Recitation 5

Announcements

- New quiz submission system

Today

- KeYmaera X demo: a time-triggered controller
- dI and differentials

Appetizer: a variance Trick

Question:

Is this axiom sound?

$$[x' = f(x) \,\&\, Q]\,P \leftarrow \forall t \geq 0\ [x := y(t)]\,P$$

['] aka "solve"

$$[x' = f(x) \,\&\, Q]$$

$$\leftarrow \forall t \geq 0 \left(\underbrace{\overset{\text{true (weakening)}}{\left(\cancel{\forall 0 \leq s \leq t\ [x := y(t)]Q} \right)}}_{\text{covariant position}} \rightarrow [x := y(t)]\,P \right)$$

— To read after the digression below

Digression: variance

P is said to be stronger than Q iff $P \rightarrow Q$ is valid.

Example:

- $A \wedge B$ stronger than A
- $A \vee B$ weaker than A
- $x = 0$ stronger than $x \geq 0$

$$\overset{-}{P} \to \overset{+}{Q}$$

contravariant

covariant

$$\overset{+}{P} \wedge \overset{+}{Q}$$

$P^{+} \vee Q^{+}$

$\neg P^{-}$

$[\alpha] P^{+}$

$\langle \alpha \rangle P^{+}$

$\forall x \, P^{+}$

$P \longleftrightarrow Q$

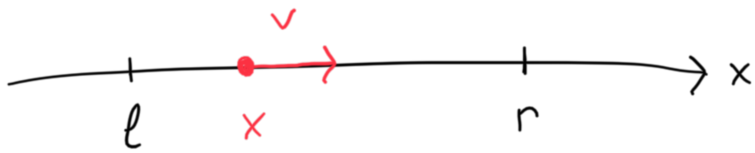
neither variant or covariant
$\rightarrow$ invariant

$(P^{+} \to Q^{-}) \to R^{+}$

Trick: if you weaken a sound axiom, it is still going to be sound.

KeYmaera X demo

Setting: 1D ping-pong controller

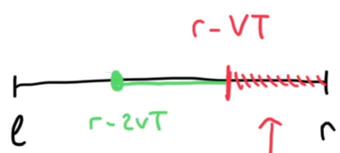


Event-triggered controller

See code (Finishing from last time)

Time-triggered controller

$v \geqslant 0$



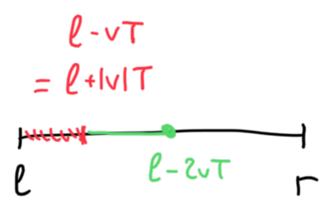
zone where the controller must react

We must have

$r - 2vT \geqslant l$

$\Rightarrow 2vT \leqslant r - l$

$v \leqslant 0$



We must have

$l - 2vT \leqslant r$

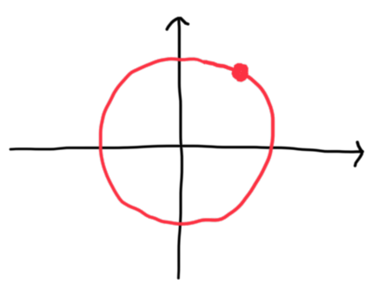
$\Rightarrow 2(-v)T \leqslant r - l$

$$\boxed{2|v|T \leqslant r - l}$$

Differential Invariants

Consider the system:
$$\begin{cases} x' = -y \\ y' = x \end{cases}$$

We want to prove:



$$x^2+y^2-1=0 \rightarrow$$
$$[x'=-y, y'=x]$$
$$x^2+y^2-1=0$$

Solution 1:

- Solve the ODE:

$$x(t) = \cos(t+\varphi), \quad y(t) = \sin(t+\varphi)$$

- $x^2+y^2-1 = \cos^2(t+\varphi) + \sin^2(t+\varphi) - 1 = 0$

BUT: dL does not have trig functions.

Also, it won't always be possible to solve ODEs.

Solution 2: dI

$$\boxed{\dfrac{\vdash [x' := f(x)]\ (e)' = 0}{e = 0 \vdash [\{x' = f(x)\}]\ e = 0}\ dI_=}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{*}{\vdash\ 2(-y)x + 2yx = 0}\ \mathbb{R}}{\vdash [x' := -y][y' := x]\ 2x'x + 2y'y = 0}\ [:=] \times 2}{\vdash [x' := -y][y' := x]\ (x^2 + y^2 - 1)' = 0}\ \cdot',\ +',\ \ldots}{}\ dI_=$$

$$x^2+y^2-1=0 \vdash [x'=-y,\ y'=x]\ x^2+y^2-1=0$$

Bonus (anticipating lecture 11)

