LFCPS Exercise Class 5
Differential Games

Exercise 1. *Recall the syntax for programs in Hybrid games*

$$\alpha, \beta ::= x := e \mid ?Q \mid x' = f(x) \& Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^\star \mid \alpha^d.$$

Hybrid games introduce the dual *operator that flips the player from demon to angel and vice versa. Find the corresponding formulas constructed with the syntax given above for the following formulas*

1. $\alpha \cap \beta \equiv ...,$
2. $\alpha^\times \equiv ...,$
3. $(x := e)^d \equiv ....$

Can you also construct such formulas for

1. $(x' = f(x) \& Q)^d,$
2. $(?Q)^d.$

Exercise 2. *Prove that the following axiom are sound:*

1. $\langle ; \rangle \quad \langle \alpha ; \beta \rangle P \leftrightarrow \langle \alpha \rangle \langle \beta \rangle P,$
2. $\langle \cup \rangle \quad \langle \alpha \cup \beta \rangle P \leftrightarrow \langle \alpha \rangle P \vee \langle \beta \rangle P.$

Exercise 3. *Recall the semantics of hybrid games given on the handout. In the lecture, we considered two explicit definitions for the semantics of loops, e.g.,*

1. $\zeta_{\alpha^\star}(X) = \bigcup_{n \in \mathbb{N}} \zeta_{\alpha^n}(X),$
2. $\zeta_{\alpha^\star}(X) = \bigcup_{n \in \mathbb{N}} \zeta_\alpha^n(X), \quad \zeta_\alpha^0(X) = X,\ \zeta_\alpha^{n+1}(X) = X \cup \zeta_\alpha(\zeta_\alpha^n(X))$ *for* $n \in \mathbb{N}.$

Now, address the following two tasks:

1. *Explore the distinctions between these two definitions and support your argument with an example.*
2. *Investigate whether these definitions stand up to scrutiny.*

Exercise 4. *Prove or disprove:*

1. $\dfrac{\vdash P \vee \langle \alpha \rangle Q \to Q}{\vdash \langle \alpha^\star \rangle P \to Q}$
2. $\dfrac{\vdash \langle \alpha \rangle (P \vee Q)}{\vdash \langle \alpha \rangle P \vee \langle \alpha \rangle Q}$

Exercise 5. *Use the* **dGL axioms to prove the following formulas:**

1. $\langle x := -x \cup (x := x + 1 \cap x := x + 2) \rangle x > 0$

2. $\langle x := x^2 \cup (x := x + 1 \cap x' = 2) \rangle x > 0$

3. $\langle x := -x \cup (x := x + 2 \cap x' = 2) \rangle x \geq 0$

4. $\langle (x := -x \cup (x := x + 2 \cap x' = 2))^\star \rangle x \geq 0$