text{R}}{A \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)}[*]}{A \vdash B \land [\alpha][\alpha^*]B}[*]}{A \vdash [\alpha^*]B}[*]$$

$$[^*]\quad [\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$$

$$\text{MR}\ \dfrac{\Gamma \vdash [\alpha]Q, \Delta \qquad Q \vdash P}{\Gamma \vdash [\alpha]P, \Delta}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{A \vdash B \qquad \cfrac{A \vdash [\alpha]J_1 \qquad \cfrac{J_1 \vdash B \qquad \cfrac{J_1 \vdash [\alpha]J_2 \qquad \cfrac{}{J_2 \vdash B \land [\alpha][\alpha^*]B}}{J_1 \vdash [\alpha](B \land [\alpha][\alpha^*]B)}\,\text{MR}}{J_1 \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)}\,\land\text{R}}{A \vdash [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)}\,\text{MR}}{A \vdash B \land [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)}\,\land\text{R}}{A \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)}\,[^*]}{A \vdash B \land [\alpha][\alpha^*]B}\,[^*]}{A \vdash [\alpha^*]B}\,[^*]$$

$$[^*]\ [\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$$

$$\text{MR}\ \frac{\Gamma \vdash [\alpha]Q, \Delta \quad Q \vdash P}{\Gamma \vdash [\alpha]P, \Delta}$$

$$\dfrac{\dfrac{A \vdash B \quad \dfrac{A \vdash [\alpha]J_1 \quad \dfrac{\dfrac{J_1 \vdash B \quad \dfrac{J_1 \vdash [\alpha]J_2 \quad \dfrac{J_2 \vdash B \quad \dfrac{}{J_2 \vdash [\alpha][\alpha^*]B}}{J_2 \vdash B \land [\alpha][\alpha^*]B}\ \land\text{R}}{J_1 \vdash [\alpha](B \land [\alpha][\alpha^*]B)}\ \text{MR}}{J_1 \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)}\ \land\text{R}}{A \vdash [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)}\ \text{MR}}{A \vdash B \land [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)}\ \land\text{R}}{\dfrac{A \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)}{\dfrac{A \vdash B \land [\alpha][\alpha^*]B}{A \vdash [\alpha^*]B}\ [^*]}\ [^*]}\ [^*]$$

$$[*]\quad [\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$$

$$\text{MR}\ \dfrac{\Gamma \vdash [\alpha]Q, \Delta \qquad Q \vdash P}{\Gamma \vdash [\alpha]P, \Delta}$$

$$
{[*]}\,\dfrac{{[*]}\,\dfrac{{[*]}\,\dfrac{\wedge\text{R}\,\dfrac{A \vdash B \qquad \text{MR}\,\dfrac{A \vdash [\alpha]J_1 \qquad \wedge\text{R}\,\dfrac{J_1 \vdash B \qquad \text{MR}\,\dfrac{J_1 \vdash [\alpha]J_2 \qquad \wedge\text{R}\,\dfrac{J_2 \vdash B \qquad \dfrac{J_2 \vdash [\alpha]J_3 \qquad \ldots}{J_2 \vdash [\alpha][\alpha^*]B}}{J_2 \vdash B \land [\alpha][\alpha^*]B}}{J_1 \vdash [\alpha](B \land [\alpha][\alpha^*]B)}}{J_1 \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)}}{A \vdash [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)}}{A \vdash B \land [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)}}{A \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)}}{A \vdash B \land [\alpha][\alpha^*]B}}{A \vdash [\alpha^*]B}
$$

$$[^*]\ \ [\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$$

$$\text{MR}\ \dfrac{\Gamma \vdash [\alpha]Q, \Delta \qquad Q \vdash P}{\Gamma \vdash [\alpha]P, \Delta}$$

$$\text{[*]}\dfrac{\text{[*]}\dfrac{\text{[*]}\dfrac{\land\text{R}\dfrac{A \vdash B \qquad \text{MR}\dfrac{A \vdash [\alpha]J \qquad \land\text{R}\dfrac{J \vdash B \qquad \text{MR}\dfrac{J \vdash [\alpha]J \qquad \land\text{R}\dfrac{J \vdash B \qquad \dfrac{J \vdash [\alpha]J \qquad \dots}{J \vdash [\alpha][\alpha^*]B}}{J \vdash B \land [\alpha][\alpha^*]B}}{J \vdash [\alpha](B \land [\alpha][\alpha^*]B)}}{J \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)}}{A \vdash [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)}}{A \vdash B \land [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)}}{A \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)}}{A \vdash B \land [\alpha][\alpha^*]B}}{A \vdash [\alpha^*]B}$$

$$\dfrac{J \vdash B}{A \vdash [\alpha^*]B}$$

$$[*]\quad [\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$$

$$\mathrm{MR}\ \dfrac{\Gamma \vdash [\alpha]Q, \Delta \qquad Q \vdash P}{\Gamma \vdash [\alpha]P, \Delta}$$

$$
[*]\dfrac{[*]\dfrac{[*]\dfrac{\land\mathrm{R}\dfrac{A \vdash B \;\mathrm{MR}\dfrac{A \vdash [\alpha]J \;\land\mathrm{R}\dfrac{J \vdash B \;\mathrm{MR}\dfrac{J \vdash [\alpha]J \;\land\mathrm{R}\dfrac{J \vdash B \qquad \dfrac{J \vdash [\alpha]J \qquad \dots}{J \vdash [\alpha][\alpha^*]B}}{J \vdash B \land [\alpha][\alpha^*]B}}{J \vdash [\alpha](B \land [\alpha][\alpha^*]B)}}{J \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)}}{A \vdash [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)}}{A \vdash B \land [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)}}{A \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)}}{A \vdash B \land [\alpha][\alpha^*]B}}{A \vdash [\alpha^*]B}
$$

