LFCPS Exercise Class 6
Complete considerations in CPS

Quiz recap

Exercise 1. *Which axioms are sound renditions of the vacuous axiom schema?*

1. $p(x) \to [a]p(x)$
2. $p() \to [?q(\bar{x})]p()$
3. $p() \to [b]p()$
4. $q(\bar{x}) \to [a]q(\bar{x})$
5. $p() \wedge q() \to [a](p() \wedge q())$

Exercise 2. *Which axioms are sound renditions of the assignment axiom schema?*

1. $[x := f(x)]p(x) \leftrightarrow p(f(x))$
2. $[x := g(x)]p(x) \leftrightarrow q(g(x))$
3. $[x := c()](p(x) \leftrightarrow p(c()))$
4. $[x := g()]p(g(x)) \leftrightarrow p(g(x))$

For each case, find conditions under which it is sound.

Completeness

Exercise 3. *In the lecture, we defined a proof calculus as sound and complete if and only if the following equivalence holds for all formulas φ:*

$$\models \varphi \text{ if and only if } \vdash \varphi.$$

1. *Explain in your own words what both directions of the equivalence mean.*
2. *Does this equivalence hold for the proof calculus of* `dL`*?*

Exercise 4. *Consider the* equational differential invariant *proof rule:*

$$\frac{\vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x)]e = 0}$$

We sat that the equation $e = 0$ is a differential invariant, *if $[x' := f(x)](e)' = 0$ is a valid formula.*
Prove that if $e_1 = 0$ and $e_2 = 0$ are both differential invariants, then the following formulas are also provable using the differential invariant proof rule:

1. $e_1 = 0 \vee e_2 = 0 \vdash [x' = f(x)](e_1 = 0 \vee e_2 = 0)$,
2. $e_1 = 0 \wedge e_2 = 0 \vdash [x' = f(x)](e_1 = 0 \wedge e_2 = 0)$.

Exercise 5. *Recall the discrete Euler approximation axiom from the lecture:*

$$\overleftarrow{\Delta} \quad [x' = f(x)]F \leftarrow \exists h_0 > 0 \forall 0 < h < h_0 [(x := x + hf(x))^*]F.$$

Can the following properties be proved using only the axiom $\overleftarrow{\Delta}$, without relying on other differential equation axioms?

1. $x \geq 0 \vdash [x' = 1]x \geq 0$,
2. $x^2 + y^2 = 1 \vdash [\{x' = -y, y' = x\}]x^2 + y^2 = 1$.