

LFCPS Exercise Class 1

Modeling cyber-physical systems

Exercise 1. Consider the following hybrid programs and compute the reachability relation $\llbracket \alpha \rrbracket$ for each one:

1. $\alpha \equiv ?\text{false}$,
2. $\alpha \equiv x := 1; (x := x + 1)^*$,
3. $\alpha \equiv x := 1; (x := x + 1)^*; ?x < 0$,
4. $\alpha \equiv x := 1; (x := x + 1; ?x < 0)^*$.

Exercise 2. Consider the following hybrid programs and provide an intuitive interpretation for each:

$$\begin{aligned}\alpha &\equiv \{x' = v, v' = a \& v \geq 0\} \\ \beta &\equiv \{x' = v, v' = a \& v \geq 0\}; ?v = 0 \\ \gamma &\equiv t := 0; \{x' = v, v' = a, t' = 1 \& t \leq T\} \\ \delta &\equiv t := 0; \{x' = v, v' = a, t' = 1 \& t \leq T\}; ?(t = T)\end{aligned}$$

Exercise 3. Consider the following model of a bouncing ball:

$$\alpha \equiv (\{x' = v, v' = -g\}; v := -cv)^*.$$

Here, the variables are defined as follows:

- x : the height of the ball above the ground.
- v : the vertical velocity of the ball, where positive values indicate upward motion and negative values indicate downward motion.
- g : the constant acceleration due to gravity, acting downward.
- c : the coefficient of restitution, representing the energy retained after each bounce, with $0 < c < 1$.

Is this a good model for a bouncing ball? Justify your answer by considering the physical behavior and limitations of the model.

Exercise 4. Consider the following system:

$$\alpha \equiv (\{x' = 1 \& x \leq 0\} \cup \{x' = -1 \& x > 0\})^*.$$

Starting with an initial condition $x \leq 0$, analyze which states are reachable under this system. Describe how the evolution of x is constrained by the dynamics specified in α .

Exercise 5. Consider the following differential equation describing the dynamics of a controlled heating system:

$$T' = k(T_s - T),$$

where T represents the current temperature, T_s denotes the desired temperature setpoint, and k is a proportional constant related to the system's response rate.

Model a controlled heating system that operates as follows:

- The heating system turns on and the setpoint is set to 22°C when the temperature falls below 19°C .
- The heating system turns off when the temperature exceeds 21°C .

Exercise 6. Consider a robot located at a point with coordinates (x, y) , facing in a direction represented by the unit vector (d_x, d_y) . The robot moves forward in the direction it is facing while simultaneously rotating with angular velocity ω .

Develop a system of differential equations to describe how the position (x, y) and direction (d_x, d_y) of the robot evolve over time. Construct your model in two steps:

1. First, model the effect of rotation alone on the direction vector (d_x, d_y) .
2. Then, consider the translation of the position (x, y) in the current direction (d_x, d_y) of the robot.

After building these components separately, combine them to form a complete system of differential equations describing the robot's dynamics.

Bonus: As an additional challenge, try to extend this model by introducing an acceleration term to describe the robot's increasing linear speed.

Exercise 7. 1. Can you find a discrete controller ctrl and a continuous program plant such that the following two HPs α, β have different behaviors?

$$\alpha \equiv (\text{ctrl}; \text{plant})^* \quad \beta \equiv (\text{ctrl} \cup \text{plant})^*.$$

2. Can you find a discrete controller ctrl and a continuous program plant with the following two properties: a) The program ctrl has the same behavior as $\text{ctrl}; \text{ctrl}$ and b) The following two HPs α, β have different behaviors?

$$\alpha \equiv \text{ctrl}; (\text{ctrl}; \text{plant})^* \quad \beta \equiv \text{ctrl}; (\text{ctrl} \cup \text{plant})^*.$$