$$\dfrac{\color{red}{J \vdash [\alpha]J} \qquad J \vdash B}{A \vdash [\alpha^*]B}$$

$$[^*]\ [\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$$

$$\text{MR}\ \dfrac{\Gamma \vdash [\alpha]Q, \Delta \qquad Q \vdash P}{\Gamma \vdash [\alpha]P, \Delta}$$

$$
\scriptsize
[^*]\dfrac{[^*]\dfrac{[^*]\dfrac{\land\text{R}\dfrac{A \vdash B \qquad \text{MR}\dfrac{A \vdash [\alpha]J \qquad \land\text{R}\dfrac{J \vdash B \qquad \text{MR}\dfrac{\color{red}{J \vdash [\alpha]J} \qquad \land\text{R}\dfrac{J \vdash B \qquad \dfrac{\color{red}{J \vdash [\alpha]J} \qquad \dots}{J \vdash [\alpha][\alpha^*]B}}{J \vdash B \land [\alpha][\alpha^*]B}}{J \vdash [\alpha](B \land [\alpha][\alpha^*]B)}}{J \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)}}{A \vdash [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)}}{A \vdash B \land [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)}}{A \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)}}{A \vdash B \land [\alpha][\alpha^*]B}}{A \vdash [\alpha^*]B}
$$

Loops of Proofs: Extracting a Proof Rule

$$\frac{A \vdash J \qquad J \vdash [\alpha]J \qquad J \vdash B}{A \vdash [\alpha^*]B}$$

$[^*]\ [\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$

$$\text{MR}\ \frac{\Gamma \vdash [\alpha]Q, \Delta \qquad Q \vdash P}{\Gamma \vdash [\alpha]P, \Delta}$$

$$\dfrac{\dfrac{A \vdash B \quad \text{MR}\dfrac{A \vdash [\alpha]J \quad \land\text{R}\dfrac{J \vdash B \quad \text{MR}\dfrac{J \vdash [\alpha]J \quad \land\text{R}\dfrac{J \vdash B \quad \dfrac{J \vdash [\alpha]J \quad \ldots}{J \vdash [\alpha][\alpha^*]B}}{J \vdash B \land [\alpha][\alpha^*]B}}{J \vdash [\alpha](B \land [\alpha][\alpha^*]B)}}{J \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)}}{A \vdash [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)}}{A \vdash B \land [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)}\ \land\text{R}$$

$$[^*]\ \dfrac{A \vdash B \land [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)}{A \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)}$$

$$[^*]\ \dfrac{A \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)}{A \vdash B \land [\alpha][\alpha^*]B}$$

$$[^*]\ \dfrac{A \vdash B \land [\alpha][\alpha^*]B}{A \vdash [\alpha^*]B}$$

Loops of Proofs: Loop Invariants

$$\text{loop}\ \frac{A \vdash J \qquad J \vdash [\alpha]J \qquad J \vdash B}{A \vdash [\alpha^*]B}$$

$$[*]\quad [\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$$

$$\text{MR}\ \frac{\Gamma \vdash [\alpha]Q, \Delta \qquad Q \vdash P}{\Gamma \vdash [\alpha]P, \Delta}$$

Invariant J generalized intermediate condition

$$
\begin{array}{c}
J \vdash [\alpha]J \qquad \dots \\
\hline
J \vdash [\alpha][\alpha^*]B
\end{array}
$$

$$
\cfrac[c]{
A \vdash [\alpha^*]B \ \text{from:}
}{}
$$

$$
\begin{array}{ll}
\wedge\text{R}: & \dfrac{J \vdash B \qquad \dfrac{J \vdash [\alpha]J \qquad \dots}{J \vdash [\alpha][\alpha^*]B}}{J \vdash B \land [\alpha][\alpha^*]B} \\[2ex]
\text{MR}: & \dfrac{J \vdash [\alpha]J \qquad J \vdash B \land [\alpha][\alpha^*]B}{J \vdash [\alpha](B \land [\alpha][\alpha^*]B)} \\[2ex]
\wedge\text{R}: & \dfrac{J \vdash B \qquad J \vdash [\alpha](B \land [\alpha][\alpha^*]B)}{J \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)} \\[2ex]
\text{MR}: & \dfrac{A \vdash [\alpha]J \qquad J \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)}{A \vdash [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)} \\[2ex]
\wedge\text{R}: & \dfrac{A \vdash B \qquad A \vdash [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)}{A \vdash B \land [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)} \\[2ex]
[*]: & \dfrac{A \vdash B \land [\alpha]\big(B \land [\alpha](B \land [\alpha][\alpha^*]B)\big)}{A \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)} \\[2ex]
[*]: & \dfrac{A \vdash B \land [\alpha](B \land [\alpha][\alpha^*]B)}{A \vdash B \land [\alpha][\alpha^*]B} \\[2ex]
[*]: & \dfrac{A \vdash B \land [\alpha][\alpha^*]B}{A \vdash [\alpha^*]B}
\end{array}
$$

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