Expressing Equilibrium in Dynamic Game Logic to Play Non Zero-Sum Games

Yiwen Yuan

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1 Abstract

The goal of the project is to express various equilibrium of a non zero-sum game in dynamic Game Logic(dGL). Our contributions includes: 1) We wrote out the formulation of three types of equilibria, Nash equilibrium, winning secure equilibrium and cooperative equilibrium in dGL for games without deadlocks and proved their semantic correctness. 2) We showed how we can use KeYmaera X to prove the existence of such equilibria in a simple box pushing game.

2 Introduction

In Differential Game Logic(dGL) [3], we study the programmatic models of zero-sum, perfect-information game between two players, called Angel and Demon. Hybrid Games in dGL are two-player, zero-sum, perfect-information games where the continuous subgame is ordinary differential equations(ODE) where the duration is an arbitrarily unbounded, but finite time chosen by the two players. However, zero-sum games do not always model the real dynamics of the world and here in this project we extends the study of game semantics to the realm of two-player, non zero-sum, perfect-information games. Specifically, we work on using the existing formulas and semantics defintion of dGL to express three types of different equilibria. Then we write out several games in KeYmarea X to prove those equilibria.

Many existing work has been done on the semantics of games. Von Neumann (1944) [4] first presented the formal description of games of strategy. Alur(2002) [1] used alternating time logic to express graph games. Chatterjee (2010) [2]'s work introduced *strategy logic* , a logic that treats strategies in two-player games as first-order objects. The authors have shown and proved that Nash equilibrium and winning secure equilibrium can be expressed using one alternation fragment of strategy logic. One alternation fragment limits the number of quantifiers($\forall$ or $\exists$) in a logic formula to be 2. This is the main inspiration to our work. The modal formulas, namely $\langle\alpha\rangle P$ and $[\alpha]Q$ uses one alternation of logic quantifiers and thus can be used to express equilibria. For example, $[\alpha]P$ in dGL expresses that Demon has a winning strategy to achieve objective P in hybrid game α, i.e., a strategy to reach any of the states satisfying P, no matter what strategy Angel chooses, which writes into $\forall$strategy Angel plays , $\exists$a strategy for Demon, that P holds true.

First we will look at a very simple non zero-sum game. We have our two robots Wall E and EVE. In the non-zero sum hybrid game setting, we can consider the following system in a 1D setting shown in Fig 1.

Here both Wall E and Eve find a treasure box and they want to push the cart containing the treasure box to their bases. Let's say that each of them gains 5 gold/mile for pushing the cart to their home and gains 1 gold/mile for helping its friend. To make the problem simpler, let Wall E and Eve's homes be infinitely far away, i.e. the two robots can push in whatever direction for whatever duration. Let's consider the winning condition for WALL E, who plays the Angel part, is that it earns 30 gold. The problem formulation is as follows: $x = 0 \land g_A = 0 \land g_D = 0 \land v = 0 \rightarrow \langle\{(w := 1 \cup w := -1); (e := 1 \cup e := -1)^{\mathrm{d}}; \{\{x' = v, v' = w + e \& v >= 0\} \cup \{x' = v, v' = w + e \& v <= 0\}\};\ \text{if } (x > 0)\{g_A := g_A + 5 * x; g_D := g_D + x;\} \text{ else } \{g_A := g_A - x; g_D := g_D - 5 * x;\}\}\rangle g_A \geq 30$.

Both WALL E and EVE can exert an acceleration to the car. For the sake of simplicity, the game only lasts for one iteration. Angel WALL E first picks its acceleration (with Angel choice $\cup$) to be towards its base($w := 1$) or towards EVE's base. Then EVE chooses (with Demon choice $\cap$) whether to push to WALL E's base ($e := 1$) or to its own base ($e := -1$). After that, based on the direction of v, the system enters into continuous evolution. The time derivative of the box's position x is the velocity of the box and and

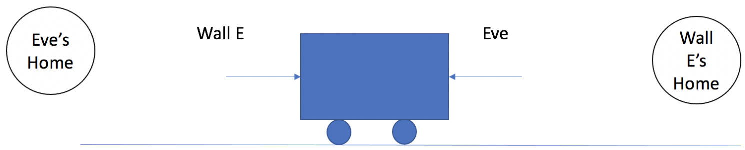

Figure 1: Box pushing game

the time derivative of the velocity v is the acceleration which is equal to $w + e$. Angel WALL E picks the duration of the evolution and the differential equation system can evolve as long as the velocity is non zero. After the continuous evolution has stopped, the two robots would receive their respective payment. Since our domain constraint has restricted that the box can only travel in one direction(switching direction would cause the domain constraint to not hold), we can calculate the payment after the evolution is over. The gold gained by each player is equivalent to their earning rate in the direction multiplied by the absolute value of distance traveled in that direction.

