Table 2.1 Statements and effects of hybrid programs (HPs)

HP Notation	Operation	Effect
$x_1 := \theta_1, \dots, x_n := \theta_n$	discrete jump set	simultaneously assigns terms θ_i to variables x_i
$x_1' = \theta_1, \dots, x_n' = \theta_n \,\&\, \chi$	continuous evolution	differential equations for x_i with terms θ_i within first-order constraint χ (evolution domain)
$?\chi$	state test / check	test first-order formula χ at current state
$\alpha; \beta$	seq. composition	HP β starts after HP α finishes
$\alpha \cup \beta$	nondet. choice	choice between alternatives HP α or HP β
α^*	nondet. repetition	repeats HP α n-times for any $n \in \mathbb{N}$

Tables 2.3, 3.4 and 4.1 Operators of differential dynamic logic (d$\mathcal{L}$), and additional operators of differential-algebraic dynamic logic (DAL) and differential temporal-dynamic logic (dTL)

d$\mathcal{L}$ Notation	Operator	Meaning
$p(\theta_1, \dots, \theta_n)$	atomic formula	true iff predicate p holds for $(\theta_1, \dots, \theta_n)$
$\neg\phi$	negation / not	true if ϕ is false
$\phi \wedge \psi$	conjunction / and	true if both ϕ and ψ are true
$\phi \vee \psi$	disjunction / or	true if ϕ is true or if ψ is true
$\phi \rightarrow \psi$	implication / implies	true if ϕ is false or ψ is true
$\phi \leftrightarrow \psi$	bi-implication / equivalent	true if ϕ and ψ are both true or both false
$\forall x\, \phi$	universal quantifier / for all	true if ϕ is true for all values of variable x
$\exists x\, \phi$	existential quantifier / exists	true if ϕ is true for some values of variable x
$[\alpha]\phi$	$[\cdot]$ modality / box	true if ϕ is true after all runs of HP α
$\langle\alpha\rangle\phi$	$\langle\cdot\rangle$ modality / diamond	true if ϕ is true after at least one run of HP α
$[\alpha]\phi$	$[\cdot]$ modality / box (DAL)	true if ϕ is true after all runs of DA-program α
$\langle\alpha\rangle\phi$	$\langle\cdot\rangle$ modality / diamond (DAL)	true if ϕ is true after some run of DA-program α
$[\alpha]\Box\phi$	$[\cdot]\Box$ modality nesting (dTL)	if ϕ is true always during all traces of HP α
$\langle\alpha\rangle\Diamond\phi$	$\langle\cdot\rangle\Diamond$ modality nesting (dTL)	if ϕ is true sometimes during some trace of HP α
$[\alpha]\Diamond\phi$	$[\cdot]\Diamond$ modality nesting (dTL)	if ϕ is true sometimes during all traces of HP α
$\langle\alpha\rangle\Box\phi$	$\langle\cdot\rangle\Box$ modality nesting (dTL)	if ϕ is true always during some trace of HP α

Table 3.2 Statements and effects of differential-algebraic programs

DA-program	Operation	Effect
$\mathcal{J}$	discrete jump	jump constraint with assignments holds for discrete jump
$\mathcal{D}$	diff.-alg. flow	differential-algebraic constraint holds during continuous flow
$\alpha; \beta$	seq. composition	DA-program β starts after DA-program α finishes
$\alpha \cup \beta$	nondet. choice	choice between alternative DA-programs α or β
α^*	nondet. repetition	repeats DA-program α n-times for any $n \in \mathbb{N}$

$$(\neg r)\ \frac{\phi \vdash}{\vdash \neg\phi} \qquad (\vee r)\ \frac{\vdash \phi, \psi}{\vdash \phi \vee \psi} \qquad (\wedge r)\ \frac{\vdash \phi \quad \vdash \psi}{\vdash \phi \wedge \psi} \qquad (\to r)\ \frac{\phi \vdash \psi}{\vdash \phi \to \psi} \qquad (ax)\ \frac{}{\phi \vdash \phi}$$

