Driven oscillations of a car suspension

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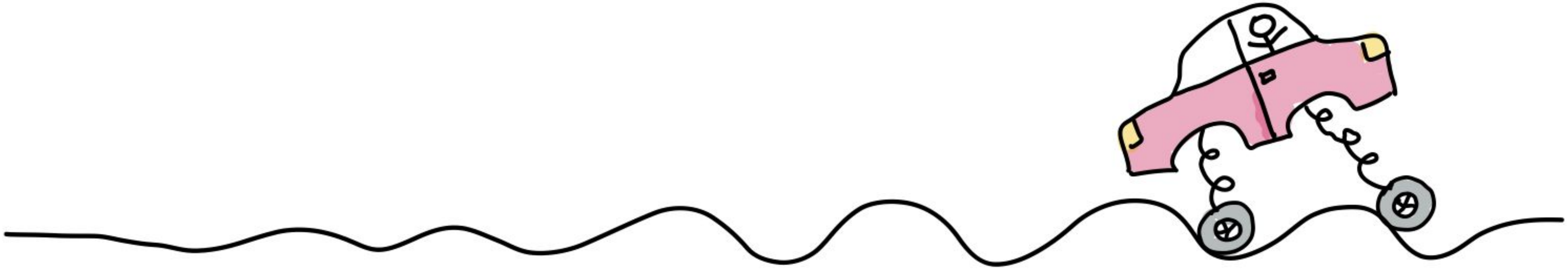
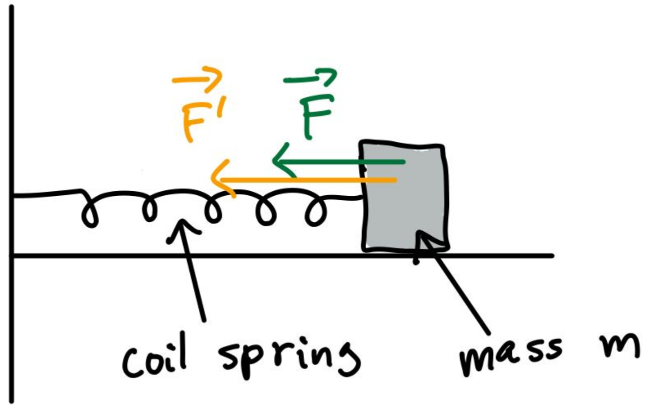


Background

- The suspension of a car is the system of tires, tire air, springs, shock absorbers and linkages that connects a vehicle to its wheels
- Due to the increasing requests from society to improve road safety, vehicle manufacturers and administrations have long been working in order to increase vehicle safety systems and vehicle regulations
- While brake system of a car has objective validation criteria, regulated by a norm that must be met for the approval of the vehicle in order to determine the system effectiveness, in the case of the suspension system, there are no official regulations