The example box pushing game is a very simple example of a two player non zero-sum game. At the end of the paper, we discuss possibilities of modeling more complicated games such as GAN(Generative Adversarial Networks) using the dGL framework. But before we look at any specific game, we look into formulating non zero-sum game playing and various equilibria in dGL.

3 Playing Non Zero-Sum Games in dGL

The conversion from non zero-sum games to zero-sum games is fairly easy in dGL. Instead of thinking Angel is playing against Demon, we can imagine the existence of God. So Angel is playing a zero-sum game against the ally of Demon and God.

In the example box pushing problem, the winning region of Angel is gold earned greater than 30. Since it is a zero-sum game between Angel and the ally, all the axioms still hold for the game.

In dGL, a special situation is when the game gets stuck. For example, $(?false)^d$ causes Demon to lose because the game would deadlock for Demon. Since it is complicated to define various equilibria in a deadlock situation, we only care about the games that does not contain a deadlock situation.

4 Equilibrium in Non Zero-Sum Games in dGL

Before we begin this section, we want to make some definitions. Let s denotes the set of states at the beginning of the game, and x, y be some strategy profiles of Angel and Demon. Consider an arbitrary game function π, that takes in the initial state and action of both players and outputs the resulting state of the game. To express that there exists a way for the two players to cooperatively ensure some winning region Ψ, we write $\exists x, \exists y, (\pi(s, x, y) \models \Psi)$. This formula says, suppose the game start with state s, there exists a strategy x for Angel and a strategy y for Demon so that the result of the game $\pi(s, x, y)$ can ensure Ψ.

4.1 Winning Region Stable Nash Equilibrium

Payoff Profile Given a game α, two players Angel and Demon, with their respective winning region Ψ and the starting state s, the *payoff* for player $l \in \{1, 2\}$ is defined as follows

$$p_l(s, x, y, \Psi_l) = \begin{cases} 1 & \text{if } \pi(s, x, y) \models \Psi_l; \\ 0 & \text{otherwise.} \end{cases}$$

The *payoff profile* (p_1, p_2) consists of the payoffs $p_1 = p_l(s, x, y, \Psi_1)$ and $p_2 = p_2(s, x, y, \Psi_2)$ for Angel and Demon.

We define the Winning Region Stable Nash Equilibrium, saying that Angel and Demon has no incentive to deviate from their current strategies if their winning regions have been reached. So a Winning Region Stable Nash

Equilibrium is reached when the two following conditions hold for strategy profile (x^*, y^*):

- $\forall x, p_1(s, x, y^*, \Psi_1) \leq p_1(s, x^*, y^*, \Psi_1)$
- $\forall y, p_2(s, x^*, y, \Psi_1) \leq p_2(s, x^*, y^*, \Psi_2)$

The state sets of the corresponding payoff profiles are defined as follows: for $i, j \in \{0, 1\}$, we have

$$\begin{aligned} NE(i,j) =& \{s \in S \mid \text{there exists a Nash equilibrium } (x^*, y^*) \text{ at } s \text{ such that} \\ & p_1(s, x^*, y^*, \Psi_1) = i \text{ and } p_2(s, x^*, y^*, \Psi_2) = j\} \end{aligned}$$

*Existence of Nash Equilibria in dGL.*Our goal is to write out the semantics of Nash equilibrium in dGL and prove them. We consider the game $\boldsymbol{\alpha}$ equivalent to α except that all the dualities are deleted. The following is the semantics of the four possible Nash equilibria expressed using formulas in dGL. $[\![\cdot]\!]$ represents the semantics of $\cdot$.

