Driven Oscillations of a Car Suspension

Star Lab (Term Paper)

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Abstract

The suspension of a car is the system of tires, tire air, springs, shock absorbers and linkages that connects a vehicle to its wheels and allows relative motion between the two. A good suspension system must assure enough adherent forces between tyre and road in order to guarantee good brake performances and steering control as well as maintaining passenger comfort within an adequate level. This project will be about modeling the suspension of a car as it drives at constant speed on a sinusoidal road.

Introduction

Due to the increasing requests from society to improve road safety, vehicle manufacturers and administrations have long been working in order to increase vehicle safety systems and vehicle regulations as well as road infrastructure. However, the natural wearing of vehicle systems, sometimes increased due to incorrect operation or lack of maintenance, raises the traffic accident risks. A few systems, such as steering, brakes or suspension, critically affects vehicle safety, thus it is necessary to check these elements up to a certain age in order to maintain the vehicle in optimal safety conditions.

The suspension of a car is the system of tires, tire air, springs, shock absorbers and linkages that connects a vehicle to its wheels and allows relative motion between the two. A good suspension system must assure enough adherent forces between tyre and road in order to guarantee good brake performances and steering control as well as maintaining passenger comfort within an adequate level.

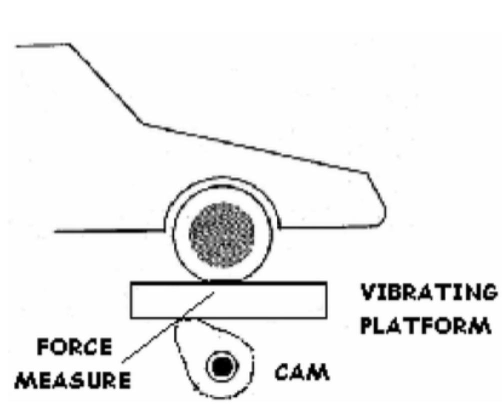
This project will be about modeling the suspension of a car as it drives at constant speed on a sinusoidal road. More specifically, the range of speeds at which the car must drive in order to keep the mudguards above the wheels will be calculated in a first step, and verified using KeYmaera X in a second step. After that, a damping ratio will be introduced and the optimal set of values for the damping ratio will be computed and verified using KeYmaera X.

Related work

While the brake system of a car has an objective check method and validation criteria, regulated by a norm that must be met for the approval of the vehicle in order to determine the system effectiveness, in the case of the suspension system, there are no official regulations. Because of that, test bench manufacturers together with shock absorbers manufacturers have proposed some test methods and validation criteria [5]. This lack of uniformity, caused by different criteria used depending on the kind of test bench utilised means that a vehicle could be accepted by one test and rejected by another.

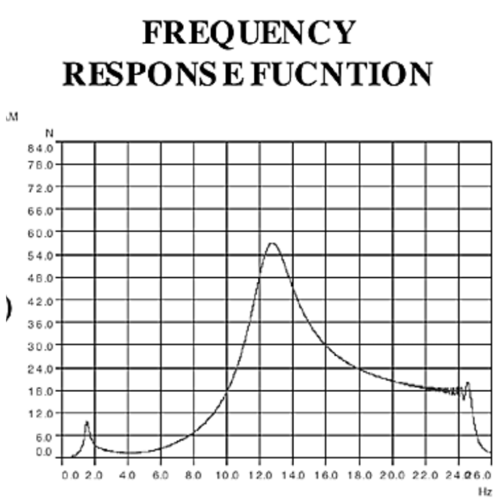
This observation is what pushed researchers at Carlos III University of Madrid [5] to conduct experiments and find the best possible parameters of a suspension system that assure safety over a wide range of vehicles.

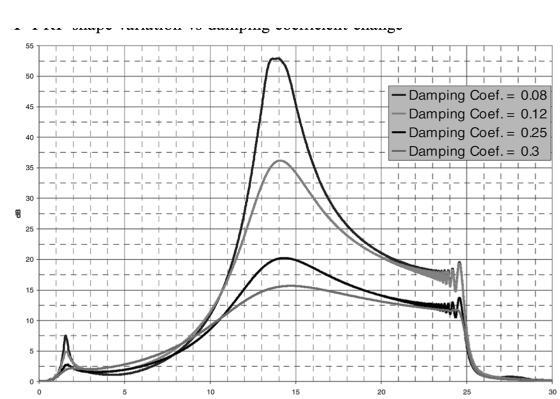
The experiment consists of a vibrating platform test bench made of an electrical motor that moves a cam. This cam transmits the movement to a platform on whose top one of the four suspension columns of the vehicle is located. The platform vibrates with constant amplitude (usually 3 mm) and a variable frequency. The force between the wheel and the platform is measured by a force sensor installed on the platform.