Driven harmonic oscillator

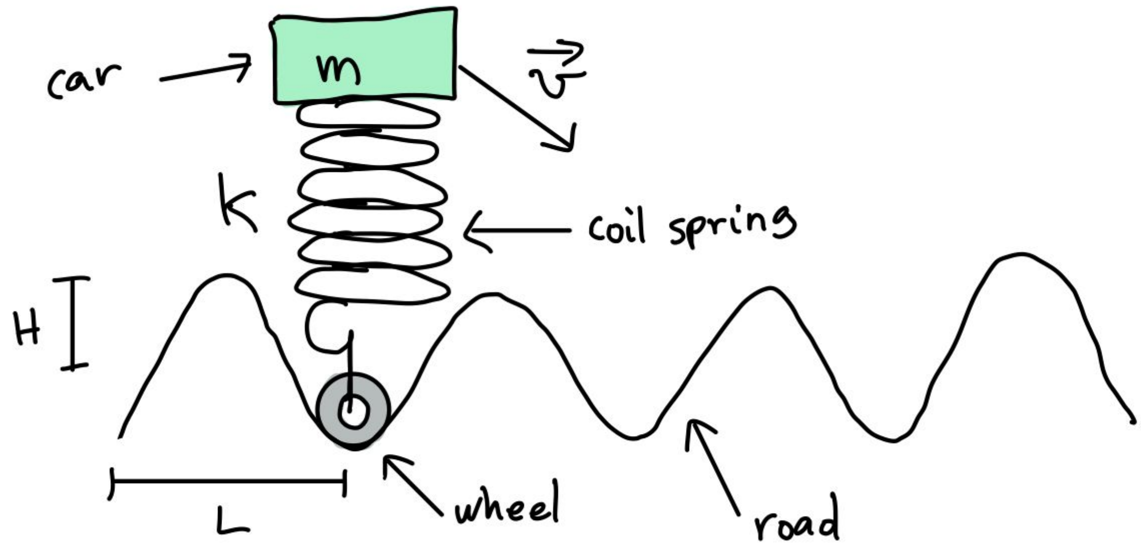


In classical mechanics, a **harmonic oscillator** is a system that, when displaced from its equilibrium position, experiences a restoring force *F* proportional to the displacement *x*:

$$\vec{F} = -k\vec{x},$$

If an external time-dependent force is present, the harmonic oscillator is described as a *driven oscillator*.

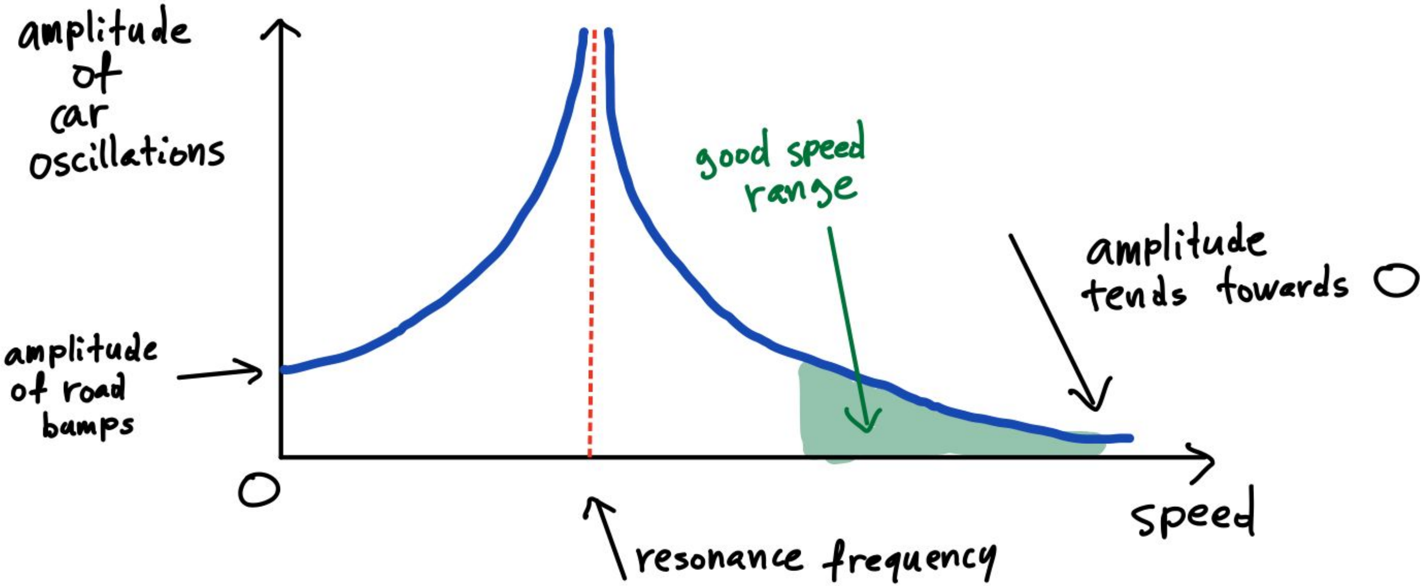
Our situation



The bumps of the road are the external driving force.

Safety condition

At what speed should the car drive so that the mudguards never hit the wheel?



