

Recitation 8 : Hybrid Games

Intro

Syntax (mostly similar)

$\alpha, \beta ::= x := e \mid ?Q \mid \{x' = f(x) \& Q\} \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \alpha^d$

$P, Q ::= \neg P \mid P \vee Q \mid \dots \mid \langle \alpha \rangle P \mid [\alpha] P$

However, the semantics is completely different

Angel and demon operators

$\wedge \quad \neg \quad \mid \quad \Box \quad \neg$

Angel	Demon
$\alpha \vee \beta$	
α^*	
$\alpha; \beta$	
$x := e$	
$\{x' = f(x) \& \alpha\}$	
$?H$	

Dual operator

α^d is same program than α , but all controls are flipped.

Winning hybrid games

$\langle \alpha \rangle P \equiv \text{"angel has a winning strategy}$
 $\text{to achieve } P"$

$\underline{\alpha} \sqcup P$ = "demon has a winning strategy
to achieve P"

Example

$$\langle x := 1 \cup x := -1 \rangle \ x \geq 0 \quad \text{valid}$$

$$[x := 1 \cup x := -1] \ x \geq 0 \quad \text{not valid}$$

angel wants this
to be false

Exercise

- $\langle (\exists H)^d \rangle P \leftrightarrow$
- $[(\exists H)^d]P \leftrightarrow$
- $\langle \alpha \cap \beta \rangle P \leftrightarrow$
- $[\alpha \cap \beta]P \leftrightarrow$
- $[\alpha] \text{ true}$

Key properties and axioms

$$(\langle \cdot \rangle) \quad \langle \alpha \rangle P \leftrightarrow \neg (\llbracket \alpha \rrbracket \neg P)$$

$$([\cdot]) \quad \llbracket \alpha \rrbracket P \leftrightarrow \neg (\langle \alpha \rangle \neg P)$$

$$\langle \alpha^d \rangle P \leftrightarrow \llbracket \alpha \rrbracket P$$

$$(\langle^d \rangle) \quad \langle \alpha^d \rangle P \leftrightarrow \neg \langle \alpha \rangle \neg P$$

Exercise

Is $\neg (\langle \alpha \rangle \text{ false})$ valid?

in dL
in dGL

Answer: $\neg (\langle \alpha \rangle \text{False}) \leftrightarrow \underbrace{[\alpha] \text{True}}_{\text{not valid, see above}}$

Semantics

Syntax

$\alpha, \beta ::= x := e \mid ?Q \mid \{x' = f(x) \& Q\} \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \alpha^d$

$P, Q ::= \neg P \mid P \vee Q \mid \dots \mid \langle \alpha \rangle P \mid [\alpha] P$

Semantics

$dL: \llbracket \alpha \rrbracket = \{(\omega, \nu) : \dots\}$ Transition semantics

$dGL: \llbracket \langle \alpha \rangle P \rrbracket = S_\alpha(\llbracket P \rrbracket) = \{\omega : \dots\}$

$\llbracket [\alpha] P \rrbracket = \delta_\alpha(\llbracket P \rrbracket)$ Winning-region semantics

$$\varsigma, \delta : \text{Game} \rightarrow \mathcal{P}(\text{state}) \rightarrow \mathcal{P}(\text{state})$$

Case of $\exists H$

- $S_{\exists H}(X) = [\exists H] \cap X$
- $\delta_{\exists H}(X) = [\exists H]^c \cup X$

Note: we expect $\delta_\alpha(X) = (S_\alpha(X^c))^c$

This is because we expect:

$$[\alpha]P \leftrightarrow \neg (\langle \alpha \rangle \neg P) \quad ([\cdot])$$

Thus:

$$\delta_{\exists H}(X) = ([\exists H] \cap X^c)^c = [\exists H]^c \cup X$$

Case of α^\dagger

$$\bullet S_{\alpha^\dagger}(X) = \delta_\alpha(X) = \underset{\uparrow}{(S_\alpha(X^c))^c}$$

$$\langle \alpha^d \rangle P \leftrightarrow [\alpha] P \quad (\vdash)$$

$$\cdot \quad \delta_{\alpha^d}(x) = S_\alpha(x) = (\delta_\alpha(x^c))^c$$

Case of α^*

Proposal: $S_{\alpha^*}(x) = \bigcup_{n \in \mathbb{N}} S_{\alpha^n}(x)$

$$\delta_{\alpha^*}(x) = \bigcap_{n \in \mathbb{N}} \delta_{\alpha^n}(x)$$

Problem:

Is the following formula valid?

$$\left\langle \underbrace{\left(\underbrace{\{x^1=1\}; \{x^1=1\}^d}_{\alpha} \right)}_{\alpha} \cup \underbrace{x := x^{-1}}_{P} \right\rangle^*$$

→ It should be.

→ However, it is not under the
 - | | -

semantics above:

$$- I \notin S_{\alpha^n}(\llbracket P \rrbracket) \text{ for all } n \in \mathbb{N}$$

Fixed-point semantics

$$\text{Let } Z = S_{\alpha^*}(X).$$

The following holds:

$$(1) \quad X \subseteq Z$$

$$(2) \quad S_{\alpha}(Z) \subseteq Z$$

We define $S_{\alpha^*}(X)$ as the smallest set that respects (1) and (2):

$$S_{\alpha^*}(X) = \bigcap \{Z : X \cup S_{\alpha}(Z) \subseteq Z\}$$

Proving Hybrid Games

Filibustering

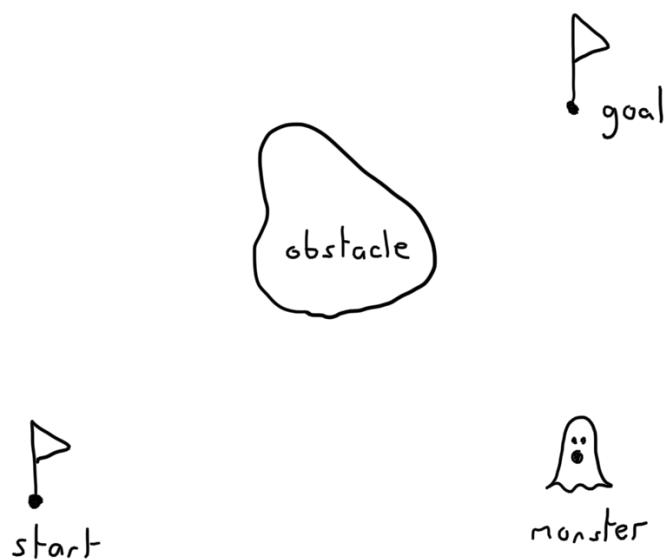
$$x = 0 \rightarrow \left\langle \left((x := 1 \cup v := 1); \{x' = v\} \right)^x \right\rangle x = 0$$

Is this valid?

Let's write an ASCII sequent proof!

(dL and dGL share most of their axioms and rules)

Reach-avoid Game



init \rightarrow <game> post

post \equiv $p_h = p_{goal}$

init \equiv ...

game \equiv ($ctrl$; dyn)

$ctrl \equiv v_h := * ; ? \|v_h\| < L_h ;$

$(v_m := * ; ? \|v_m\| < L_m)^d$

$$\text{dyn} \equiv \vdash := o ; \\ \{ p_h' = v_h, \quad p_m' = v_m, \quad t' = 1 \\ \& \quad \vdash \ll \top \wedge \quad p_h \neq p_m \}$$