

15812 28 Relative Completeness

Thm (Relative Completeness)

DL is sound and complete

i.e. $\models \phi \iff \vdash \phi$ using $\frac{F \models \phi}{\vdash \phi} \quad \frac{\vdash \phi}{F \models \phi}$

Lem: For every DL formula ϕ , there is effective ϕ^b in FOL $_Z$

Proof: $\models \phi \Rightarrow$ "By induction on FOL $_Z$ "

• $\models \phi$ wlog ϕ in NNF so ϕ atomic formula
 $\vdash \phi$ thus $\vdash \phi$ and $\vdash \psi$ by sem. thus (IH) $\vdash \phi$ and $\vdash \psi$

• $\models \phi \wedge \psi$ thus $\vdash \phi \wedge \psi$ so $\vdash \phi \wedge \psi$ by IR

• $\models \phi \vee \psi$ thus $\vdash \phi \vee \psi$

• $\models F \rightarrow \forall x G$ thus $\vdash F \rightarrow G$ so $\vdash F \rightarrow G$

$\vdash F \rightarrow G$ so $\vdash F \rightarrow G$

VR $\frac{F \rightarrow G}{F \rightarrow \forall x G}$ wlog $x \notin F$

• $\models F \rightarrow \exists x G$

$\vdash F \rightarrow \exists x(G^b)$ and $\vdash (G^b) \rightarrow G$ by Lem

Because $F \rightarrow \exists x(G^b) \prec F \rightarrow \exists x G$ so by IH $\vdash F \rightarrow \exists x(G^b)$

Because $(G^b) \rightarrow G \prec F \rightarrow \exists x G$ so by IH $\vdash (G^b) \rightarrow G$

$\vdash F \rightarrow \exists x(G^b)$

MP $\frac{F \rightarrow \exists x(G^b) \quad \exists M \frac{(G^b) \rightarrow G}{\vdash \exists x(G^b) \rightarrow \exists x G}}{\vdash F \rightarrow \exists x G}$

• $\vdash F \rightarrow [\alpha] G$

• $\vdash F \rightarrow [x := \theta] G$

$\vdash \frac{F \rightarrow \forall y (y = \theta \rightarrow G_x)}{F \rightarrow [y := \theta] G_x}$ wlog $y \neq x$

• $\vdash F \rightarrow [\alpha \cup \beta] G$ then $\vdash F \rightarrow [\alpha] G$ and $\vdash F \rightarrow [\beta] G$

As $F \rightarrow [\alpha] G \prec F \rightarrow [\alpha \cup \beta] G \hookrightarrow$ IH $\vdash F \rightarrow [\alpha] G$

\hookrightarrow IH $\vdash F \rightarrow [\beta] G$

so $\vdash (F \rightarrow [\alpha] G) \wedge (F \rightarrow [\beta] G)$

so $\vdash F \rightarrow [\alpha] G \wedge [\beta] G$ \hookrightarrow prop

so $\vdash F \rightarrow [\alpha \cup \beta] G$ \hookrightarrow C \cup

• $\vdash F \rightarrow [\alpha; \beta] G$ then $\vdash F \rightarrow [\alpha] [\beta] G$

As $F \rightarrow [\alpha] [\beta] G \prec F \rightarrow [\alpha; \beta] G$ (!)

$\vdash F \rightarrow [\alpha] [\beta] G$

$\vdash \frac{F \rightarrow [\alpha] [\beta] G}{F \rightarrow [\alpha; \beta] G}$

• $\vdash F \rightarrow [\exists Q] G$ then $\vdash F \sim Q \rightarrow G$

by IH $\vdash F \sim Q \rightarrow G$

$\vdash \frac{\text{prop}}{F \rightarrow (Q \rightarrow G)}$

$\vdash \frac{C \exists}{F \rightarrow [\exists Q] G}$

• $\vdash F \rightarrow [\alpha^*] G$ then

$\vdash \frac{*(\text{IH})}{F \rightarrow ([\alpha^*] G)^b}$

$\vdash \frac{([\alpha^*] G)^b \rightarrow [\alpha \oplus (\alpha^*)] G}{F \rightarrow [\alpha \oplus (\alpha^*)] G}$

$\vdash \frac{([\alpha^*] G)^b \rightarrow G}{F \rightarrow [\alpha^*] G}$

and $\vdash \frac{F \rightarrow \Psi}{F \rightarrow \Psi}$

$\vdash \frac{\Psi \rightarrow [\alpha] \Psi}{F \rightarrow [\alpha] \Psi}$

$\vdash \frac{\Psi \rightarrow G}{F \rightarrow G}$

$$\vdash F \rightarrow [\alpha^*]G$$

\Downarrow

$$\vdash F \rightarrow ([\alpha^*]G) \leftarrow F \rightarrow [\alpha^*]G \rightarrow \text{IH}$$

