LFCPS Exercise Class 4
Differential Invariants

1 Quiz recap

Exercise 1. *Determine whether the following proof rules are sound:*

$$\frac{\Gamma \vdash J, \Delta \qquad J \vdash [\alpha]J, \Delta \qquad J \vdash P}{\Gamma \vdash [\alpha^*]P, \Delta}$$

$$\frac{\Gamma \vdash P, \Delta \qquad P \vdash [\alpha]P}{\Gamma \vdash [\alpha^*]P, \Delta}$$

2 Differential reasoning

Exercise 2. *Recall the following axioms from the lecture:*

$$DW\,[x' = f(x)\ \&\ Q]P \leftrightarrow [x' = f(x)\ \&\ Q](Q \to P)$$
$$DI\,([x' = f(x)\ \&\ Q]P \leftrightarrow [?Q]P) \leftarrow [x' = f(x)\ \&\ Q](P)'$$
$$DC\,([x' = f(x)\ \&\ Q]P \leftrightarrow [x' = f(x)\ \&\ Q \wedge C]P) \leftarrow [x' = f(x)\ \&\ Q]C$$

1. *Choose a simple differential equation (e.g. x' = 1), a domain constraint Q, and a formula P.*
2. *Verify that each axiom holds for your example.*
3. *Reflect on the intuition behind each axiom: How do these axioms simplify reasoning in hybrid systems?*

Exercise 3. *Recall that:*

- $(P \wedge Q)' \leftrightarrow (P)' \wedge (Q)'$
- $(P \vee Q)' \leftrightarrow (P)' \wedge (Q)'$
- $(e \geq 0)' \equiv (e)' \geq 0$
- $(e > 0)' \equiv (e)' \geq 0$
- $(e = 0)' \equiv (e)' = 0$
- $(e \neq 0)' \equiv (e)' = 0$

Simplify the following formulas using the rules for the differential operator.

1. $((v^2 + w^2 < r^2 \wedge v \leq 0) \vee v = 0)'$
2. $(x^2 + 2x < 0 \vee (x \neq 0 \wedge x > 1))'$

Exercise 4. *Prove the following formulas using differential invariants, differential cuts, and differential weakening as required:*

1. $\omega \geq 0 \wedge x = 0 \wedge y = 3 \to [\{x' = y, y' = -\omega^2 x - 2\omega y\}]\omega^2 x^2 + y^2 \leq 9$
2. $xy^2 + x \geq 7 \to [\{x' = -2xy, y' = 1 + y^2\}]xy^2 + x \geq 7$
3. $x \geq 2 \wedge y = 1 \to [\{x' = x^2 y + x^4, y' = 1 + y^2\}]x^3 \geq 1$
4. $x \geq 2 \wedge y \geq 2 \wedge z = 1 \to [\{x' = x^2 z + x^4, y' = y^2 + y^4 z, z' = z^2 + 1\}](x^3 \geq 1 \wedge y^3 \geq 1)$

5. $x^2 + y^2 = 0 \rightarrow [x' = 4y^3, y' = -4x^3]x^2 + y^2 = 0$

Exercise 5. *Prove that the following dL formula is a sound axiom:*

$$[x' = f(x)](e)' = 0 \rightarrow ([x' = f(x)]e = 0 \leftrightarrow e = 0).$$

Exercise 6. *Prove or disprove the following proof rule:*

$$\frac{F \wedge Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x)\&Q]F}$$