$$\begin{aligned} NE(1,1) &= [\![\langle\boldsymbol{\alpha}\rangle(\Psi_1 \wedge \Psi_2)]\!] \\ NE(0,0) &= [\![\langle\alpha\rangle\neg\Psi_2 \wedge [\alpha]\neg\Psi_1]\!] \\ NE(1,0) &= [\![\langle\boldsymbol{\alpha}\rangle\Psi_1 \wedge \langle\alpha\rangle\neg\Psi_2]\!] \\ NE(0,1) &= [\![\langle\boldsymbol{\alpha}\rangle\Psi_2 \wedge [\alpha]\neg\Psi_1]\!] \end{aligned}$$

Proof. Now let's construct formal proof for the semantics of each Nash Equilibrium.

For $NE(1,1)$, we want to show that the set of states where the two players form a Nash Equilibrium with utility $(1,1)$ is equivalent to $[\![\langle\boldsymbol{\alpha}\rangle\Psi_1 \wedge \Psi_2]\!]$. By definition,

$$[\![\langle\boldsymbol{\alpha}\rangle\Psi_1 \wedge \Psi_2]\!] = \{u : v \in [\![\Psi_1 \wedge \Psi_2]\!] \text{ for some states } v \text{ s.t. } (u,v) \in [\![\boldsymbol{\alpha}]\!]\}$$

$$\begin{aligned} NE(1,1) =& \{s \in S \mid \exists \text{ a Nash Equilibrium } (x^*, y^*) \text{ s.t. } p_1(s, x^*, y^*, \Psi_1) = 1 \\ & \text{and } p_2(s, x^*, y^*, \Psi_2) = 1\} \\ =& \{s \in S \mid \exists \text{ a Nash Equilibrium } (x^*, y^*) \text{ s.t. } \pi(s, x^*, y^*) \models \Psi_1 \\ & \text{and } \pi(s, x^*, y^*) \models \Psi_2\} \qquad (*) \\ =& \{s \in S \mid \exists \text{ some transition } (s,t) \in [\![\boldsymbol{\alpha}]\!] \text{ s.t. } t \models \Psi_1 \text{ and } t \models \Psi_2\} \\ =& [\![\langle\boldsymbol{\alpha}\rangle(\Psi_1 \wedge \Psi_2)]\!] \end{aligned}$$

$\boldsymbol{\alpha}$ is the game α where duality symbols are eliminated. Therefore, $[\![\boldsymbol{\alpha}]\!]$ is the set of state transitions (s,t) of playing an arbitrary strategy profile from state s and result in t in game α. To justify that step $(*)$ is correct, we know that by playing (x^*, y^*) in the game $\boldsymbol{\alpha}$ from state s, we will be able to get to state $\pi(s, x^*, y^*)$. Then this easily infer that there exists some state transition $(s, t = \pi(s, x^*, y^*))$ where both Ψ_1 and Ψ_2 are valid. Since both Ψ_1 and Ψ_2 are valid, the utility of each player is 1, and therefore neither of them has an incentive to deviate. This leads to a Nash equlibirum.

$$
\begin{aligned}
NE(1,0) =& \{s \in S \mid \exists \text{ a Nash Equilibrium } (x^*, y^*), \text{s.t. } p_1(s, x^*, y^*, \Psi_1) = 1 \\
& \text{ and } p_2(s, x^*, y^*, \Psi_2) = 0\} \\
=& \{s \in S \mid \exists \text{ a Nash Equilibrium } (x^*, y^*), \text{s.t. } \pi(s, x^*, y^*) \models \Psi_1 \\
& \text{ and } \pi(s, x^*, y^*) \models \neg\Psi_2\} \\
=& \{s \in S \mid \exists \text{ a pair of strategies for the two players} (x^*, y^*), \text{s.t.} \\
& \pi(s, x^*, y^*) \models \Psi_1 \text{ and } \forall \text{ strategy } y \text{ of Player } 2, \pi(s, x^*, y) \models \neg\Psi_2\} \\
=& [\![\langle\boldsymbol{\alpha}\rangle\Psi_1 \wedge \langle\alpha\rangle\neg\Psi_2]\!] \qquad (\dagger)
\end{aligned}
$$

We know that if a Nash Equilibrium exists with the utility $(1,0)$, then there's no incentive for Demon to deviate as he cannot push his utility to 1. This means that for all strategy Demon plays, Angel can always ensure $\neg\Psi_2$.