The Math

ODE

$$u'' + \omega_0^2 u = \omega_0^2 \frac{H}{2} sin(\omega t)$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$\omega = \frac{2\pi vx}{L}$$

Solution

$$car\_y\,(t) = A\ sin(\omega t) + l0 - \frac{mg}{k}$$

$$A = \frac{H}{2}\frac{1}{1-\frac{\omega^2}{\omega_0^2}}$$

Safety condition

$$wheel\_y(t) < car\_y(t)$$

$$vx > \sqrt{\frac{k}{m}}\frac{L}{2\pi}\sqrt{\frac{1}{1-\frac{H}{2(l0-\frac{mg}{k})}}}$$

The Model

The key observation is that position of wheels and car are 2 sinuses with same pulse and phase

Preconditions

```
( /* constants are positive in the real world */
   k>0 & m>0 & l0>0 & g>0 & H>0 & vx>0 &
    /* sufficient speed */
 (k/m)^(0.5) * (1/(1-H/(2*(l0-m*g/k))))^(0.5) < vx &
 /* weight of the car does not completely compress the spring */
 l0 - m*g/k > H/2 &
 /* initial conditions for the wheel position and speed */
 wheely^2 + (wheelvy/vx)^2 = (H/2)^2 &
 /* synchrnize wheels and car */
 (1-vx^2*m/k)*(cary-(l0-m*g/k))=wheely
 )
```

Continuous Dynamics

```
[
  {
      wheely' = wheelvy,
      wheelvy' = -vx^2*wheely,
      cary' = 1/(1-vx^2*m/k)*wheelvy,

  }
]
/* Safety condition. The wheels never hit the car. */
(  wheely < cary  )
```

