5.0 Winning and Proving Hybrid Games

Truth Identification

5.1. In which states is the following dGL formula true and what is Demon's winning strategy in those states

$$[\big(((\{x'=1\}\cup\{x'=-1\});(\{y'=1\}^{\mathrm{d}}\cap\{y'=-1\}^{\mathrm{d}})\big)^*]\,x<y$$

In which states is this variation true and what is Demon's winning strategy?

$$[\big(((\{x'=1\}\cup\{x'=-1\});(\{y'=1\}^{\mathrm{d}}\cap\{y'=-1\}^{\mathrm{d}})\big)^*]\,(x-y)^2<5$$

These dGL formulas have disconnected physics, where the duration of evolution of Angel's differential equation may have nothing to do with the duration of evolution of Demon's differential equation. Most games synchronize in time, however. The following dGL formula has different control choices for the different players but the differential equations are combined into a single differential equation system under Angel's control of time. In which states is the following formula true and what is Demon's winning strategy?

$$[\big((v:=1\cup v:=-1);(w:=1\cap w:=-1)\{x'=v,y'=w\}\big)^*]\,(x-y)^2<5$$

5.2 (Say when). For each of the following dGL formulas identify the set of states in which it is true and characterize this set by a formula of real arithmetic. For each case, briefly sketch the player's winning strategy when it is true and explain why the dGL formula is false in all other states by giving a counterstrategy:

$$\langle x:=-1\cup(x:=0\cap x:=y)\rangle x\geq 0$$
$$\langle\big(x:=x+2\cup(x:=x-1;\{x'=-1\}^{\mathrm{d}})\big)^*\rangle 0<x\leq 2$$
$$[(x:=x\cap x'=-2)^*]x\geq 0$$
$$[(x:=-x\cap x'=-x^2)^*]x\geq 0$$
$$\langle\big(x:=0\cup\big((x:=x+1;\{x'=1\}^{\mathrm{d}})\cup x:=x-1\big)\big)^*\rangle 0<x\leq 1$$
$$\langle\big(x:=x^2\cup(x:=x+1\cap x'=2)\big)^*\rangle x>0$$

5.3. Explain how often you will have to repeat the winning-region construction from the inflationary semantics to show that the following dGL formulas are valid:

$$\langle(x:=x+1;x'=1^{\mathrm{d}}\cup x:=x-1)^*\rangle\,(0\leq x<1)$$
$$\langle(x:=x-1;y'=1^{\mathrm{d}}\cup y:=y-1;z'=1^{\mathrm{d}}\cup z:=z-1)^*\rangle(x<0\wedge y<0\wedge z<0)$$

Formal Proofs with dGL

5.4 (Diamond proofs). Use the dGL axioms to prove the following formulas:

$$\langle x := -x \cup (x := x+1 \cap x := x+2)\rangle x > 0$$
$$\langle x := x^2 \cup (x := x+1 \cap x' = 2)\rangle x > 0$$
$$\langle x := -x \cup (x := x+2 \cap x' = 2)\rangle x \geq 0$$
$$\langle \big(x := -x \cup (x := x+2 \cap x' = 2)\big)^*\rangle x \geq 0$$

5.5 (Demon's controls). Show that the following dGL axiom for Demon's controls is sound.

$$\langle\cap\rangle\ \langle \alpha \cap \beta\rangle P \leftrightarrow \langle\alpha\rangle P \wedge \langle\beta\rangle P$$

5.6 (Demon's controls). Show that the following dGL axiom for Demon's controls is derived from the definition of Demon's control operators.

$$\langle{}^\times\rangle\ \langle \alpha^\times\rangle P \leftrightarrow P \wedge \langle\alpha\rangle\langle\alpha^\times\rangle P$$

5.7 (Demon's repetition). Show that the following proof rules for Demon's repetition are derived rules:

$$\text{ind}^\times\ \frac{P \rightarrow \langle\alpha\rangle P}{P \rightarrow \langle\alpha^\times\rangle P} \qquad \text{FP}^\times\ \frac{P \vee [\alpha]Q \rightarrow Q}{[\alpha^\times]P \rightarrow Q}$$

5.8 (Unsound axioms). Not all hybrid systems axioms can be used for hybrid games. Prove that the following perfectly valid hybrid systems axiom is unsound for hybrid games by giving a counterexample, i.e., an instance of the axiom that is not a valid dGL formula:

$$[]\wedge\ [\alpha](P \wedge Q) \leftrightarrow [\alpha]P \wedge [\alpha]Q$$