Showing "$\Rightarrow$" direction of †. It's obvious that if there exist a set of strategies (x^*, y^*) to reach a state where Ψ_1 holds true then $s \in [\![\langle\boldsymbol{\alpha}\rangle\Psi_1]\!]$. On the other hand, if for all Demon's strategies, Angel playing x^* can result in a state where $\neg\Psi_2$ holds true, then $s \in [\![\langle\alpha\rangle\neg\Psi_2]\!]$.
Showing "$\Leftarrow$" direction of †. Suppose $s \in [\![\langle\boldsymbol{\alpha}\rangle\Psi_1 \wedge \langle\alpha\rangle\neg\Psi_2]\!]$, then there exists some transition $(s,t) \in [\![\boldsymbol{\alpha}]\!]$, s.t. $t \in [\![\Psi_1]\!]$. This implies that there exists some (x', y'), $\pi(x', y', s) \models \Psi_1$. The second part of the conjunction states that $\forall y$ that Demon chooses, Angel can achieve $\neg\Psi_2$ with some strategy x'. This implies that $\forall y$, $\pi(x', y, s) \models \neg\Psi_2$. Therefore, if $s \in [\![\langle\boldsymbol{\alpha}\rangle\Psi_1 \wedge \langle\alpha\rangle\neg\Psi_2]\!]$, it is also true that $s \in NE(1,0)$. The proof of $NE(0,1)$ is similar to that of $NE(1,0)$, except that we swap the two players.

$$
\begin{aligned}
NE(0,0) =& \{s \in S \mid \exists \text{ a Nash Equilibrium } (x^*, y^*), \text{s.t. } p_1(s, x^*, y^*, \Psi_1) = 0 \\
& \text{ and } p_2(s, x^*, y^*, \Psi_2) = 0\} \\
=& \{s \in S \mid \exists \text{ a Nash Equilibrium } (x^*, y^*), \text{s.t. } \pi(s, x^*, y^*) \models \neg\Psi_1 \text{ and} \\
& \pi(s, x^*, y^*) \models \neg\Psi_2\} \\
=& \{s \in S \mid \exists \text{ a strategy pair } (x^*, y^*), \text{s.t. } \forall y, \pi(s, x^*, y) \models \neg\Psi_2 \text{ and} \\
& \forall x, \pi(s, x, y^*) \models \neg\Psi_1\} \\
=& [\![\langle\alpha\rangle\neg\Psi_2 \wedge [\alpha]\neg\Psi_1]\!]
\end{aligned}
$$

The proof is quite straightforward, if there exists a Nash Equilibrium with utility $(0,0)$, meaning that no player can increase his/her utility by deviating from the current strategy, then this is equivalent to saying that for Angel, she can always ensure $\neg\Psi_2$ and for Demon, he can always ensure $\neg\Psi_1$.

4.2 Winning Region Secure Equilibrium

The Winning Region Stable Nash Equilibrium express the states where either both Angel and Demon win/lose, or exactly one of them win/lose. However, it cannot capture the notion of conditional competitiveness.
Lexicographic ordering of payoff profiles We define the lexicographic ordering $\preceq_1$ and $\preceq_2$ on payoff profiles.

$$(p_1, p_2) \preceq_1 (p_1', p_2') \text{ iff } (p_1 < p_1') \vee (p_1 = p_1' \wedge p_2 \geq p_2')$$

$$(p_1, p_2) \preceq_2 (p_1', p_2') \text{ iff } (p_2 < p_2') \vee (p_2 = p_2' \wedge p_1 \geq p_1')$$

The Winning Region Secure Equilibrium can be defined as follows,

- $\forall x, (p_1(s, x, y^*, \Psi_1), p_2(s, x, y^*, \Psi_2)) \preceq_1 (p_1(s, x^*, y^*, \Psi_1), p_2(s, x^*, y^*, \Psi_2))$
- $\forall y, (p_1(s, x^*, y, \Psi_1), p_2(s, x^*, y, \Psi_2)) \preceq_2 (p_1(s, x^*, y^*, \Psi_1), p_2(s, x^*, y^*, \Psi_2))$