And "likewise" for \leftrightarrow (see LICS'12, JAR'17)

Note \leq is a well order (!) ACM TOCL'15

$$\bullet \models F \rightarrow \langle \alpha \circ \beta \rangle G \quad x = \text{FV}(\langle \alpha^* \rangle G)$$

$$\bullet \models F \rightarrow \langle \alpha^* \rangle G$$

Note $\models \forall x(G \vee \langle \alpha \rangle \psi \rightarrow \psi) \rightarrow (\langle \alpha^* \rangle G \rightarrow \psi)$
for any DL formula ψ with $\text{FV}(\psi) \subseteq x$ because games

since $[\langle \alpha^* \rangle G] \subseteq \text{LFP } \llbracket \alpha \rrbracket_0 \llbracket \alpha \rrbracket(Z) \leq Z$
 $G \vee \langle \alpha \rangle \psi \rightarrow \psi$ hence $[\psi] \leq \text{FP}$

$[\langle \alpha^* \rangle G] \hookrightarrow \text{least fixpoint}$ so $[\langle \alpha^* \rangle G] \leq [\psi]$

Hence $\models \forall x(G \vee \langle \alpha \rangle p(x) \rightarrow p(x)) \rightarrow (\langle \alpha^* \rangle G \rightarrow p(x))$

By assm $\models \forall x(G \vee \langle \alpha \rangle p(x) \rightarrow p(x)) \rightarrow (F \rightarrow p(x))$

since $\forall x(G \vee \langle \alpha \rangle p(x) \rightarrow p(x)) \rightarrow (F \rightarrow p(x)) \prec F \rightarrow \langle \alpha^* \rangle G$

By IH $\vdash \forall x(G \vee \langle \alpha \rangle p(x) \rightarrow p(x)) \rightarrow (F \rightarrow p(x))$

$$\text{US } \frac{\forall x(G \vee \langle \alpha \rangle \langle \alpha^* \rangle G \rightarrow \langle \alpha^* \rangle G) \rightarrow (F \rightarrow \langle \alpha^* \rangle G)}{\forall x(G \vee \langle \alpha \rangle \langle \alpha^* \rangle G \rightarrow \langle \alpha^* \rangle G) \rightarrow (F \rightarrow \langle \alpha^* \rangle G)}$$

$$\text{Cut MP } \frac{\text{US } \frac{\forall x(G \vee \langle \alpha \rangle \langle \alpha^* \rangle G \rightarrow \langle \alpha^* \rangle G) \rightarrow (F \rightarrow \langle \alpha^* \rangle G)}{\forall x(G \vee \langle \alpha \rangle \langle \alpha^* \rangle G \rightarrow \langle \alpha^* \rangle G)} * \forall R \frac{\forall x(G \vee \langle \alpha \rangle \langle \alpha^* \rangle G \rightarrow \langle \alpha^* \rangle G)}{\forall x(G \vee \langle \alpha \rangle \langle \alpha^* \rangle G \rightarrow \langle \alpha^* \rangle G)}}{F \rightarrow \langle \alpha^* \rangle G}$$

□

In other words we proved

Thm (Relative Completeness, schematically)

If logic L is expressive for DL i.e.
for all $\phi \in \text{DL}$ associate $\phi \in L$ such that $\models \phi \leftrightarrow \phi$

then DL is sound and complete relative to L, i.e.

$$\models \phi \text{ iff } \vdash \phi \text{ using Reg } \overline{\phi} \text{ (y } \models \phi \text{ valid in L)}$$

Equivalently

$$\models \phi \text{ iff Taut}_L \vdash \phi$$

Proof: [ACM TOCL'15, JAR'17] "see above"

Note proof only uses (\rightarrow) for $\exists x G$ and $[\alpha^*]G$

Corollary: $F \rightarrow \langle \alpha \rangle G \oplus \ominus$ Meyer Halper JACM '82

Corollary: $[\alpha]$ -free $\hookrightarrow \oplus \ominus$ Schmitt Inf. Cap. '84