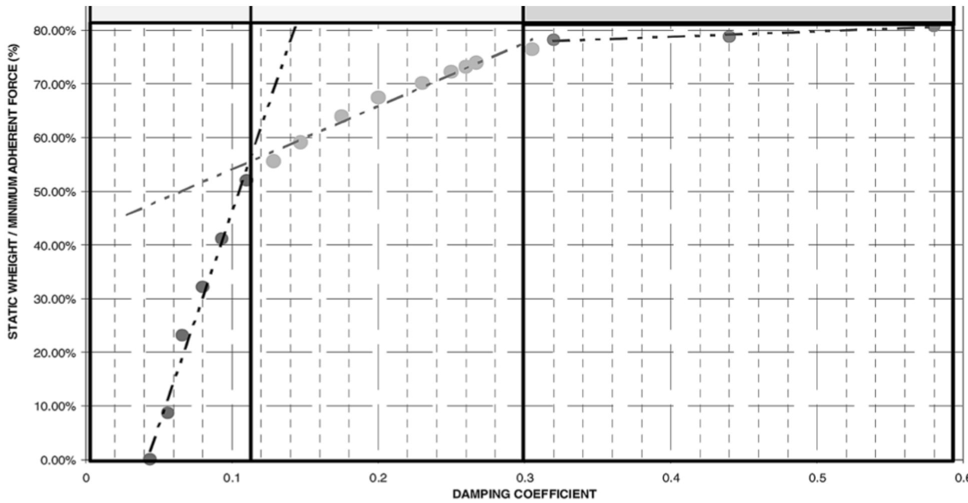
The validation criteria is simply the ratio between the minimal measured force between the wheels and the platform, and the static force, which is simply the weight of the car. As this ratio tends towards 0, the vehicle tends to lift off the platform and the tires stop transmitting the force of the motor to the road because of the lack of friction (e.g. the force between wheels and platform becomes too small).

The first thing they found out is that the system had two resonance frequencies. The first one (at about 1 Hz) was the resonance frequency of the unsprung mass, which means the excitement of the mass not supported by the suspension (e.g. tires, suspension itself etc...). The second resonance frequency is much higher and corresponds to the sprung mass of the system (the mass supported by the suspension). Both peaks are shown on the left picture below. Obviously, as the damping ratio increases, the peak becomes smaller (right picture below), but they determined that the worst case scenario is represented by the second resonance peak. In a followup experiment, they varied the damping ratio while maintaining the resonance frequency and measured the force between wheels and platform again.





The result show that the minimum adherent force becomes dangerously close to 0 as the damping ratio drops below 0.1.



The conclusions of the study are that the component that has the most influence on dynamic behaviour of the suspension system is the shock absorber (e.g. the damping ratio), and that a 'Limit Damping Coefficient' as validation criteria can be established, below which the dynamic behaviour of the suspension system demonstrates an outstanding loss of performance. That means an excessive wearing out of the shock absorber and a need to change it in order to maintain proper vehicle safety conditions.

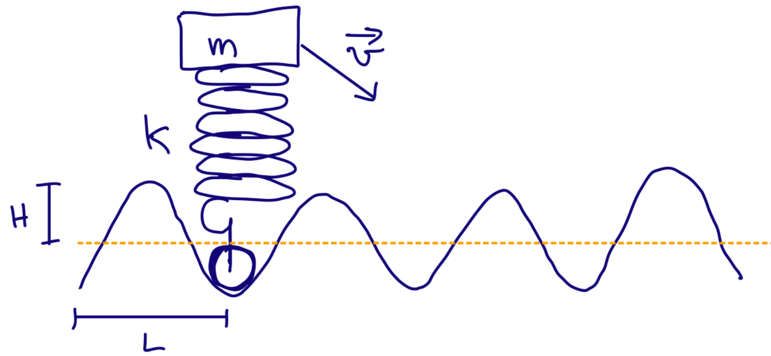
Driven oscillations, no damping

Problem statement

In our model, we will ignore the unsprung mass as it is negligible compared to the sprung mass. We will also model only 1 wheel according to 2 dimensions. First let's consider the case of a car driving on a sinusoidal road of wave length L and amplitude H. The car is modelled by a wheel touching the ground at any point in time. The car of mass m is directly above the wheel and connected to the wheel via a coil spring of stiffness k and initial length l0. The car drives at constant speed vx. There is no damping in the coil spring. The goal is to find an expression for the height of the car at a given time t, then compute the possible values for vx such that the car never hits the mudguards (more formally, the car must always be above the wheels).

