

LFCPS Exercise Class 7

Comprehensive CPS Correctness

Uniform substitution

Exercise 1. Determine the free variables in the following formulas:

1. $[x := 1 \cup y := 2] x \geq 1$
2. $[z := y; \{x := 1 \cup y := 2\}] x \geq z$

Determine the free variables in the following substitutions:

1. $\sigma = \{a \mapsto x' := 1, p(\cdot) \mapsto \cdot \geq 1\}$
2. $\sigma = \{q \mapsto \forall x(x \leq 1), p(\cdot) \mapsto \cdot + y \geq 3, a \mapsto \{?z = 0\}\}$

Exercise 2. Find a uniform substitution σ that causes a clash in the formula:

$$p \mapsto [a]p.$$

Explain why the clash occurs.

Exercise 3. A uniform substitution σ is called *U-admissible* for a formula φ if:

$$U \cap \text{FV}(\sigma|_{\Sigma(\varphi)}) = \emptyset.$$

Consider the following uniform substitution in the context of the bouncing ball model:

$$\sigma = \{a \mapsto \{x' = v, v' = -g \ \& \ x \geq 0\}, \quad b \mapsto \{?x = 0; v := -cv\}, \quad p(\bar{x}) \mapsto 2gx \leq 2gH - v^2\}.$$

Recall the slogan: “If you bind a free variable, you go to logic jail.” Apply the substitution σ to the following formulas and determine whether any variable clashes occur:

1. $[a \cup b]p(\bar{x}) \leftrightarrow [a]p(\bar{x}) \wedge [b]p(\bar{x})$
2. $[a; b]p(\bar{x}) \leftrightarrow [a][b]p(\bar{x})$

Does the slogan accurately capture the formal definition of clashes in uniform substitutions? Why or why not?

Exercise 4. Prove the following statement using uniform substitutions:

$$(x = x_0 \wedge y = y_0) \rightarrow [x := x + y][y := x - y][x := x - y](x = y_0 \wedge y = x_0).$$