Corollary: GL will even d in $\langle \alpha \rangle$ is semi

Corollary: Full DL is $\oplus \ominus$ wrt. FOL_Z Cook + Harel '76

Corollary: Full DL is $\oplus \ominus$ wrt. FOL_Z

Can't be better because

semidecidable if $G \in \text{FOL}$ Gödel

decidable if $G \in \text{FOL}_R$ Tarski

Π_1^1 -complete if $G \in [\alpha^*]H$

arithmetic hierarchy $\sum_n^0 \{ z : \exists y_1 \forall y_2 \exists y_3 \dots y_n \psi(y_1, y_2, \dots, y_n) \text{ with } \psi \text{ decidable} \}$

$$\sum_n^0 \{ z : \forall y_1 \exists y_2 \forall y_3 \dots y_n \psi(y_1, y_2, \dots, y_n) \text{ - II - } \}$$

$$\prod_n^0 \{ z : \exists y_1 \forall y_2 \exists y_3 \dots y_n \psi(y_1, y_2, \dots, y_n) \text{ - III - } \}$$

analytic hierarchy

$$\sum_1^1 \{ z : \exists f \forall y \varphi(z, y, f) \text{ wff } \varphi \text{ dec, } f \text{ fd} \}$$

$$\prod_1^1 \{ z : \forall f \exists y \varphi(z, y, f) \text{ - } \text{ - } \text{ - } \text{ - } \}$$

(om: $\text{FOL}_{\mathbb{Z}}$ \hookrightarrow expressive for DL

$$\vdash \phi \leftrightarrow \vdash_{\text{FOL}_{\mathbb{Z}}} \phi$$

Proof:

- $(\langle \alpha \rangle \phi)^{\downarrow} \equiv \exists \vec{v} (S_{\alpha}(\vec{x}, \vec{v}) \wedge (\phi)^{\downarrow}_{\vec{x}, \vec{v}})$
- $([\alpha] \phi)^{\downarrow} \equiv \forall \vec{v} (S_{\alpha}(\vec{x}, \vec{v}) \rightarrow (\phi)^{\downarrow}_{\vec{x}, \vec{v}})$
- $(\langle x := \theta \rangle \phi)^{\downarrow} \equiv \exists v (v = \theta \wedge (\phi)^{\downarrow}_x)$

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$$S_{x_i = \theta}(\vec{x}, \vec{v}) \equiv v_i = \theta \wedge \bigwedge_{j \neq i} v_j = x_j$$

$$S_{\in Q}(\vec{x}, \vec{v}) \equiv (Q)^{\downarrow} \wedge \vec{x} = \vec{v}$$

$$S_{\alpha \vee \beta}(\vec{x}, \vec{v}) \equiv S_{\alpha}(\vec{x}, \vec{v}) \vee S_{\beta}(\vec{x}, \vec{v})$$

$$S_{\alpha \wedge \beta}(\vec{x}, \vec{v}) \equiv \exists \vec{z} (S_{\alpha}(\vec{x}, \vec{z}) \wedge S_{\beta}(\vec{z}, \vec{v}))$$

$$S_{\alpha^*}(\vec{x}, \vec{v}) \equiv \exists \vec{z} (\vec{x} = \vec{z}_0 \wedge \vec{v} = \vec{z}_1 \wedge \forall j (0 \leq j < 3 \wedge S_{\alpha}(\vec{z}_{j+4}, \vec{z}_{j+5})))$$

$$\wedge \bigwedge_{j=0}^3 v_j = f(z_j)$$

$$\begin{cases} \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z} \\ (a, b) \mapsto 2^a 3^b \mapsto (a, b) \end{cases} \quad \text{Gödel}$$

$$\mathbb{Z} \hookrightarrow \mathbb{Z} \times \mathbb{Z}$$

$$f(n, 0) = 1$$

$$f(n, k+1) = n \cdot f(n, k)$$

(om: Gödel) There is $\text{FOL}_{\mathbb{Z}}$ formula $a^r(n, k, r)$ such that $a^r(n, k, r)$ is true iff r is the k -th component of the sequence with Gödel encoding n .

$$\mathbb{Z}^{(N)} \rightarrow \mathbb{Z}$$

Notation $n_k = r$

Any formula $\phi(n_k)$ is read $\forall r (a^r(n_k, r) \rightarrow \phi(r))$

Corollary (Gödel incompleteness)

FOL $_{\mathbb{Z}}$ does not have sound + complete (effective) calc

Proof: DL is complete relative to $\text{FOL}_{\mathbb{Z}}$

(recall R99). If $\text{FOL}_{\mathbb{Z}}$ with R99 were sound + complete, then DL would be. \checkmark
 DL incompleteness.

Corollary: $\text{FOL}_{\mathbb{Z}}$ is not semi-decidable and not c-semi.

