Exercise 7: Completeness in dL

Soundness: Every provable formula is valid.

Completeness: Every valid formula is provable.

Gödel's First Incompleteness Theorem: Any sufficiently powerful, consistent, effective formal system is incomplete.

A riddle: "Let us define a logician to be *accurate* if everything he can prove is true; he never proves anything false. One day, an accurate logician visited the Island of Knights and Knaves, in which each inhabitant is either a knight or a knave, and knights make only true statements and knaves make only false ones. The logician met a native who made a statement from which it follows that the native must be a knight, but the logician can never prove that he is! What statement would work?" [4, p. XXIII.1]

Differential Dynamic Logic can *not* be complete. The best thing one can hope for is *relative completeness.*

Definition 1. *A logical system L is* complete relative *to a syntactic fragment $F \subseteq L$, if every valid formula is derivable from the valid formulas of F.*

Theorem 2 (dL relatively complete, [2])**.** *Differential Dynamic Logic is complete relative to its continuous and discrete fragment:*

1. $\mathsf{Taut}_{\mathsf{FOD}} \vdash \varphi$ *iff* $\vDash \varphi$
2. $\mathsf{Taut}_{\mathsf{DL}} \vdash \varphi$ *iff* $\vDash \varphi$

Key Idea: Expressiveness (differential expressiveness).

Continuous Completeness: Need to express every hybrid program with only continuous programs. The difficulty is with loops! You can use real Gödel encodings to encode an *entire* possible run of the program into a single real number. A loop in a diamond modality becomes an existentially quantified formula and a loop in a box modality becomes a universally quantified formula.

Discrete Completeness: Need to express every hybrid program without continuous programs. Use Euler-approximation steps in a loop to approximate the trajectory.

Equiexpressible and Relatively Decidable: Full dL, its continuous fragment and its discrete fragment are all equiexpressible. Full dL is decidable relative to an oracle for these two fragments.

Theorem 3 (Invariant Completeness [3])**.** *All valid (semi-algebraic)* differential invariants *are provable in the* dL *calculus.*

The Hilbert Basis Theorem plays a role in the proof of Theorem 3.

Theorem 4 (Hilbert Basis Theorem for $\mathbb{R}$ [1, Theorem 4])**.** *Suppose $p_1, p_2, \ldots \in \mathbb{R}[x_1, \ldots, x_k]$ are polynomials. There is some N such that for all $m > N$ there are polynomials $g_1, \ldots, g_N \in \mathbb{R}[x_1, \ldots, x_k]$ such that*

$$p_m = \sum_{i=1}^{N} g_i \cdot p_i$$

For a term e the *differential radical formula* $\dot{e}^{(*)} = 0$ is

$$\bigwedge_{i=0}^{N} \dot{e}^{(i)}$$

where N is as in Theorem 4 for polynomials $e^{(0)}, e^{(1)}, \ldots$

Theorem 5 (Algebraic Invariant Completeness [3, Theorem 4.5])**.** *The differential radical invariant axiom:*

$$[x' = f(x) \& Q]e = 0 \leftrightarrow (Q \rightarrow \dot{e}^{(*)} = 0)$$

is derivable in dL*.*

A dL program is *algebraic* iff every formula in a test or evolution domain constraint is of the form $g = 0$.

Theorem 6 (Algebraic Hybrid Systems Completeness [3, Corollary 4.6])**.** *For any formula* $[\alpha]e = 0$ *describing an algebraic hybrid system there is a term* g *such that*

$$[\alpha]e = 0 \leftrightarrow g = 0.$$

In particular $\varphi \rightarrow [\alpha]e = 0$ *is provable iff it is valid.*

References

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[2] *Logic in Computer Science (LICS), 2012 27th Annual IEEE Symposium on.* Los Alamitos: IEEE, 2012. ISBN: 978-1-4673-2263-8.

[3] André Platzer and Yong Kiam Tan. "Differential Equation Invariance Axiomatization". In: *J. ACM* 67.1 (2020), 6:1–6:66. DOI: 10.1145/3380825.

[4] R.M. Smullyan. *The Riddle of Scheherazade and Other Amazing Puzzles, Ancient & Modern.* A harvest book. Harcourt Brace, 1998. ISBN: 9780156006064.