LFCPS Exercise Class 1
Modeling cyber-physical systems

First-order logic

Exercise 1. *Add the omitted parentheses:*

1. $\forall \varepsilon \, \exists \delta \, \forall y \, (x-y)^2 < \delta \rightarrow (f(x) - f(y))^2 < \varepsilon$
2. $\varphi \wedge \psi \wedge \rho$
3. $\varphi \wedge \psi \vee \rho$
4. $\alpha; \beta; \gamma$
5. $\alpha \cup \beta; \gamma$

Exercise 2. *In which of the following states is the formula* $\exists x \;\, x > 0 \wedge x = y$ *satisfied?*

1. $\omega(x) = 1$ *and* $\omega(y) = 1$.
2. $\omega(x) = 1$ *and* $\omega(y) = -1$.
3. $\omega(x) = -1$ *and* $\omega(y) = 1$.

Exercise 3. *Find a quantifier-free formula* $\varphi(a, b, c)$ *such that*

$$\exists x (ax^2 + bx + c) = 0 \leftrightarrow \varphi(a, b, c)$$

is valid (over $\mathbb{R}$*).*

Differential Equations

Exercise 4. *Solve the following initial value problems:*

1. $x'' = 3$ *and* $x(2) = 4$
2. $x' = 2x$ *and* $x(0) = -1$
3. $x' = 2y, y' = -2x$ *and* $(x(0), y(0)) = (1, 0)$

Exercise 5. *(*) Suppose* $x : (a, b) \rightarrow \infty$ *is a total differentiable function with* $x' = 1 + x^2$ *and* $x(0) = 0$*. Show that* $x(t) \geq 0$ *for* $t \in [0, b)$*.*

Exercise 6. *Consider a robot at a point with coordinates* (x, y) *that is facing in direction* (d_x, d_y)*. While the robot is moving in the direction in which it is facing, it is simultaneously rotating with angular velocity* ω*.*

Develop a differential equation system describing how the position and direction of the robot change over time. Build your way up to that differential equation by first considering just the rotation of (d_x, d_y)*, then considering the motion of* (x, y) *in a fixed direction* (d_x, d_y)*, and then putting both types of behavior together. Can you subsequently generalize the dynamics to also include an acceleration of the linear ground speed when the robot is speeding up?*

Hybrid Programs

Exercise 7. *For the following hybrid programs compute the transition relation they define, i.e. compute* $[\![\alpha]\!]$.

1. $\alpha \equiv x := 1; (x := x + 1)^*$.
2. $\alpha \equiv x := 1; (x := x + 1)^*; ?x < 0$.
3. $\alpha \equiv x := 1; (x := x + 1; ?x < 0)^*$.

Exercise 8. *Find a hybrid program* α *such that* α *has runs only from states* ω *in which* $\omega(x)$ *is a positive even integer.*

Exercise 9. *What is the difference between the continuous programs* α *and* β*?*

1. $\alpha \equiv x' = x$ *and* $\beta \equiv x' = x \& x > 0$
2. $\alpha \equiv x' = y, y' = -x$ *and* $\beta \equiv (x' = y, y' = -x \& x^2 + y^2 = 1)$

Exercise 10. *1. Can you find a discrete controller* ctrl *and a continuous program* plant *such that the following two HPs* α, β *have different behaviors?*

$$\alpha \equiv (\mathsf{ctrl}; \mathsf{plant})^* \qquad \beta \equiv (\mathsf{ctrl} \cup \mathsf{plant})^*.$$

2. Can you find an initialization program init*, a discrete controller* ctrl *and a continuous program* plant *with the following two properties: a) The program* init *has the same behavior as* init; ctrl *and b) The following two HPs* α, β *have different behaviors?*

$$\alpha \equiv \mathsf{init}; (\mathsf{ctrl}; \mathsf{plant})^* \qquad \beta \equiv \mathsf{init}; (\mathsf{ctrl} \cup \mathsf{plant})^*.$$

Exercise 11. *Are there terms* $f(x), g(y)$ *such that the hybrid programs*

$$\alpha \equiv y := x; (x' = f(x), y' = g(y)); y := x \qquad \beta \equiv x' = f(x); y := x$$

have different behaviors?

Modeling Pitfalls

Exercise 12. *We want to model a bowling ball being rolled on a bowling lane. (Modeled as the interval* $[0, L]$*.) Because we are only interested in what happens while the ball is on the lane we add an evolution domain constraint. Now we can start rolling the ball at some point* $x_0 \in [-P, 0)$ *before the lane begins. We assume we release the ball with some velocity* $v_r \in (a, b)$*. For simplicity we assume that the bowling ball maintains constant velocity (ignoring friction).*

1. Is

$$x := *; ?(-P \leq x < 0); v := *; ?(a < v < b); (x' = v \& 0 \leq x \leq L)$$

a good model?

2. We now want to extend this model to be able to roll a second ball in case we haven't knocked over all the pins. What would be a good hybrid program model?

Exercise 13. *We consider a modification of the robot of Exercise 6. It is still located at position* (x, y) *and facing in direction* d_x, d_y*. Now the robot can only either rotate or move straight in the direction it is facing. In order to rotate the robot must first stand* *still and have set the acceleration a to 0. To model the two separate physical processes we define*

$$\mathsf{move} \equiv (x' = vd_x, y' = vd_y, v' = a \& v > 0)$$
$$\mathsf{rotate} \equiv (d_x' = d_y, d_y = -d_x \& v = 0)$$

A protype for the controller of the robot could be:

$$(a := *; ?(0 \leq a < A); (\mathsf{move} \cup ?(a = 0); \mathsf{rotate}))^*.$$

Is this a good model?

Exercise 14. *We want to model a remotely controlled car. The car receives a signal c telling it when to brake (c = 0) and when to accelerate (c = 1). Is*

$$(c := *; (?c = 0; (x'' = -A \& v \geq 0) \cup ?c = 1; (x'' = A \& v \geq 0)))^*$$

a good model? (A > 0)

Modeling

Exercise 15. *We model the floodgates of a dam.*

1. *We model the reservoir as a cuboid with base area A.*
2. *The dam gates have area a. Opening them causes the water to flow out at a rate of av where v is the velocity at which the water is flowing out.*
3. *By Bernoulli's principle, the water flows at a velocity of $\sqrt{2gh}$ where h is the current height of the water in the reservoir.*
4. *Water flows into the reservoir from a river at a rate of $r(t)$, where $0 \leq r(t) \leq M$.*

Write a hybrid program model for the dam controller.