The model consists of

- a sinusoidal road of amplitude H (the height of the road with respect to the horizontal position is modeled by an equation of type
  $h(x) \; = \; \frac{H}{2}\, sin(\frac{2\pi}{L}\; x)$
- a coil spring of stiffness k (verifying to Hooke's law [1]), and initial length *l0*
- a car of mass m moving at horizontal speed vx
- a wheel connected to the spring, touching the ground at any point in time and following the sinusoide



Differential equation

The goal is to calculate the differential equation of the vertical position of the car, as it moves at constant speed over the sinusoide. We expect the ODE to be the ODE of a driven harmonic oscillator (a harmonic oscillator being subject to an external force, in this case the car's velocity "pushing" the wheel against the road, thus extending and compressing the spring). Recall that the ODE of a driven oscillator with no damping has the following shape.

$$u'' + \omega_0^2 u = \alpha_0^2 sin(\omega t)$$

where

- u is the position of the car along the vertical axis
- ω_0 is the original pulse of the oscillator
- ω is the pulse of the applied force

Let's first compute the position wheel_y of the wheel at time t. We already know that the road has a sinusoidal shape and that the wheels are always touching the road. Thus

$$wheel\_y = \frac{H}{2} sin\left(\frac{2\pi}{L}\, wheel\_x\right)$$

We also know that the car is driving at constant speed vx. Thus

$$wheel\_x = vx\ t \quad \text{and} \quad wheel\_y = \frac{H}{2} sin\left(\frac{2\pi}{L}\ vx\ t\right)$$

Let car_y be the vertical position of the car. According to Newton's second law [2],

$$\sum F = m\ a \leftrightarrow P + F = m\ a$$

where a is the acceleration of the car, m is the mass of the car, P is the weight of the car and F is the force that the coil spring applies to the car.

Projecting these forces on the vertical axis, we get

$$P = -mg \quad \text{and} \quad F = -k\,\Delta z \text{ [1]}$$

where m is the mass of the car, g is the gravitational acceleration of the earth, Δz is the distension of the coil spring and k is the stiffness of the coil spring [1] According to Hooks's law, we know that a force needed to compress or extend a spring is proportional to its displacement Δz. Δz can be written as $car\_y - wheel\_y - l0$, since $car\_y - wheel\_y$ is the actual length of the coil spring. So $car\_y - wheel\_y - l0$ is the distension of the spring with respect to its length at equilibrium l0.

Note that if $car\_y - wheel\_y - l0 > 0$ then the coil is distended (and not compressed), and pulls the car towards the road. Indeed $F =- k\ \Delta z < 0$ is directed towards the road.

Injecting these results into Newton's second law [2], we get

$$m\ car\_y'' = -mg - k(car\_y - wheel\_y - l0)$$
$$m\ car\_y'' = -mg - k\,car\_y + k\,l0 + k\,\frac{H}{2}\,sin(\frac{2\pi}{L}\,vx\ t)$$

If $\omega = \frac{2\pi vx}{L}$, $\omega_0 = \sqrt{\frac{k}{m}}$, then after dividing by m,

$$car\_y'' + g + \omega_0^2\,car\_y - \omega_0^2\,l0 = \omega_0^2\,\frac{H}{2}\ sin(\omega t)$$

iff

$$car\_y'' + \omega_0^2\,(car\_y + \frac{g}{\omega_0^2} - l0) = \omega_0^2\,\frac{H}{2}sin(\omega t)$$

If we define the new variable $u = car\_y + \frac{g}{\omega_0^2} - l0$, then $car\_y'' = u''$ and

$$u'' + \omega_0^2 u = \omega_0^2\frac{H}{2}sin(\omega t)$$

and we get exactly the equation that we were looking for. The fact that $\omega_0 = \sqrt{\frac{k}{m}}$ is not surprising at all since according to Hooke's law [1], it is the pulse of a harmonic oscillator.

Intuitively, we expect the vertical position of the car car_y to be the same as the vertical position of the wheels wheel_y, scaled by some factor and shifted by some constant. Let's try a solution of the shape

$$car\_y\,(t) = A\ sin(\omega t) + l0 - \frac{mg}{k}$$

where A is the amplitude of the resulting oscillations. The constant shifting is simply the length of the coil spring minus the length that is already compressed by the weight of the car.
By taking the second derivative and injecting the result in the ODE, we get

$$-A\omega^2\,sin(\omega t) + \omega_0^2 A\,sin(\omega t) = \omega_0^2\,\frac{H}{2}sin(\omega t)$$