The state sets of the corresponding payoff profiles are defined as follows: for $i, j \in \{0, 1\}$, we have

$$
\begin{aligned}
SE(i,j) =& \{s \in S \mid \text{there exists a secure equilibrium } (x^*, y^*) \text{ at } s \text{ such that} \\
& p_1(s, x^*, y^*, \Psi_1) = i \text{ and } p_2(s, x^*, y^*, \Psi_2) = j\}
\end{aligned}
$$

Existence of Winning Secure Equilibira in dGL. Let U be the set of all initial states s. We consider the game $\boldsymbol{\alpha}$ equivalent to α except that all the dualities are deleted. Similarly, we define the winning secure equilibrium in dGL.

$$\begin{aligned}
SE(1,1) &= \llbracket\langle\boldsymbol{\alpha}\rangle(\Psi_1 \wedge \Psi_2) \wedge \langle\alpha\rangle(\Psi_2 \Rightarrow \Psi_1) \wedge [\alpha](\Psi_1 \Rightarrow \Psi_2)\rrbracket \\
SE(1,0) &= \llbracket\langle\alpha\rangle(\Psi_1 \wedge \neg\Psi_2)\rrbracket \\
SE(0,1) &= \llbracket[\alpha](\Psi_2 \wedge \neg\Psi_1)\rrbracket \\
SE(0,0) &= U \setminus \{SE(1,1) \cup SE(1,0) \cup SE(0,1)\}
\end{aligned}$$

Proof

$$\begin{aligned}
SE(1,1) =& \{s \in S \mid \text{there exists a secure equilibrium } (x^*, y^*) \text{ at } s \text{ such that} \\
&\quad p_1(s, x^*, y^*, \Psi_1) = 1 \text{ and } p_2(s, x^*, y^*, \Psi_2) = 1\} \\
=& \{s \in S \mid \text{there exists a secure equilibrium } (x^*, y^*) \text{ at } s \text{ such that} \\
&\quad \pi(s, x^*, y^*) \models \Psi_1 \text{ and } \pi(s, x^*, y^*) \models \Psi_2\} \qquad (*) \\
=& \llbracket\langle\boldsymbol{\alpha}\rangle(\Psi_1 \wedge \Psi_2) \wedge \langle\alpha\rangle(\Psi_2 \Rightarrow \Psi_1) \wedge [\alpha](\Psi_1 \Rightarrow \Psi_2)\rrbracket
\end{aligned}$$

We prove step $(*)$ from two directions. "$\Rightarrow$" direction. Let $s \in SE(1,1)$, we want to show that $s \in \llbracket\langle\boldsymbol{\alpha}\rangle(\Psi_1 \wedge \Psi_2) \wedge \langle\alpha\rangle(\Psi_2 \Rightarrow \Psi_1) \wedge [\alpha](\Psi_1 \Rightarrow \Psi_2)\rrbracket$. Since secure equilibrium is a special type of Nash equilibrium, we know that $\exists(s,t) \in \llbracket\boldsymbol{\alpha}\rrbracket$, $t \in \llbracket\Psi_1 \wedge \Psi_2\rrbracket$. This follows directly from our proof on the semantics of $NE(1,1)$ in the previous section. Now we want to show the two other term, $\langle\alpha\rangle(\Psi_2 \Rightarrow \Psi_1)$ and $[\alpha](\Psi_1 \Rightarrow \Psi_2)$. By definition of secure equilibrium, $\forall x, p_1(s,x,y^*) < p_1(s,x^*,y^*) \vee (p_1(s,x,y^*) = p_1(s,x^*,y^*) \wedge p_2(s,x,y^*) \geq p_2(s,x^*,y^*))$. Since the utility of each player is 1, meaning $\pi(s,x^*,y^*) \models \Psi_1$ and $\pi(s,x^*,y^*) \models \Psi_2$, this is equivalent to

$$\begin{aligned}
\forall x, &(\pi(s,x,y^*) \models \neg\Psi_1 \wedge \pi(s,x^*,y^*) \models \Psi_1) \vee \\
&(\pi(s,x,y^*) \models \Psi_1 \wedge \pi(s,x^*,y^*) \models \Psi_1 \wedge \pi(s,x,y^*) \models \Psi_2)
\end{aligned}$$