The Proof

Proof statistics
tacticsize=3
budget=0[s]
duration=1658[ms]
qe=0[ms]
steps=1110

Proof by differential cut of 2 invariants

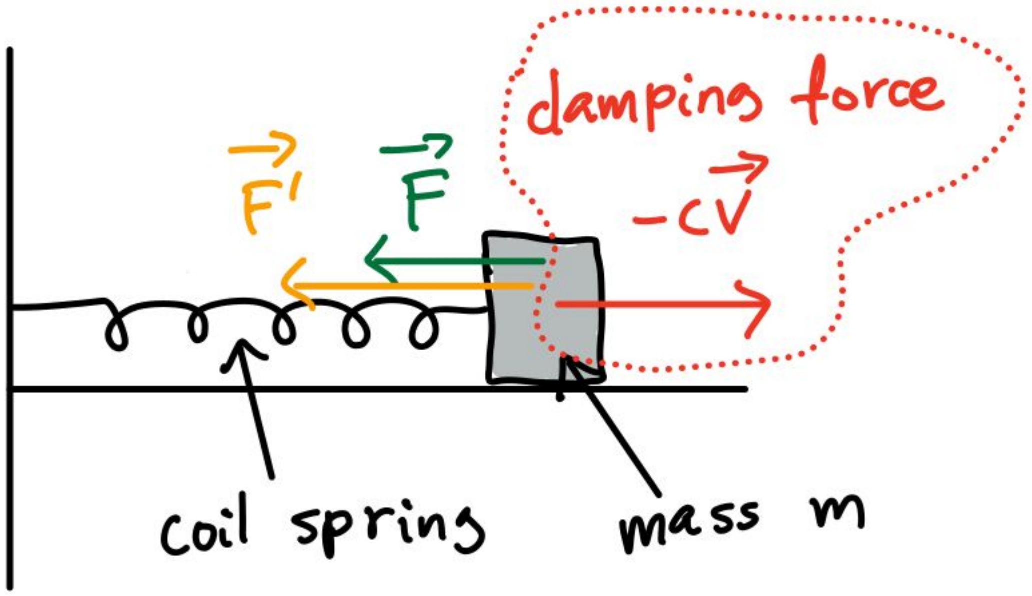
Wheels behave like sinus

$$wheely^2 + (wheelvy/vx)^2 = \frac{H^2}{4}$$

Wheels and car are multiples of each other

$$(1 - vx^2 m/k)\,(cary - (l0 - \frac{mg}{k})) = wheely$$

Damping factor

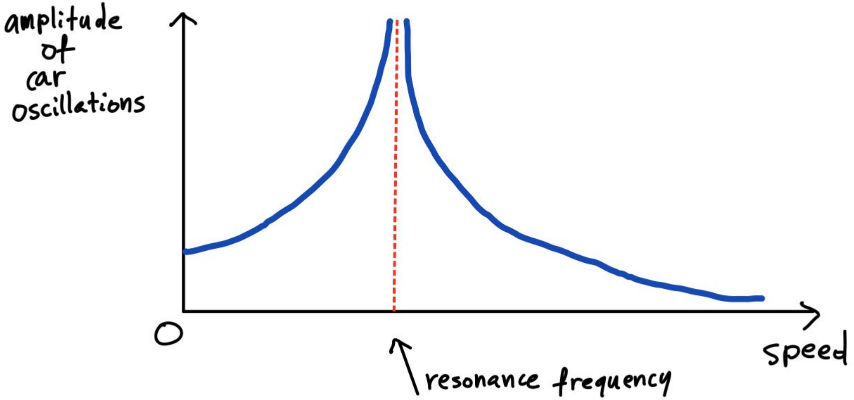


The damping force is proportional to the opposite of the speed

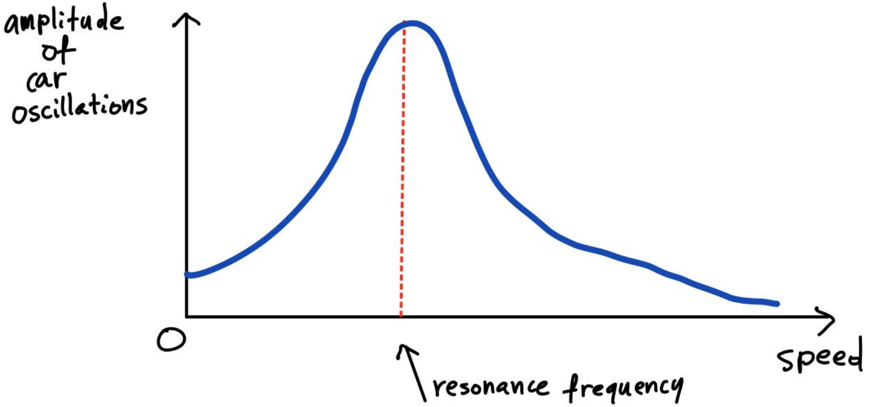
Safety condition

What is the minimal damping factor such that the mudguards never hit the car at resonance speed?

No damping



Damping



The Math

ODE

$$u'' + 2\zeta\omega_0 u' + \omega_0^2 u = \omega_0^2 \frac{H}{2} sin(\omega t)$$

$$\zeta = \frac{c}{2\sqrt{mk}} \quad \omega_0 = \sqrt{\frac{k}{m}} \quad \omega = \frac{2\pi v x}{L}$$

Solution

$$u(t) = A\, sin(\omega\, t + \varphi)$$

$$A = \frac{H}{2}\frac{1}{\sqrt{\left(\frac{2\omega\zeta}{\omega_0}\right)^2 + \left(1 - \frac{\omega^2}{\omega_0^2}\right)^2}}$$

Safety condition

$$wheel\_y(t) < car\_y(t)$$

$$\zeta > \frac{1}{\frac{4}{H}(l0 - \frac{mg}{k}) - 2} \Leftrightarrow c > \frac{\sqrt{mk}}{\frac{2}{H}(l0 - \frac{mg}{k}) - 1}$$

The Model

This time, the wheels and the car's sinuses have different phase which is induced by the damping factor

Preconditions

```
( /* constants are positive in the real world */
   k>0 & m>0 & l0>0 & g>0 & H>0 & c>0 &
/* sufficient damping */
c>(m*k)^0.5/((2/H)*(l0-m*g/k)-1) &
/* weight of the car does not completely compress the spring */
l0 - m*g/k > H/2 &
/* initial conditions for the wheel position and speed */
wheely^2 + wheelvy^2*m/k = (H/2)^2 &
/* initial conditions for the car position and speed */
(cary-(l0-m*g/k))^2 + carvy^2*m/k = H^2*m*k/(4*c^2)
)
```

Continuous Dynamics

```
[
  {
      wheely' = wheelvy,
      wheelvy' = -(k/m)*wheely,
      cary' = carvy,
      carvy' = -(k/m)*(cary-(l0-m*g/k))
  }
]
/* Safety condition. The wheels never hit the car. */
(  wheely < cary  )
```

The Proof

Proof by differential cut of 2 invariants. Then use differential weakening.
For each branch, proof is similar as before.

Wheels always below top of road bumps

$$wheely < \frac{H}{2}$$

Car always above top of road bumps

$$cary > \frac{H}{2}$$

Proof statistics
tacticsize=3
budget=0[s]
duration=1459[ms]
qe=0[ms]
steps=1266

Summary

- Modeled and proved safety for undamped harmonic oscillations of a car suspension
- Modeled and proved safety for damped harmonic of a car suspension at resonance frequency

Thanks. Are there any questions?