$$(\neg l)\ \frac{\vdash \phi}{\neg\phi \vdash} \qquad (\vee l)\ \frac{\phi \vdash \quad \psi \vdash}{\phi \vee \psi \vdash} \qquad (\wedge l)\ \frac{\phi, \psi \vdash}{\phi \wedge \psi \vdash} \qquad (\to l)\ \frac{\vdash \phi \quad \psi \vdash}{\phi \to \psi \vdash} \qquad (cut)\ \frac{\vdash \phi \quad \phi \vdash}{\vdash}$$

$$(\langle;\rangle)\ \frac{\langle\alpha\rangle\langle\beta\rangle\phi}{\langle\alpha;\beta\rangle\phi} \qquad (\langle^{*n}\rangle)\ \frac{\phi \vee \langle\alpha\rangle\langle\alpha^*\rangle\phi}{\langle\alpha^*\rangle\phi} \qquad (\langle:=\rangle)\ \frac{\phi_{x_1}^{\theta_1} \cdots {}_{x_n}^{\theta_n}}{\langle x_1 := \theta_1, .., x_n := \theta_n\rangle\phi}$$

$$([;])\ \frac{[\alpha][\beta]\phi}{[\alpha;\beta]\phi} \qquad ([^{*n}])\ \frac{\phi \wedge [\alpha][\alpha^*]\phi}{[\alpha^*]\phi} \qquad ([:=])\ \frac{\langle x_1 := \theta_1, .., x_n := \theta_n\rangle\phi}{[x_1 := \theta_1, .., x_n := \theta_n]\phi}$$

$$(\langle\cup\rangle)\ \frac{\langle\alpha\rangle\phi \vee \langle\beta\rangle\phi}{\langle\alpha\cup\beta\rangle\phi} \qquad (\langle?\rangle)\ \frac{\chi \wedge \psi}{\langle ?\chi\rangle\psi} \qquad (\langle'\rangle)\ \frac{\exists t{\geq}0\,\big((\forall 0{\leq}\tilde{t}{\leq}t\,\langle\mathscr{S}_{\tilde{t}}\rangle\chi) \wedge \langle\mathscr{S}_t\rangle\phi\big)}{\langle x_1' = \theta_1, .., x_n' = \theta_n \,\&\, \chi\rangle\phi}\,^{1}$$

$$([\cup])\ \frac{[\alpha]\phi \wedge [\beta]\phi}{[\alpha\cup\beta]\phi} \qquad ([?])\ \frac{\chi \to \psi}{[?\chi]\psi} \qquad ([']) \ \frac{\forall t{\geq}0\,\big((\forall 0{\leq}\tilde{t}{\leq}t\,\langle\mathscr{S}_{\tilde{t}}\rangle\chi) \to \langle\mathscr{S}_t\rangle\phi\big)}{[x_1' = \theta_1, .., x_n' = \theta_n \,\&\, \chi]\phi}\,^{1}$$

$$(\forall r)\ \frac{\vdash \phi(s(X_1, .., X_n))}{\vdash \forall x\,\phi(x)}\,^{2} \qquad (\exists r)\ \frac{\vdash \phi(X)}{\vdash \exists x\,\phi(x)}\,^{4}$$

$$(\exists l)\ \frac{\phi(s(X_1, .., X_n)) \vdash}{\exists x\,\phi(x) \vdash}\,^{2} \qquad (\forall l)\ \frac{\phi(X) \vdash}{\forall x\,\phi(x) \vdash}\,^{4}$$

$$(i\forall)\ \frac{\vdash \mathrm{QE}(\forall X\,(\Phi(X) \vdash \Psi(X)))}{\Phi(s(X_1, .., X_n)) \vdash \Psi(s(X_1, .., X_n))}\,^{3} \qquad (i\exists)\ \frac{\vdash \mathrm{QE}(\exists X \bigwedge_i (\Phi_i \vdash \Psi_i))}{\Phi_1 \vdash \Psi_1 \ \ \ldots \ \ \Phi_n \vdash \Psi_n}\,^{5}$$