Consider Demon's perspective, we want to show whatever strategy Angel plays, we can always ensure $\Psi_1 \Rightarrow \Psi_2$. If Angel plays x^*, then $x = x^*$ and $\pi(s,x,y^*) \models \Psi_1$. According to the above formula, if $\pi(s,x,y^*) \models \Psi_1$, then the right side of the disjunction is true, which also ensures $\pi(s,x,y^*) \models \Psi_2$. If Angel plays x, where $x \neq x^*$, then the utility of playing x must be 1 because

otherwise, Angel always has an incentive to deviate, as playing x^* produces a higher utility. Therefore, the right hand side of the disjunction must be true, which ensures $\pi(s, x, y^*) \models \Psi_2$. We have shown that by playing y^*, for whatever strategy that Angel plays, Demon can always ensure $\Psi_1 \Rightarrow \Psi_2$. This corresponds to the modal formula $[\alpha](\Psi_1 \Rightarrow \Psi_2)$. Showing $\langle\alpha\rangle(\Psi_2 \Rightarrow \Psi_1)$ is similar as we only need to swap the roles of Angel and Demon.
The "$\Leftarrow$" direction is more straight forward as the semantics itself corresponds to the definition of winning secure equilibrium. If $s \in [\![\langle\boldsymbol{\alpha}\rangle]\!](\Psi_1 \wedge \Psi_2)$, then there exists some strategy profile (x^*, y^*) for Angel and Demon where both players' utility is 1. (This follows from our previous proof on $NE(1,1)$). Since both players' utility is 1, meaning they both win the game, then there exists a secure equilibrium.

$$\begin{aligned}
SE(1,0) =& \{s \in S \mid \text{there exists a secure equilibrium } (x^*, y^*) \text{ at } s \text{ such that} \\
& p_1(s, x^*, y^*, \Psi_1) = 1 \text{ and } p_2(s, x^*, y^*, \Psi_2) = 0\} \\
=& \{s \in S \mid \text{there exists a secure equilibrium } (x^*, y^*) \text{ at } s \text{ such that} \\
& \pi(s, x^*, y^*) \models \Psi_1 \text{ and } \pi(s, x^*, y^*) \models \neg\Psi_2\} \qquad (*) \\
=& [\![\langle\alpha\rangle(\Psi_1 \wedge \neg\Psi_2)]\!]
\end{aligned}$$

"$\Rightarrow$" direction. By definition of secure equilibrium, $\forall y, (p_1(s, x^*, y, \Psi_1), p_2(s, x^*, y, \Psi_2)) \preceq_2 (p_1(s, x^*, y^*, \Psi_1), p_2(s, x^*, y^*, \Psi_2))$. Since the utility of Angel is 1 and the utility of Demon is 0, we have

$$\forall y, (\pi(s, x^*, y) \models \neg\Psi_2 \wedge \pi(s, x^*, y^*) \models \neg\Psi_2 \wedge \pi(s, x^*, y) \models \Psi_1)$$

From Angel's perspective, it's easy to infer from the above formula that $\langle\alpha\rangle(\Psi_1 \wedge \neg\Psi_2)$.
"$\Leftarrow$" direction. If for every strategy Demon chooses, Angel can always ensure $\Psi_1 \wedge \neg\Psi_2$, this is the same as having a secure equilibrium of $(1,0)$ because both Angel and Demon have no incentive to deviate.
The proof of $SE(0,1)$ is similar and we consider the complement set to be $SE(0,0)$.

4.3 Cooperative Equilibrium

Cooperative Equilibrium captures the cooperative external choice. Like Winning Region Secure Equilibrium, it is a special case of Nash Equilibrium.

Lexicographic ordering of payoff profiles We define the lexicographic ordering $\preceq_1^{co}$ and $\preceq_2^{co}$ on payoff profiles.

$$(p_1, p_2) \preceq_1^{co} (p_1', p_2') \text{ iff } (p_1 < p_1') \vee (p_1 = p_1' \wedge p_2 < p_2')$$

$$(p_1, p_2) \preceq_2^{co} (p_1', p_2') \text{ iff } (p_2 < p_2') \vee (p_2 = p_2' \wedge p_1 < p_1')$$

A cooperative equilibrium is a Nash equilibrium that respects the lexicographic ordering $\preceq_1^{co}$ and $\preceq_2^{co}$ on payoff profiles.