Dividing by $sin(\omega t)$ yields

$$(\omega_0^2 - \omega^2)A = \omega_0^2\ \frac{H}{2}$$

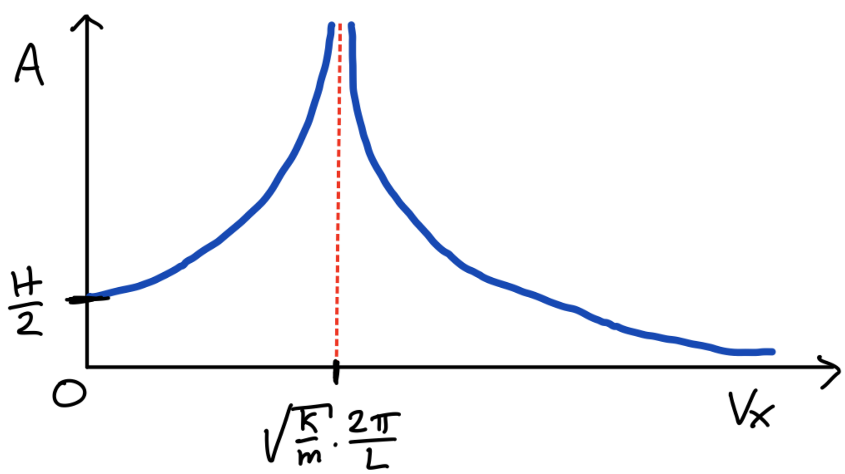
so

$$A = \frac{H}{2}\frac{1}{1-\frac{\omega^2}{\omega_0^2}}$$

Let's look at this expression more closely. A is the amplitude of the oscillations of the car. We want to keep the amplitude low and most of all, it must always be low enough that the car does not touch the wheel. We need to make the distinction between 3 different cases here. Remember that $\omega = \frac{2\pi vx}{L}$ and $\omega_0 = \sqrt{\frac{k}{m}}$.

- The speed vx is such that $\omega = \omega_0$. This is true if $vx = \sqrt{\frac{k}{m}\frac{L}{2\pi}}$. In that case the amplitude tends towards infinity and the ride will become extremely uncomfortable. This phenomenon is called resonance [3]. It happens when a harmonic oscillator is driven by the pulse of the oscillator itself. For example, it happens when pushing a kid on a swing. Or when the wind destroyed the Tacoma bridge [4].
- The speed vx is such that $\omega < \omega_0$. This is true if $vx < \sqrt{\frac{k}{m}\frac{L}{2\pi}}$. In that case the amplitude is a finite real number, but can never be lower than some positive limit: if vx tends towards 0, then A tends towards H/2. Intuitively, this means that if the car drives extremely slowly, there will barely be any oscillations and the length of the coil spring remains almost constant. The amplitude of the car is almost the same as the amplitude of the road.
- The speed vx is such that $\omega > \omega_0$. This is true if $vx > \sqrt{\frac{k}{m}\frac{L}{2\pi}}$. In that case, the higher the speed, the lower the amplitude. In the extreme case where vx tends towards infinity, A tends towards 0.

Let's sketch the amplitude with respect to the speed vx.



Verification using KeYmaera X

After performing the above calculations, we can now model such a system in KeYmaera X, and show that if the speed is fast enough, then the car will never hit the wheels. Let's try to find the different speeds for which the condition

$$wheel\_y(t) < car\_y(t)$$

is true for every t.

$$
\begin{aligned}
& car\_y(t) \; > \; wheel\_y(t) \\
\Leftrightarrow \; & \frac{H}{2}\frac{1}{1-\frac{\omega^2}{\omega_0^2}} sin(\omega t) \; + \; l0 - \frac{mg}{k} > \frac{H}{2} sin(\omega t) \\
\Leftarrow \; & \frac{H}{2}\frac{1}{1-\frac{\omega^2}{\omega_0^2}} + \; l0 - \frac{mg}{k} > \frac{H}{2} \\
\Leftrightarrow \; & \ldots \\
\Leftrightarrow \; & vx > \sqrt{\frac{k}{m}\frac{L}{2\pi}} \sqrt{\frac{1}{1-\frac{H}{2(l0-\frac{mg}{k})}}}
\end{aligned}
$$

Recall that the condition to avoid resonance was $vx > \sqrt{\frac{k}{m}} \frac{L}{2\pi}$. This condition is fulfilled if

$$\sqrt{\frac{1}{1-\frac{H}{2(l0-\frac{mg}{k})}}} > 1$$

which holds when

$$l0 - \frac{mg}{k} > \frac{H}{2}$$

Intuitively, this means that the length of the coil spring at equilibrium minus the length that is already compressed by the weight of the car must be above $\frac{H}{2}$. Indeed, if the length of the coil is too short, then there is no way to keep the car above the bumps at all times. This important initial condition will be added later in our proof.

For the modelling, the idea is not to use a controller, but simply to show that for any speed above the threshold, at any point in time, the car stays above the wheels.

To get rid of the annoying π constant, we will simply make the assumption that $L = 2\pi$.