$$([]gen)\ \frac{\vdash \forall^\alpha(\phi \to \psi)}{[\alpha]\phi \vdash [\alpha]\psi} \qquad (\langle\rangle gen)\ \frac{\vdash \forall^\alpha(\phi \to \psi)}{\langle\alpha\rangle\phi \vdash \langle\alpha\rangle\psi}$$

$$(ind)\ \frac{\vdash \forall^\alpha(\phi \to [\alpha]\phi)}{\phi \vdash [\alpha^*]\phi} \qquad (con)\ \frac{\vdash \forall^\alpha \forall v{>}0\,(\varphi(v) \to \langle\alpha\rangle\varphi(v-1))}{\exists v\,\varphi(v) \vdash \langle\alpha^*\rangle\exists v{\leq}0\,\varphi(v)}\,^{6}$$

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[1] t and $\tilde{t}$ are fresh logical variables and $\langle\mathscr{S}_t\rangle$ is the jump set $\langle x_1 := y_1(t), .., x_n := y_n(t)\rangle$ with simultaneous solutions $y_1, .., y_n$ of the respective differential equations with constant symbols x_i as symbolic initial values.

[2] s is a new (Skolem) function symbol and $X_1, .., X_n$ are all free logical variables of $\forall x\,\phi(x)$.

[3] X is a new logical variable. Further, QE needs to be defined for the formula in the premise.

[4] X is a new logical variable.

[5] Among all open branches, free logical variable X only occurs in the branches $\Phi_i \vdash \Psi_i$. Further, QE needs to be defined for the formula in the premise, especially, no Skolem dependencies on X can occur.

[6] Logical variable v does not occur in α.

Fig. 2.11 Proof calculus for differential dynamic logic ($\mathsf{d}\mathscr{L}$)

$$(\mathrm{r}\forall)\ \frac{\mathrm{QE}(\forall x \bigwedge_i (\Gamma_i \vdash \Delta_i))}{\Gamma \vdash \Delta, \forall x\, \phi}\,{}^{1} \qquad (\mathrm{r}\exists)\ \frac{\mathrm{QE}(\exists x \bigwedge_i (\Gamma_i \vdash \Delta_i))}{\Gamma \vdash \Delta, \exists x\, \phi}\,{}^{1}$$

$$(\mathrm{l}\forall)\ \frac{\mathrm{QE}(\exists x \bigwedge_i (\Gamma_i \vdash \Delta_i))}{\Gamma, \forall x\, \phi \vdash \Delta}\,{}^{1} \qquad (\mathrm{l}\exists)\ \frac{\mathrm{QE}(\forall x \bigwedge_i (\Gamma_i \vdash \Delta_i))}{\Gamma, \exists x\, \phi \vdash \Delta}\,{}^{1}$$

$$(\langle;\rangle)\ \frac{\langle\alpha\rangle\langle\beta\rangle\phi}{\langle\alpha;\beta\rangle\phi} \qquad (\langle\exists\rangle)\ \frac{\exists x \langle\mathscr{J}\rangle\phi}{\langle\exists x\, \mathscr{J}\rangle\phi} \qquad (\langle:=\rangle)\ \frac{\chi \wedge \phi_{x_1}^{\theta_1} \cdots {}_{x_n}^{\theta_n}}{\langle x_1 := \theta_1 \wedge .. \wedge x_n := \theta_n \wedge \chi\rangle\phi}\,{}^{3}$$

$$([;])\ \frac{[\alpha][\beta]\phi}{[\alpha;\beta]\phi} \qquad ([\exists])\ \frac{\forall x [\mathscr{J}]\phi}{[\exists x\, \mathscr{J}]\phi} \qquad ([:=])\ \frac{\chi \rightarrow \phi_{x_1}^{\theta_1} \cdots {}_{x_n}^{\theta_n}}{[x_1 := \theta_1 \wedge .. \wedge x_n := \theta_n \wedge \chi]\phi}\,{}^{3}$$