Existence of Cooperative Equilibria in dGL.

$$\begin{aligned}
CE(1,1) &= \llbracket \langle \boldsymbol{\alpha} \rangle(\Psi_1 \wedge \Psi_2) \wedge [\alpha](\Psi_2 \Rightarrow \Psi_1) \wedge \langle \alpha \rangle(\Psi_1 \Rightarrow \Psi_2)]\!] \\
CE(1,0) &= \llbracket \langle \boldsymbol{\alpha} \rangle \Psi_1 \wedge \langle \alpha \rangle(\Psi_1 \Rightarrow \neg\Psi_2)]\!] \\
CE(0,1) &= \llbracket \langle \boldsymbol{\alpha} \rangle \Psi_2 \wedge [\alpha](\Psi_2 \Rightarrow \neg\Psi_1)]\!] \\
CE(0,0) &= U \setminus \{CE(1,1) \cup CE(1,0) \cup CE(0,1)\}
\end{aligned}$$

5 Equilibrium Points in the Box-Pushing Game

Now we go back to the box pushing game we discussed earlier and check for the existence of various equilibrium using our formulation.
First, we will manually identify the equilibrium points in the cart-pushing game defined as follows.

$x = 0 \wedge g_A = 0 \wedge g_D = 0 \rightarrow \langle \{(w := 1 \cup w := -1); (e := 1 \cup e := -1)^{\mathrm{d}};$
$\{\{x' = v, v' = w + e \& v >= 0\} \cup \{x' = v, v' = w + e \& v <= 0\}\};$
if $(x > 0)\{g_A := g_A + 5 * x; g_D := g_D + x;\}$ else $\{g_A := g_A - x; g_D := g_D - 5 * x;\}\}\rangle$
$g_A \geq 30$

Assume that both Angel and Demon's initial gold are 0, the cart starts at $x = 0$ i.e. $g_A = 0 \wedge g_D = 0 \wedge x = 0$. A winning secure equilibrium with the payoff $(1, 0)$(assuming Angel is player 1 and Demon is player 2) does not exist with precondition $v = 0$ because Demon can just push in the opposite direction of Angel and since total acceleration is 0, neither Angel or Demon can gain any gold. A winning secure equilibrium with payoff $(1, 0)$ does exist if $v > 0$ because Angel can always maintain a positive velocity and stop evolving the ODE as soon as she reaches 30 gold. Since Demon earns much less gold per distance unit traveling in the positive direction, he won't be able to reach the 30 gold. (See Pos_v&SE(1,0)_Proof.kyx).

A winning secure equilibrium with the payoff $(1, 1)$ is quite tricky. It's obvious that when $v = 0$ or $v > 0$, no such equilibrium exists because in the first case no one would be able to move and in the second case Angel always can stop before Demon reaches his winning region. However, a winning secure equilibrium does exist when the initial velocity negative because ODE can evolve long enough in the negative direction so that Angel can earn her gold. Since in the negative direction Demon earns gold faster, when Angel has reached her objective, Demon must have reached his objective (See Neg_v&SE(1,1)_Proof).(Note: I split the proof into three parts because KeYmaera X complains when I try to prove them all together. However, since the format of $SE(1, 1)$ is the conjunction of three formulas, it's equivalent to prove each of them separately.)

6 Conclusion and Future Steps

We introduced playing non zero-sum games in dynamic Game Logic where previously only zero-sum games are supported. We showed and proved that we can express various equlibria using the modal formulas in dGL.

Given the limit of time, we are only able to explore the simple box pushing game in this project, in the future, we aim at looking more complicated games such as GANs(Generative Adversarial Nets). GAN is a generative model where a discriminator and a generator seek to reach a Nash equilibrium where each player cannot reduce their cost without changing the other player's parameters. However, one major limit is that dGL can only express those equilibria that are achieved through pure strategies and cannot be extended to expressing equilibria achieved using mixed strategies. Therefore, incorporating stochasticity to play more complicated games like GANs would be our future focus.

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