4.0 Proving Unsolvable Differential Equations

Differential Equations With Proofs

4.1 (Differential invariant practice). Prove the following formulas using differential invariants, differential cuts, and differential weakening as required:

$$\omega \geq 0 \land x = 0 \land y = 3 \rightarrow [x' = y, y' = -\omega^2 x - 2\omega y]\, \omega^2 x^2 + y^2 \leq 9$$
$$xy^2 + x \geq 7 \rightarrow [x' = -2xy, y' = 1 + y^2]\, xy^2 + x \geq 7$$
$$x \geq 2 \land y = 1 \rightarrow [x' = x^2 y + x^4, y' = 1 + y^2]\, x^3 \geq 1$$
$$x \geq 2 \land y \geq 2 \land z = 1 \rightarrow [x' = x^2 z + x^4, y' = y^2 + y^4 z, z' = z^2 + 1]\, (x^3 \geq 1 \land y^3 \geq 1)$$

4.2 (Differential ghost practice). Prove the following formulas using differential ghosts, differential invariants, differential cuts, and differential weakening as required. When using differential ghosts also intuitively explain why the differential ghost was necessary.

$$x > 0 \rightarrow [x' = -x]\, x > 0$$
$$x > 0 \rightarrow [x' = -x^2]\, x > 0$$
$$x > 0 \rightarrow [x' = x^2] x > 0$$
$$x > 0 \rightarrow [x' = x] x > 0$$
$$x > 4 \rightarrow [x' = x] x > 4$$
$$x > 1 \rightarrow [x' = -x]\, x > 1$$

Proof Principles for Differential Equations

4.3 (Weak differentials of strong inequations). Prove that both of the following alternative definitions yield a sound differential invariant proof rule:

$$(e < k)' \equiv ((e)' < (k)')$$
$$(e < k)' \equiv ((e)' \leq (k)')$$

4.4 (Disequalities). We have defined

$$(e \neq k)' \equiv ((e)' = (k)')$$

Suppose you remove this definition so that you can no longer use the differential invariant proof rule for formulas involving $\neq$. Can you derive a proof rule to prove such differential invariants regardless? If so, how? If not, why not?

4.5 (Derivation lemma proof). Prove the other cases of the derivation lemma where the term is a variable x or a subtraction $e - k$.