LFCPS Exercise Class 5
Differential Games

Exercise 1. *Recall the syntax for programs in Hybrid games*

$$\alpha, \beta ::= x := e \mid ?Q \mid x' = f(x) \& Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \alpha^d.$$

Hybrid games introduce the dual *operator that flips the player from demon to angel and vice versa. Find the corresponding formulas constructed with the syntax given above for the following formulas*

1. $\alpha \cap \beta \equiv \ldots,$
2. $\alpha^\times \equiv \ldots,$
3. $(x := e)^d \equiv \ldots.$

Can you also construct such formulas for

1. $(x' = f(x) \& Q)^d,$
2. $(?Q)^d.$

Exercise 2. *Prove that the following axiom are sound:*

1. $\langle ; \rangle \quad \langle \alpha ; \beta \rangle P \leftrightarrow \langle \alpha \rangle \langle \beta \rangle P,$
2. $\langle \cup \rangle \quad \langle \alpha \cup \beta \rangle P \leftrightarrow \langle \alpha \rangle P \vee \langle \beta \rangle P.$

Recall the definition of the iterations by recursion on ordinals.

$$\zeta_\alpha^0(X) = X$$
$$\zeta_\alpha^{\gamma+1}(X) = X \cup \zeta_\alpha(\zeta_\alpha^\gamma(X))$$
$$\zeta_\alpha^\lambda(X) = \bigcup_{\gamma < \lambda} \zeta_\alpha^\gamma(X)$$

Exercise 3. *In the lecture, the following four potential definitions for the semantics of repetition games were given:*

1. *Advance-notice semantics:* $\zeta_{\alpha^*}(X) = \bigcup_{n \in \mathbb{N}} \zeta_{\alpha^n}(X),$
2. *ω-semantics:* $\zeta_{\alpha^*}(X) = \bigcup_{n \in \mathbb{N}} \zeta_\alpha^n(X)$
3. *Inflationary semantics:* $\zeta_{\alpha^*}(X) = \bigcup_{\gamma \in \mathsf{Ord}} \zeta_\alpha^n(X)$
4. *Intersection semantics:* $\zeta_{\alpha^*}(X) = \bigcap \{Z : X \cup \zeta_\alpha(Z) \subseteq Z\}$

Compare these semantics. That is for any pair of semantics either show that they are the same or find an example where they disagree.

Exercise 4. *Which of the four semantics best captures the spirit of gameplay and why?*

Exercise 5. *How many times to you need to repeat the winning region construction to show semantically validity of the following formulas:*

1. $\langle (x := x + 1; x' = 1^{\mathrm{d}} \cup x := x - 1)^* \rangle 0 \leq x < 1$
2. $\langle (x := x - 1; y' = 1^{\mathrm{d}} \cup y := y - 1; z' = 1^{\mathrm{d}} \cup z := z - 1)^* \rangle 0 \leq x < 0 \wedge y < 0 \wedge z < 0$

Exercise 6. *Are the following rules sound or unsound? Prove this.*

1. $\dfrac{\vdash P \vee \langle\alpha\rangle Q \to Q}{\vdash \langle\alpha^*\rangle P \to Q}$

2. $\dfrac{\vdash \langle\alpha\rangle P \vee \langle\alpha\rangle Q}{\vdash \langle\alpha\rangle(P \vee Q)}$

3. $\dfrac{\vdash \langle\alpha\rangle(P \vee Q)}{\vdash \langle\alpha\rangle P \vee \langle\alpha\rangle Q}$

Exercise 7. *Use the **dGL** axioms to prove the following formulas:*

1. $\langle x := -x \cup (x := x+1 \cap x := x+2)\rangle x > 0$
2. $\langle x := x^2 \cup (x := x+1 \cap x' = 2)\rangle x > 0$
3. $\langle x := -x \cup (x := x+2 \cap x' = 2)\rangle x \geq 0$
4. $\langle (x := -x \cup (x := x+2 \cap x' = 2))^*\rangle x \geq 0$