Proof: not semi-decidable ✓
 not c-semi: $\text{FOL}_{\mathbb{Z}}$ is a complete theory i.e.
 for every closed sentence ϕ is either valid or its
 negation $\neg\phi$ is. $\models \phi$ or $\models \neg\phi$
 Consider any state w . Then $w \in \llbracket \phi \rrbracket$ or $w \notin \llbracket \phi \rrbracket$.
 $w \in \llbracket \phi \rrbracket$ or $w \in \llbracket \neg\phi \rrbracket$.
 So $w \in \llbracket \phi \rrbracket$ iff $\neg w \in \llbracket \phi \rrbracket$ follows
 $\models \phi$ or $\models \neg\phi$ \square

Remember: $[\alpha]$ -free fragment uninterpreted
 alias termination $F \rightarrow (\alpha) b$
 is complete relative FOL i.e.
 fct + pred symbols, no \mathbb{Z}

↳ FOL has sound + complete effective calc
 (Gödel's)

DL uninterpreted $[\alpha]$ -free $\text{FOL}_{\text{uninterp}}$
 DL $\text{FOL}_{\mathbb{Z}}$

Theorem: (Decidable)
 For every algebraic formula P (all equations)
 and every "co-algebraic" program α (all tests are
 complement of alg.)
 a polynomial q is computable such that

$$[\alpha] P \leftrightarrow q = 0$$

is provable in DL.

Proof: By induction
 Observe (7) $p = 0 \vee q = 0 \leftrightarrow pq = 0$ over \mathbb{Z}
 (9) $p = 0 \wedge q = 0 \leftrightarrow p^2 + q^2 = 0$ over \mathbb{Z}

WLOG P post condition is $p = 0$

- $[\alpha := \theta] P(x) = 0 \leftrightarrow P(\theta) = 0$ by $[\alpha :=]$
- $[\exists q \neq 0] P = 0 \leftrightarrow (q \neq 0 \rightarrow p = 0)$ by $[\exists]$
- $\neg q = 0 \vee p = 0$ so $pq = 0$ by (7)
- $[\alpha \cup \beta] P = 0 \leftrightarrow \underbrace{[\alpha] P = 0} \wedge \underbrace{[\beta] P = 0} \text{ by } [\cup]$
 $q_1 = 0 \quad q_2 = 0 \text{ by IH}$

$$\text{So } [\alpha \cup \beta] P = 0 \leftrightarrow q_1^2 + q_2^2 = 0 \text{ by (9)}$$

$$\cdot [\alpha; \beta] P = 0 \leftrightarrow \underbrace{[\alpha] P = 0}_{\sim \sim} \wedge \underbrace{[\beta] P = 0}_{\sim \sim} \text{ by } [\alpha; \beta]$$

$$\begin{array}{c}
 \overbrace{q_0, q_1, q_2, \dots}^{q_i \in \mathbb{Z}[x]} \\
 \vdash q_0 = 0 \quad \text{by IH} \\
 \vdash q_1 = 0 \quad \text{by IH} \\
 \vdash q_2 = 0 \quad \text{by IH} \\
 \vdash \dots = 0 \quad \text{by IH}
 \end{array}$$

$\vdash [x] p = 0$
 $p = 0$
 $[x] p = 0$
 $[x] [x] p = 0$
 $[x] [x] [x] p = 0$
 $[x] q_i = 0 \quad \text{apply IH obtain } q_{i+1}$
 $\text{such that } [x] q_i = 0 \Leftrightarrow q_i = 0$
 provide

$q_0, q_1, q_2, q_3, q_4, \dots$ Noetherian ring

$$(q_0) \subseteq (q_0, q_1) \subseteq (q_0, q_1, q_2) \subseteq \dots = (q_0, q_1, \dots, q_n)$$

Ideal generated by q_1, \dots, q_n is

$$(q_1, \dots, q_n) = \left\{ \sum_i g_i q_i : g_i \in \mathbb{Z}[x] \text{ are polynomials} \right\}$$

By Noetherian ring

$$q_n = \sum_{i=0}^{n-1} g_i q_i$$

Loop invariant

$$\frac{\text{IH} \quad \bigwedge_{i=0}^{n-1} q_i = 0 \vdash \bigwedge_{i=0}^{n-1} q_i = 0}{\bigwedge_{i=0}^n q_i = 0 \vdash [x] \bigwedge_{i=0}^{n-1} q_i = 0}$$

$$q_i = 0$$

$$\bigwedge_{i=0}^{n-1} q_i = 0 \vdash q = 0$$