LFCPS Exercise Class 1
Modeling cyber-physical systems

Exercise 1. *Consider the following hybrid programs and compute the reachability relation* $[\![\alpha]\!]$ *for each one:*

1. $\alpha \equiv ?\,\mathrm{false}$,
2. $\alpha \equiv x := 1; (x := x + 1)^*$,
3. $\alpha \equiv x := 1; (x := x + 1)^*; ?x < 0$,
4. $\alpha \equiv x := 1; (x := x + 1; ?x < 0)^*$.

Exercise 2. *Consider the following hybrid programs and provide an intuitive interpretation for each:*

$$\alpha \equiv \{x' = v, v' = a \& v \geq 0\}$$
$$\beta \equiv \{x' = v, v' = a \& v \geq 0\}\,; ?v = 0$$
$$\gamma \equiv t := 0; \{x' = v, v' = a, t' = 1 \& t \leq T\}$$
$$\delta \equiv t := 0; \{x' = v, v' = a, t' = 1 \& t \leq T\}\,; ?(t = T)$$

Exercise 3. *Consider the following model of a bouncing ball:*

$$\alpha \equiv (\{x' = v,\ v' = -g\}\,;\ v := -cv)^*\,.$$

Here, the variables are defined as follows:

- x*: the height of the ball above the ground.*
- v*: the vertical velocity of the ball, where positive values indicate upward motion and negative values indicate downward motion.*
- g*: the constant acceleration due to gravity, acting downward.*
- c*: the coefficient of restitution, representing the energy retained after each bounce, with* $0 < c < 1$.

Is this a good model for a bouncing ball? Justify your answer by considering the physical behavior and limitations of the model.

Exercise 4. *Consider the following system:*

$$\alpha \equiv (\{x' = 1 \ \&\ x \leq 0\} \cup \{x' = -1 \ \&\ x > 0\})^*\,.$$

Starting with an initial condition $x \leq 0$*, analyze which states are reachable under this system. Describe how the evolution of* x *is constrained by the dynamics specified in* α.

Exercise 5. *Consider the following differential equation describing the dynamics of a controlled heating system:*

$$T' = k(T_s - T),$$

where T *represents the current temperature,* T_s *denotes the desired temperature setpoint, and* k *is a proportional constant related to the system's response rate.*

Model a controlled heating system that operates as follows:

- *The heating system turns on and the setpoint is set to 22°C when the temperature falls below 19°C.*
- *The heating system turns off when the temperature exceeds 21°C.*

Exercise 6. *Consider a robot located at a point with coordinates* (x, y)*, facing in a direction represented by the unit vector* (d_x, d_y)*. The robot moves forward in the direction it is facing while simultaneously rotating with angular velocity* ω*.*

Develop a system of differential equations to describe how the position (x, y) *and direction* (d_x, d_y) *of the robot evolve over time. Construct your model in two steps:*

1. *First, model the effect of rotation alone on the direction vector* (d_x, d_y)*.*
2. *Then, consider the translation of the position* (x, y) *in the current direction* (d_x, d_y) *of the robot.*

After building these components separately, combine them to form a complete system of differential equations describing the robot's dynamics.

Bonus*: As an additional challenge, try to extend this model by introducing an acceleration term to describe the robot's increasing linear speed.*

Exercise 7.

1. *Can you find a discrete controller* ctrl *and a continuous program* plant *such that the following two HPs* α, β *have different behaviors?*

$$\alpha \equiv (\mathrm{ctrl}; \mathrm{plant})^* \quad \beta \equiv (\mathrm{ctrl} \cup \mathrm{plant})^*.$$

2. *Can you find a discrete controller ctrl and a continuous program plant with the following two properties: a) The program ctrl has the same behavior as ctrl; ctrl and b) The following two HPs* α, β *have different behaviors?*

$$\alpha \equiv \mathrm{ctrl}; (\mathrm{ctrl}; \mathrm{plant})^* \quad \beta \equiv \mathrm{ctrl}; (\mathrm{ctrl} \cup \mathrm{plant})^*.$$