Model

The model will make use of the following constants

- k, the stiffness of the spring according to Hooke's law [1]
- m, the mass of the car
- $l0$, the length of the coil spring at equilibrium position
- g, the gravitational acceleration of the earth
- H, the height of the bumps on the road
- vx, the horizontal speed of the car and the wheels

Let's talk about how we will model the movement of the wheel and the movement of the car. Recall that their equations are given by

$$
\begin{aligned}
car\_y\,(t) \; &= \; A \; sin(\frac{2\pi}{L}\, vx\, t) \; + \; l0 \; - \; \frac{mg}{k} \\
wheel\_y(t) \; &= \frac{H}{2}\, sin(\frac{2\pi}{L}\; vx \;\; t)
\end{aligned}
$$

Then $wheel\_y(t)' \ = \frac{H}{2}\frac{2\pi}{L}vx\,cos(\frac{2\pi}{L}\ \ vx\ \ t)$ and $wheel\_y(t)'' \ =-\ (\frac{2\pi}{L}vx)^2\frac{H}{2}sin(\frac{2\pi}{L}vx\ \ t)$. Combined with the fact that $L\ =\ 2\pi$,

$$wheel\_y(t)' \ = wheel\_vy(t)$$
$$wheel\_vy(t)' =-\ vx^2 wheel\_y(t)$$

Next, we note that $car\_y\ (t)$ and $wheel\_y(t)$ have the same phase and pulse, which means that both are sinus functions that are multiples of each other. More formally,

$$\frac{H}{2}(car\_y(t)\ -\ (l0\ -\ \frac{mg}{k}))=A\,wheel\_y(t)$$
$$\frac{H}{2}car\_y(t)'=A\,wheel\_y(t)'$$
$$car\_y(t)' \ = \frac{1}{1-\frac{vx^2m}{k}}wheel\_vy(t)$$

This is the final model

```
Definitions
        Real k;     /* stiffness of the spring */
        Real m;     /* mass of the car */
        Real l0;    /* length of the spring at equilibrum */
        Real g;     /* gravitational acceleration */
        Real H;     /* height of the bumps of the road */
        Real vx;    /* horizontal speed of the car and wheels */
    End.

    ProgramVariables
      Real wheely;  /* veritcal position of the wheels */
      Real wheelvy; /* vertical speed of the wheels */
      Real cary;    /* vertical position of the car */
      Real carvy;   /* vertical speed of the car */
    End.

    Problem
      /* Initial conditions */

      ( /* constants are positive in the real world */
          k>0 & m>0 & l0>0 & g>0 & H>0 & vx>0 &
           /* sufficient speed */
        (k/m)^(0.5) * (1/(1-H/(2*(l0-m*g/k))))^(0.5) < vx &
        /* weight of the car does not completely compress the spring */
        l0 - m*g/k > H/2 &
        /* initial conditions for the wheel position and speed */
        wheely^2 + (wheelvy/vx)^2 = (H/2)^2 &
        /* synchronize wheels and car */
        (1-vx^2*m/k)*(cary-(l0-m*g/k))=wheely
        )
      ->
      [
        {
            wheely' = wheelvy,
            wheelvy' = -vx^2*wheely,
            cary' = 1/(1-vx^2*m/k)*wheelvy,
            carvy' = -vx^2*(cary-(l0-m*g/k))
        }
      ]
      /* Safety condition. The wheels never hit the car. */
      (  wheely < cary  )
    End.
```

Proof strategy

The proof strategy consists of showing that the initial conditions remain true (e.g. they are invariants). First, we show that the wheel position and speed behave like sinus and cosinus (the sum of their squares are constant) by differential cut.

$$wheely^2 + (wheelvy/vx)^2 = \frac{H^2}{4}$$

The next invariant to prove is that the car's position is a multiple of the wheel's position, which means that the sinuses stay aligned (same phase, same pulse). The dL expression is cut a second time using

$$(1 - vx^2 m/k)\,(cary - (l0 - \frac{mg}{k})) = wheely$$

Once we have that it is trivial to prove that, if the speed and coil length are sufficient, the car is always above the wheels using real logic.

Driven oscillations with a damping factor

Problem statement

For now, we were concentrating our efforts on a pure harmonic oscillator. However, in the real world, there is always a damping factor due to friction and adding a damping force would eliminate the unrealistic case of infinitely large oscillations in the case of resonance. With a damping force, the car is able to drive at lower speeds, and even at the speed corresponding to the resonance frequency, provided that the damping is strong enough. In this case, a damping force would simply be a force proportional to the vertical speed, but in the opposite direction. If we include a damping force, then the oscillations of our coil spring would slowly fade during a transition period of overlapping oscillations as the oscillations of the applied force would take over. This means that eventually, it will be like having no suspension at all.