$$(\langle\cup\rangle)\ \frac{\langle\alpha\rangle\phi \vee \langle\beta\rangle\phi}{\langle\alpha\cup\beta\rangle\phi} \qquad (\langle J\rangle)\ \frac{\langle\mathscr{J}_1 \cup \ldots \cup \mathscr{J}_n\rangle\phi}{\langle\mathscr{J}\rangle\phi}\,{}^{2} \qquad (\langle D\rangle)\ \frac{\langle(\mathscr{D}_1 \cup \ldots \cup \mathscr{D}_n)^*\rangle\phi}{\langle\mathscr{D}\rangle\phi}\,{}^{4}$$

$$([\cup])\ \frac{[\alpha]\phi \wedge [\beta]\phi}{[\alpha\cup\beta]\phi} \qquad ([J])\ \frac{[\mathscr{J}_1 \cup \ldots \cup \mathscr{J}_n]\phi}{[\mathscr{J}]\phi}\,{}^{2} \qquad ([D])\ \frac{[(\mathscr{D}_1 \cup \ldots \cup \mathscr{D}_n)^*]\phi}{[\mathscr{D}]\phi}\,{}^{4}$$

$$([DR])\ \frac{\vdash [\mathscr{E}]\phi}{\vdash [\mathscr{D}]\phi}\,{}^{5} \qquad (\langle DR\rangle)\ \frac{\vdash \langle\mathscr{D}\rangle\phi}{\vdash \langle\mathscr{E}\rangle\phi}\,{}^{5} \qquad (DS)\ \frac{\vdash [\mathscr{D}]\chi \quad \vdash [\mathscr{D} \wedge \chi]\phi}{\vdash [\mathscr{D}]\phi}$$

$$([]gen)\ \frac{\vdash \forall^\alpha(\phi \rightarrow \psi)}{[\alpha]\phi \vdash [\alpha]\psi} \qquad (\langle\rangle gen)\ \frac{\vdash \forall^\alpha(\phi \rightarrow \psi)}{\langle\alpha\rangle\phi \vdash \langle\alpha\rangle\psi}$$

$$(ind)\ \frac{\vdash \forall^\alpha(\phi \rightarrow [\alpha]\phi)}{\phi \vdash [\alpha^*]\phi} \qquad (con)\ \frac{\vdash \forall^\alpha \forall v{>}0\, (\varphi(v) \rightarrow \langle\alpha\rangle\varphi(v-1))}{\exists v\, \varphi(v) \vdash \langle\alpha^*\rangle \exists v{\leq}0\, \varphi(v)}\,{}^{6}$$

$$(DI)\ \frac{\vdash \forall^\alpha \forall y_1 .. \forall y_k\, (\chi \rightarrow F'{}_{x_1'}^{\theta_1} \cdots {}_{x_n'}^{\theta_n})}{[\exists y_1 .. \exists y_k\, \chi]F \vdash [\exists y_1 .. \exists y_k\, (x_1' = \theta_1 \wedge .. \wedge x_n' = \theta_n \wedge \chi)]F}\,{}^{7}$$

$$(DV)\ \frac{\vdash \exists \varepsilon{>}0\, \forall^\alpha \forall y_1 .. y_k\, (\neg F \wedge \chi \rightarrow (F' \geq \varepsilon)_{x_1'}^{\theta_1} \cdots {}_{x_n'}^{\theta_n})}{[\exists y_1 .. y_k\, (x_1' = \theta_1 \wedge .. \wedge x_n' = \theta_n \wedge {\sim}F)]\chi \vdash \langle\exists y_1 .. y_k\, (x_1' = \theta_1 \wedge .. \wedge x_n' = \theta_n \wedge \chi)\rangle F}\,{}^{8}$$

[1] $\Gamma_i \vdash \Delta_i$ are obtained from the subgoals of side deduction $(\star)$ in Fig. 3.10, in which x is assumed to occur in first-order formulas only, as QE is then applicable. The side deduction starts from goal $\Gamma \vdash \Delta, \phi$ at the bottom (or $\Gamma, \phi \vdash \Delta$ for l∀ and l∃), where x does not occur in Γ, Δ using renaming.