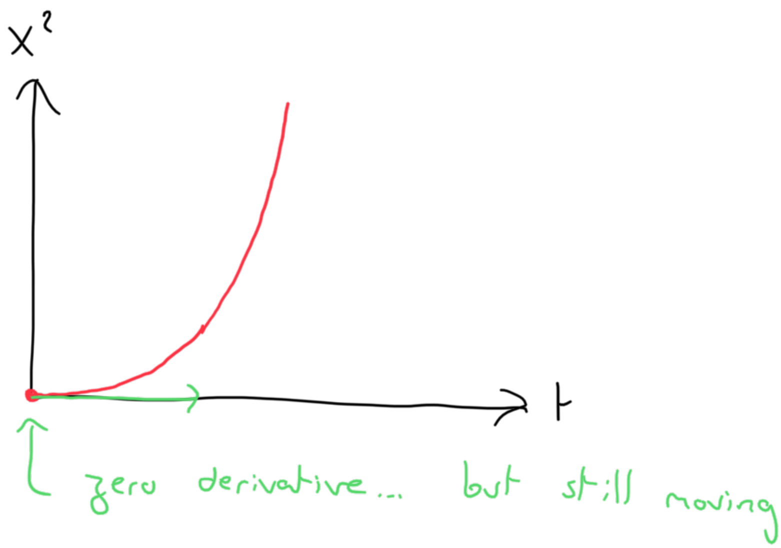
$$\dfrac{e=0 \vdash [x':=f(x)]\ (e)'=0}{e=0 \vdash [\{x'=f(x)\}]\ e=0}$$

Is this sound?

(oops!)

$$\dfrac{\dfrac{\dfrac{*}{x^2=0 \vdash 2x=0}\ \mathbb{R}}{x^2=0 \vdash [x':=1]\ 2x'x=0}}{}$$

$$x^2 = 0 \vdash [x' = 1]\ x^2 = 0$$



Differentials

So many primes everywhere ...

We should add them to the syntax and give them a semantics.

$$e ::= c \mid x \mid x + y \mid x \cdot y \mid x' \mid (e)'$$

Semantics

$$\omega[\![x']\!] = \omega(x') \qquad State = Var \cup Var' \to \mathbb{R}$$

$\in \mathbb{R}$ makes no sense!

$$\omega [\![(e)']\!] \doteq \frac{d(\omega [\![e]\!])}{dt}$$

$$\boxed{\omega [\![(e')]\!] = \sum_{x} \omega(x') \frac{\partial [\![e]\!]}{\partial x} (\omega)}$$

$$[\![(\overbrace{x^2 + y^2}^{e})']\!] = ?$$

$$[\![e]\!] = \overbrace{\underbrace{[\![x^2 + y^2]\!]}_{\in \text{State} \to \mathbb{R}}}^{F} : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$(x, y) \longmapsto x^2 + y^2$$

$$\omega[\![e']\!] = \frac{d(F(x(t), y(t)))}{dt} = \frac{dF}{dt} \text{ for short}$$

$$\frac{dF}{dt} = \underbrace{\frac{dx}{dt}}_{x'} \underbrace{\frac{\partial F}{\partial x}}_{2x} + \underbrace{\frac{dy}{dt}}_{y'} \underbrace{\frac{\partial F}{\partial y}}_{2y}$$

$$\Rightarrow \omega[\![(x^2+y^2)']\!] = \omega(x') \cdot \underbrace{\omega[\![2x]\!]}_{\frac{\partial[\![e]\!]}{\partial x}(\omega)} + \omega(y') \cdot \underbrace{\omega[\![2y]\!]}_{\frac{\partial[\![e]\!]}{\partial y}(\omega)}$$

$$= \sum_x \omega(x') \cdot \frac{\partial[\![e]\!]}{\partial x}(\omega)$$

Next steps

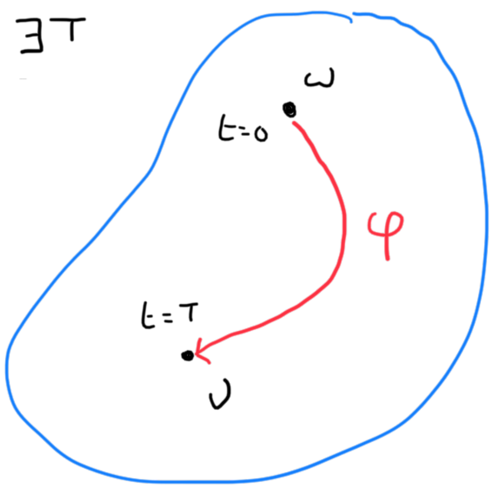
- Updating the ODE semantics
- Differential lemma

If we do not have time to cover this, the textbook has <u>great</u> explanations (~3 pages)

$$(\omega, \nu) \in [\![x' = f(x) \;\&\; Q]\!]$$

$$\Longleftrightarrow$$

$\exists T$



$$\exists \varphi : [0,T] \to \underbrace{\text{State}}_{\text{Var} \cup \text{Var}' \to \mathbb{R}}$$

$$\varphi(t)[\![x']\!] \overset{\text{def}}{=} \frac{d\varphi(t)(x)}{dt}$$

$$\text{s.t. } \varphi(t) \in [\![\underbrace{x' = f(x) \land Q}_{\text{FORMULA (not HP)}}]\!] \quad \forall t$$

...

If φ is a trajectory respecting all these conditions, then for any term e:

$$\boxed{\varphi(t)[\![e']\!] = \frac{d(\varphi(t)[\![e]\!])}{dt}}$$

aka: prime terms have the right value along the ODE trajectory.