Differential equation

The ODE for a damped driven harmonic oscillator has the following shape:

$$u'' + 2\zeta\omega_0 u' + \omega_0^2 u = \alpha_0 sin(\omega t)$$

where

- u is the position of the car along the vertical axis
- ω_0 is the original undamped pulse of the oscillator
- w is the pulse of the applied force
- ζ is the damping ratio
- α_0 is some constant

Let's see if we can construct this equation in order to find the missing constants. We follow the steps described above by applying Newton's second law [2], but this time we add a force that is proportional to the opposite of the vertical speed (the damping force) and derive the following result.

$$m\ car\_y'' = \ -mg - k\,car\_y + k\,l0 + k\,\frac{H}{2}\,sin(\frac{2\pi}{L}\,vx\ t) - c\,car\_y'$$

If $\omega\ = \frac{2\pi vx}{L}$, $\omega0\ =\ \sqrt{\frac{k}{m}}$, $\zeta = \frac{c}{2\sqrt{mk}}$ then after dividing by m,

$$car\_y'' + 2\zeta\omega_0 car\_y' + \omega_0^2\,(car\_y + \frac{g}{\omega_0^2} - l0) = \omega_0^2\frac{H}{2}\,sin(\omega t)$$

If we define the new variable $u\ =\ car\_y + \frac{g}{\omega_0^2} - l0$, then $car\_y''\ =\ u''$, $car\_y'\ =\ u'$

$$u'' + 2\zeta\omega_0 u' + \omega_0^2 u = \omega_0^2\frac{H}{2} sin(\omega t)$$

and we get exactly the equation that we were looking for. Again, this equation can be solved exactly for any driving force, using the solutions u(t) that satisfy the unforced equation, and adding the force on top of it. More formally, The general solution is a sum of a transient solution that depends on initial conditions, and a steady state that is independent of initial conditions. The transient solution will gradually fade away as time passes because it will be weakened by the damping coefficient. Eventually, only the steady state solution will remain [6].

Steady state solution

Let's try to solve the steady solution first. Like last time, our intuition dictates us to try a solution of the shape

$$u(t) = A\,sin(\omega\,t + \varphi)$$

Deriving this expression and injecting it into the ODE will give us

$$-\omega^2 A\,sin(\omega t + \varphi) + 2\zeta\omega_0\omega\frac{H}{2}cos(\omega t + \varphi) + \omega_0^2 A\,sin(\omega t + \varphi) = \omega_0^2\frac{H}{2}sin(\omega t)$$

By using the identity $sin(A - B) = sin(A)cos(B) - cos(A)sin(B)$ on $sin(\omega t)$ with $A = \omega + \varphi$ and $B = \varphi$, then rearranging terms, we get

$$[A(\omega_0^2 - \omega^2) - \omega_0^2\frac{H}{2}cos(\varphi)]sin(\omega t + \varphi) + [2A\zeta\omega_0\omega + \omega_0^2\frac{H}{2}sin(\varphi)]cos(\omega t + \varphi) = 0$$

We want the above equation to hold for all t. In particular, the equation is true when both constant coefficients of the sin and cos functions are zero. That way, we will hopefully be able to derive a solution which does not depend on t. This means that

$$A(\omega_0^2 - \omega^2) - \omega_0^2\frac{H}{2}cos(\varphi) = 0$$

and

$$2A\zeta\omega_0\omega + \omega_0^2\frac{H}{2}sin(\varphi) = 0$$

Summing the squares of both equations will give us

$$A = \frac{H}{2}\frac{1}{\sqrt{\left(\frac{2\omega\zeta}{\omega_0}\right)^2+\left(1-\frac{\omega^2}{\omega_0^2}\right)^2}}$$

Note that if the damping ratio ζ is zero, then this corresponds exactly to the amplitude we found earlier in the case of no damping. On the other hand, if the damping ratio tends towards infinity, then A tends towards 0. This makes sense as the coil spring will simply act like a cushion and will absorb the shock entirely.

Transient state solution

The transient state solution is obtained by solving the above differential equation but without the driving force.

$$u'' + 2\zeta\omega_0 u' + \omega_0^2 u = 0$$

This equation is more complex to solve, but we will skip the steps to derive it and jump directly to the solution. To express the solution, the distinction between three cases must be made here [7]

- $\zeta > 1$. This is called *overdamping*. It happens when the damping is so strong that it inhibits the oscillations. It is the same as falling into a cushion. The cushion absorbs the shock entirely. In the case of overdamping, the solution is given by

$$u(t) = Ae^{-\omega_0 t\,(\zeta+\sqrt{\zeta^2-1})}$$

where A is determined by the initial conditions.

