LFCPS Exercise Class 5
Differential Games

Exercise 1. *Recall the syntax for programs in Hybrid games*

$$\alpha, \beta ::= x := e \mid ?Q \mid x' = f(x)\&Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^d.$$

Hybrid games introduce the dual *operator that flips the player from demon to angel and vice versa. Find the corresponding formulas constructed with the syntax given above for the following formulas*

1. $\alpha \cap \beta \equiv ...,$

2. $\alpha^\times \equiv ...,$

3. $(x := e)^d \equiv ....$

Can you also construct such formulas for

1. $(x' = f(x)\&Q)^d,$

2. $(?Q)^d.$

Exercise 2. *Prove that the following axiom are sound:*

1. $\langle;\rangle \quad \langle \alpha; \beta \rangle P \leftrightarrow \langle \alpha \rangle \langle \beta \rangle P,$

2. $\langle \cup \rangle \quad \langle \alpha \cup \beta \rangle P \leftrightarrow \langle \alpha \rangle P \vee \langle \beta \rangle P.$

Recall the definition of the iterations by recursion on ordinals.

$$\zeta_\alpha^0(X) = X$$
$$\zeta_\alpha^{\gamma+1}(X) = X \cup \zeta_\alpha(\zeta_\alpha^\gamma(X))$$
$$\zeta_\alpha^\lambda(X) = \bigcup_{\gamma < \lambda} \zeta_\alpha^\gamma(X)$$

Exercise 3. *In the lecture, the following four potential definitions for the semantics of repetition games were given:*

1. *Advance-notice semantics:* $\zeta_{\alpha^*}(X) = \bigcup_{n \in \mathbb{N}} \zeta_{\alpha^n}(X),$

2. *ω-semantics:* $\zeta_{\alpha^*}(X) = \bigcup_{n \in \mathbb{N}} \zeta_\alpha^n(X)$

3. *Inflationary semantics:* $\zeta_{\alpha^*}(X) = \bigcup_{\gamma \in \mathsf{Ord}} \zeta_\alpha^n(X)$

4. *Intersection semantics:* $\zeta_{\alpha^*}(X) = \bigcap\{Z : X \cup \zeta_\alpha(Z) \subseteq Z\}$

Compare these semantics. That is for any pair of semantics either show that they are the same or find an example where they disagree.

Exercise 4. *Which of the four semantics best captures the spirit of gameplay and why?*

Exercise 5. *How many times to you need to repeat the winning region construction to show semantically validity of the following formulas:*

1. $\langle (x := x + 1; x' = 1^d \cup x := x - 1)^* \rangle 0 \leq x < 1$

2. $\langle (x := x - 1; y' = 1^d \cup y := y - 1; z' = 1^d \cup z := z - 1)^* \rangle 0 \leq x < 0 \wedge y < 0 \wedge z < 0$

Exercise 6. *Are the following rules sound or unsound? Prove this.*

1. $$\dfrac{\vdash P \vee \langle \alpha \rangle Q \rightarrow Q}{\vdash \langle \alpha^* \rangle P \rightarrow Q}$$

2. $$\dfrac{\vdash \langle \alpha \rangle P \vee \langle \alpha \rangle Q}{\vdash \langle \alpha \rangle (P \vee Q)}$$

3. $$\dfrac{\vdash \langle \alpha \rangle (P \vee Q)}{\vdash \langle \alpha \rangle P \vee \langle \alpha \rangle Q}$$

Exercise 7. *Use the **dGL** axioms to prove the following formulas:*

1. $\langle x := -x \cup (x := x + 1 \cap x := x + 2) \rangle x > 0$

2. $\langle x := x^2 \cup (x := x + 1 \cap x' = 2) \rangle x > 0$

3. $\langle x := -x \cup (x := x + 2 \cap x' = 2) \rangle x \geq 0$

4. $\langle (x := -x \cup (x := x + 2 \cap x' = 2))^* \rangle x \geq 0$