Recitation 10
Games, convergence and US.

Announcements

- **Next week:** midterm.
  Don’t forget to look at the practice exam.

- **Advice for labs/class project:**
  Never give up on a proof/conjecture before you have figured out why exactly it does not work out.

Warm-up exercise
\[(P \rightarrow Q) \rightarrow ([\alpha]P \rightarrow [\alpha]Q)\]

Sound in dL?

Answer: NO

Take \[P \equiv \text{true} : \quad Q \rightarrow [\alpha]Q\]
\[Q \equiv x = 0 \quad x = 0 \rightarrow [x := 1] x = 0\]
\[\alpha \equiv (x := 1)\]

\[\vdash P \rightarrow Q\]

\[\vdash [\alpha]P \rightarrow [\alpha]Q\]

Sound in dL? In dGL?
Answer: yes to both (this is the monotonicity rule).

Proving Hybrid Games

Filibustering

\[
\left\langle \left( (x := 1 \lor \neg i := 1); \{x' = \nu\} \right)^x \right\rangle \ x = 0
\]

\[\leftrightarrow\] ???
Convergence

Exercise 1: prove this formula:

\[ \langle (y := x-1; x := 1)^* \rangle \gamma = 0 \]

Answer: use \((\ast)^*\) three times.

Exercise 2:

\[ \langle (y := y-1; x := x+1)^* \rangle x \geq \gamma \]

Answer: see KeyMaer X proof using can.
Uniform Substitution

Intro

How can we apply \[ [:=] \quad [x := e] p(x) \iff p(e) \]
in this example:

\[
[:=] \quad \begin{array}{c}
\frac{x = -3}{x = -3} \quad \vdash \quad \boxed{[x := (x+1)+2] x = 0} \\
\end{array}
\]

Adding function symbols to dL
Syntax

\[ e ::= \ldots | f(e_1, \ldots, e_k) \]
\[ p, q ::= \ldots | p(e_1, \ldots, e_k) \]
\[ \alpha ::= \ldots | \alpha \]

Example:
\[ [x := c()] \ p(x) \iff p(c()) \]

Semantics:

\[ [\cdot] : \text{Term} \to \text{Interpretation} \to \text{State} \to \text{IR} \]

\[ I(f) \in IR^k \to IR \quad \text{for some} \ k \]

\[ \omega \ [f(e_1, \ldots, e_k)] = I(f)(I\omega[e_1], \ldots, I\omega[e_k]) \]

\[ \omega \ [p(e_1, \ldots, e_k)] = I(p)(I\omega[e_1], \ldots, I\omega[e_k]) \]

Exercise:
\[ p() \rightarrow [\alpha] p() \quad \text{Sound?} \]

**Uniform substitution:**

\[
\begin{align*}
\text{US} & \quad \phi \\
\Rightarrow & \quad \sigma(\phi) \\
\text{(if } \sigma \text{ is admissible for } \phi) & \quad \uparrow \\
\sigma \text{ should never introduce a free variable in an environment where it is bound.}
\end{align*}
\]

**Example**

\[
\begin{align*}
\text{US} & \quad [x := c()] \; p(x) \leftrightarrow p(c()) \\
& \quad [x := x+1] \; x = 1 \leftrightarrow x + 1 = 1 \quad \text{IR} \\
& \quad x = 0 \vdash x + 1 = 1 \\
\text{CER} & \quad x = 0 \vdash [x := x+1] \; x = 1
\end{align*}
\]
Looking closer at the US step:

\[
\phi \\
\frac{\left[ x:=c(1) \right] p(x) \iff p(c(1))}{\left[ x:=x+1 \right] x=1 \iff x+1=1} \\
\sigma^{-}(\phi)
\]

\[
\sigma^{-} = \begin{cases} 
  c(1) & \mapsto x+1 \\
  p(\cdot) & \mapsto \cdot = 1 
\end{cases}
\]
\[ [x := c()] \ p(x) \iff p(c()) \]

\[ [x := x+1][x := x+2] \ x = 0 \iff \]

\[ \sigma : \left\{ \begin{array}{l}
    c() \mapsto x+1 \\
    p(*) \mapsto [x := x+2] \ x = 0 \\
    [x := y+2] \ y = 0 \\
    [x := x+2] \ y = 0
\end{array} \right. \]

which of these do not clash?
Exercise:

\[ [x := c(); x := c()] \ p(x) \iff [x := c()] \ p(x) \]

Sound?

\[ [x := c(); x := c()] \ p(x) \iff [x := c()] \ p(x) \]

\[ [x := x+1; x := \text{\textcircled{} x+1}] \ x = 0 \iff [x := x+1] \ x = 0 \]

\text{introduced in a bound context}

\text{CLASH!}
Exercise:

\[
p(\tau) \rightarrow [\alpha] p(\tau)
\]

\[
\begin{align*}
\text{US} \\
\tau = 1 \rightarrow [\tau = 1] \tau = 1
\end{align*}
\]

Clash?

Exercise

\[
[\alpha; \beta] p() \leftrightarrow [\alpha][\beta] p()
\]

\[
\begin{align*}
\text{US} \\
[x = 1; y = 2] x + 1 = y \leftrightarrow [x = 1][y = 2] x + 1 = y
\end{align*}
\]

Clash?
YES! How to fix [;] so that the clash does not happen?