- $\zeta < 1$. This is called *underdamping*. It happens when the damping is weak enough to allow some oscillations that will become smaller until they eventually fade. In practice, this form of damping is preferred because the ideal case of critical damping is hard to achieve and we want to avoid overdamping when constructing cars. Remember that in the real life experiment shown above damping factors between 0.05 and 0.6 were experimented. In the case of underdamping, the solution is given by

$$u(t) = Ae^{-\zeta\omega_0 t} sin(\varphi + \omega_0 t\sqrt{1-\zeta^2})$$

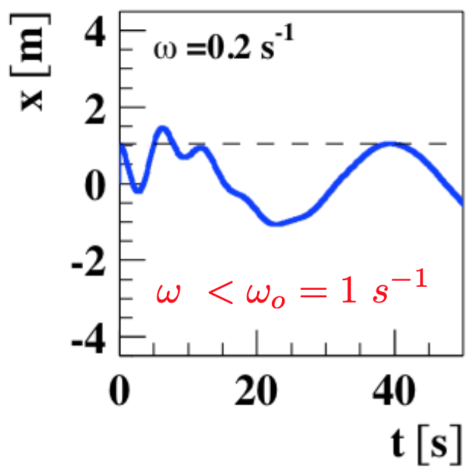
where the amplitude A and the phase φ are determined by the initial conditions.

- $\zeta = 1$. This is called *critical damping*. This is the limit between overdamping and overdamping. It means that there are no oscillations but if the damping was just a little bit weaker there would be some. In the case of critical damping, the solution is given by

$$u(t) = Ae^{-\zeta\omega_0 t}\omega_0 t$$

where A is determined by the initial conditions.

As mentioned before, the transient solution is added on top of the steady state solution. And since in all three cases (overdamping, critical damping and underdamping) the solution tends towards 0 as time gets big enough, the final solution will tend towards the steady state solution. This is shown in the figure below:



Verification using KeYmaera X

In the previous model, we showed that without any damping factor, if the speed was above some threshold, then we could bound the amplitude (e.g. avoid resonance and prevent the wheels from hitting the mudguards). Here, the amplitude is always bound. In the following model, we will try to show that if the damping is sufficiently high, and the car drives at resonance speed, then the car will never hit the wheels.

Using the observations above, we know that the transient solution of the damped driven harmonic oscillator will eventually fade away, so let's ignore it in our model. The solution of the steady state, as established earlier, is given by

$$u(t) = A\,sin(\omega\, t + \varphi)$$

where

$$A = \frac{H}{2}\frac{1}{\sqrt{\left(\frac{2\omega\zeta}{\omega_0}\right)^2 + \left(1 - \frac{\omega^2}{\omega_0^2}\right)^2}}$$

We will try to prove that the car never hits the wheels by proving the weaker fact that the length of the coil spring minus the car's amplitude is always higher than the highest bumps. In other words

$$l0 - \frac{mg}{k} - A > \frac{H}{2}$$

Unfortunately, the truth of the above expression does not only depend on c, but also on the length of the coil spring and the ratio of ω and ω_0 (below resonance, resonance and above resonance). The derivations of the results are complicated, so they will be skipped. The set of values for which the above expression is true are summarized in the table below.

	$l0 - \frac{mg}{k} < \frac{H}{2\left\lvert\frac{\omega^2}{\omega_0^2} - 1\right\rvert} + \frac{H}{2}$	$l0 - \frac{mg}{k} >= \frac{H}{2\left\lvert\frac{\omega^2}{\omega_0^2} - 1\right\rvert} + \frac{H}{2}$

$\omega \neq \omega_0$	$\zeta > \frac{1}{4}\sqrt{\frac{\frac{1}{\frac{4}{H}(l0-\frac{mg}{k})-2}-4}{\frac{\omega^2}{\omega_0^2}} - 4\frac{\omega^2}{\omega_0^2} + 8}$	$\zeta > 0$

	$l0 - \frac{mg}{k} > \frac{H}{2}$
$\omega = \omega_0$	$\zeta > \frac{1}{\frac{4}{H}(l0-\frac{mg}{k})-2}$

The case $\omega \neq \omega_0$ seems quite complicated, so we will focus our efforts on the case where the car drives at resonance speed. That is, $\omega = \omega_0 \Leftrightarrow vx = \sqrt{\frac{k}{m}\frac{L}{2\pi}}$.
To get rid of the annoying π constant, we will simply make the assumption that $L = 2\pi$.

model

The model will make use of the following constants
- k, the stiffness of the spring according to Hooke's law [1]
- m, the mass of the car
- $l0$, the length of the coil spring at equilibrium position
- g, the gravitational acceleration of the earth
- H, the height of the bumps on the road
- c, the magnitude of the damping force, in the opposite direction as the vertical speed of the car

When $\omega = \omega_0$, the car never hits the wheels if

$$l0 - \frac{mg}{k} > \frac{H}{2}$$

and

$$\zeta > \frac{1}{\frac{4}{H}(l0-\frac{mg}{k})-2} \Leftrightarrow c > \frac{\sqrt{mk}}{\frac{2}{H}(l0-\frac{mg}{k})-1}$$

Note that the denominator in the second equation is never 0 because of the first condition.