[2] $\mathscr{J}_1 \vee \cdots \vee \mathscr{J}_n$ is a disjunctive normal form of the DJ-constraint $\mathscr{J}$.

[3] Rule applicable for any reordering of the conjuncts of the DJ-constraint where χ is jump-free.

[4] $\mathscr{D}_1 \vee \cdots \vee \mathscr{D}_n$ is a disjunctive normal form of the DA-constraint $\mathscr{D}$.

[5] $\mathscr{D}$ implies $\mathscr{E}$, i.e., satisfies the assumptions of Lemma 3.3.

[6] Logical variable v does not occur in α.

[7] Applicable for any reordering of the conjuncts where χ is non-differential. F is first-order without negative equalities, and F' abbreviates $D(F)$, with z' replaced with 0 for unchanged variables.

[8] Like *DI*, but F contains no equalities and the differential equations are Lipschitz continuous.

Fig. 3.9 Proof calculus for differential-algebraic dynamic logic (DAL)

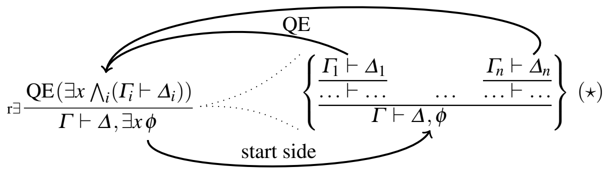

Fig. 3.10 Side deduction

$$([\cup]\Box)\ \frac{[\alpha]\pi \land [\beta]\pi}{[\alpha \cup \beta]\pi}\,{}^{1} \qquad (\langle\cup\rangle\Diamond)\ \frac{\langle\alpha\rangle\pi \lor \langle\beta\rangle\pi}{\langle\alpha \cup \beta\rangle\pi}\,{}^{1}$$

$$([;]\Box)\ \frac{[\alpha]\Box\phi \land [\alpha][\beta]\Box\phi}{[\alpha;\beta]\Box\phi} \qquad (\langle;\rangle\Diamond)\ \frac{\langle\alpha\rangle\Diamond\phi \lor \langle\alpha\rangle\langle\beta\rangle\Diamond\phi}{\langle\alpha;\beta\rangle\Diamond\phi}$$

$$([?]\Box)\ \frac{\phi}{[?\chi]\Box\phi} \qquad (\langle?\rangle\Diamond)\ \frac{\phi}{\langle?\chi\rangle\Diamond\phi}$$

$$([:=]\Box)\ \frac{\phi \land [x:=\theta]\phi}{[x:=\theta]\Box\phi} \qquad (\langle:=\rangle\Diamond)\ \frac{\phi \lor \langle x:=\theta\rangle\phi}{\langle x:=\theta\rangle\Diamond\phi}$$

$$([']\Box)\ \frac{[x'=\theta]\phi}{[x'=\theta]\Box\phi} \qquad (\langle'\rangle\Diamond)\ \frac{\langle x'=\theta\rangle\phi}{\langle x'=\theta\rangle\Diamond\phi}$$

$$([^{*n}]\Box)\ \frac{[\alpha;\alpha^*]\Box\phi}{[\alpha^*]\Box\phi} \qquad (\langle^{*n}\rangle\Diamond)\ \frac{\langle\alpha;\alpha^*\rangle\Diamond\phi}{\langle\alpha^*\rangle\Diamond\phi}$$

$$([^{*}]\Box)\ \frac{[\alpha^*][\alpha]\Box\phi}{[\alpha^*]\Box\phi} \qquad (\langle^{*}\rangle\Diamond)\ \frac{\langle\alpha^*\rangle\langle\alpha\rangle\Diamond\phi}{\langle\alpha^*\rangle\Diamond\phi}$$

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[1] π is a trace formula and—unlike the state formulas ϕ and ψ—may thus begin with a temporal modality $\Box$ or $\Diamond$.

Fig. 4.3 Proof calculus for differential temporal dynamic logic (dTL)