Remember that the movement of the wheels is given by $wheel\_y(t) = \frac{H}{2}\,sin(\frac{2\pi}{L}\ vx\ t)$. Then $wheel\_y(t)' = \frac{H}{2}\frac{2\pi}{L}vx\,cos(\frac{2\pi}{L}\ vx\ t)$ and $wheel\_y(t)'' =- (\frac{2\pi}{L}vx)^2\frac{H}{2}sin(\frac{2\pi}{L}vx\ t)$. Combined with the fact that $vx = \sqrt{\frac{k}{m}\frac{L}{2\pi}}$ and $L = 2\pi$,

$$wheel\_y(t)' = wheel\_vy(t)$$
$$wheel\_vy(t)' =- \frac{k}{m}wheel\_y(t)$$

The derivation of the equations for the movement of the car works in a similar way. Remember that the movement of the car is given by $car\_y(t) \ = A\ sin(\frac{2\pi}{L}\ \ vx\ \ t\ \ +\ \ \varphi)\ +\ l0\ -\ \frac{mg}{k}$. Then $car\_y(t)' \ = A\frac{2\pi}{L}vx\ cos(\frac{2\pi}{L}\ \ vx\ \ t\ +\ \varphi)$ and $car\_y(t)'' \ =-\ (\frac{2\pi}{L}vx)^2 A\ sin(\frac{2\pi}{L}vx\ t\ +\ \varphi)$. Combined with the fact that $vx\ =\ \sqrt{\frac{k}{m}\frac{L}{2\pi}}$ and $L\ =\ 2\pi$,

$$car\_y(t)' \ = car\_vy(t)$$
$$car\_vy(t)' =-\frac{k}{m}(car\_y(t)\ -\ (l0\ -\frac{mg}{k}))$$

The final model is

```
Definitions
        Real k;     /* stiffness of the spring */
        Real m;     /* mass of the car */
        Real l0;    /* length of the spring at equilibrum */
        Real g;     /* gravitational acceleration */
        Real H;     /* height of the bumps of the road */
        Real c;     /* magnitude of the damping force */
    End.

    ProgramVariables
      Real wheely;  /* veritcal position of the wheels */
      Real wheelvy; /* vertical speed of the wheels */
      Real cary;    /* vertical position of the car */
      Real carvy;   /* vertical speed of the car */
    End.

    Problem
      /* Initial conditions */

      ( /* constants are positive in the real world */
          k>0 & m>0 & l0>0 & g>0 & H>0 & c>0 &
        /* sufficient damping */
        c>(m*k)^0.5/((2/H)*(l0-m*g/k)-1) &
        /* weight of the car does not completely compress the spring */
        l0 - m*g/k > H/2 &
        /* initial conditions for the wheel position and speed */
        wheely^2 + wheelvy^2*m/k = (H/2)^2 &
        /* initial conditions for the car position and speed */
        (cary-(l0-m*g/k))^2 + carvy^2*m/k = H^2*m*k/(4*c^2)
        )
      ->
      [
        {
            wheely' = wheelvy,
            wheelvy' = -(k/m)*wheely,
            cary' = carvy,
            carvy' = -(k/m)*(cary-(l0-m*g/k))
        }
      ]
      /* Safety condition. The wheels never hit the car. */
      (  wheely < cary  )
    End.
```

Proof strategy

The proof strategy consists of showing that the position of the wheels are always below $\frac{H}{2}$ and that the position of the car is always above $\frac{H}{2}$ by using two differential cuts. After that, it is trivial to show by differential weakening that $cary\ <\ wheely$. In order to show the two differential cuts, we will use the fact that for car and wheels, the sum of the squares of their position and speed (sum of squares of a sinus and a cosinus) is constant as a differential invariant (like we did in the previous proof).

Discussion

For this project, I had to brush up my physics skills by doing some research about things like Hooke's law, Newton's equations of motion, harmonic oscillators etc… While this is basic physical knowledge for a physics major, few CS majors (like myself) even know what a differential equation is.

Regarding the first model, I did not use the differential equation directly. I derived the sinus based solution and used the solution to model the movement of the car and the wheels. This is because I wasn't able to prove the relation between the raw differential equation and a sinus using KeyMaeraX. I wasn't able to use the classic invariant of the conservation of energy, since the energy is not constant in the system.

Regarding the second model, it was simplified to model a car driving exactly at resonance frequency. This made the model easier, and prevented it from overlapping with the first model, where the speed was strictly above resonance frequency. Ideally, the second model should have been proved for any speed by separating different cases in the proof (below, equal and above resonance frequency).

Deliverables

- Model + Proof of the undamped harmonic oscillator
- Model + Proof of the damped harmonic oscillator

References

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