Expressing Equilibrium in Differential Game Logic to Play Non Zero-Sum Games

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Our Contribution

• Designed ways to play two-player perfect information non zero-sum game in dGL.

• Used simple modal formulas to express three types of equilibria in those games and wrote proofs for them.
dGL Grammar

**Definition 14.1 (Hybrid games).** The *hybrid games of differential game logic* dGL are defined by the following grammar (α, β are hybrid games, x is a vector of variables, f(x) is a vector of (polynomial) terms of the same dimension, and Q is a dGL formula or just a formula of first-order real arithmetic):

\[
\alpha, \beta ::= x := e \mid x' = f(x) \& Q \mid ?Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^d
\]
Definition 14.2 (dGL formulas). The formulas of differential game logic dGL are defined by the following grammar ($P, Q$ are dGL formulas, $e, \tilde{e}$ are terms, $x$ is a variable, and $\alpha$ is a hybrid game):

\[ P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \land Q \mid \exists x P \mid \langle \alpha \rangle P \mid [\alpha] P \]
Logic Foundation

• Nash Equilibrium can be expressed using one alternation fragment of logic. [1][2]

• Modal formulas in dGL is one-alternation. ($<\alpha>P$ means for all strategies Demon can play, Angel can ensure P)
Nash Equilibrium

• No player has an incentive to deviate from their current strategy, i.e. changing strategy doesn’t increase payoff.
Nash Equilibrium-Payoff Profile

• $\pi$ is the function that takes into state $s$, strategy of Angel $x$ and strategy of Demon $y$.
• $\Psi_l$ is the winning region of player $l$.

\[
p_l(s, x, y, \Psi_l) = \begin{cases} 
1 & \text{if } \pi(s, x, y) \models \Psi_l; \\
0 & \text{otherwise.} 
\end{cases}
\]
Nash Equilibrium-Semantics

• The set of states that can lead to a gameplay of Nash Equilibrium

\[ NE(i, j) = \{ s \in S \mid \text{there exists a Nash equilibrium } (x^*, y^*) \text{ at } s \text{ such that } p_1(s, x^*, y^*, \Psi_1) = i \text{ and } p_2(s, x^*, y^*, \Psi_2) = j \} \]
Nash Equilibrium in dGL

- $\alpha$ is the game $\alpha$ with duality symbols deleted.

\[
\begin{align*}
NE(1, 1) &= [[\langle \alpha \rangle (\Psi_1 \land \Psi_2)]] \\
NE(0, 0) &= [[\langle \alpha \rangle \neg \Psi_2 \land [\alpha] \neg \Psi_1]] \\
NE(1, 0) &= [[\langle \alpha \rangle \Psi_1 \land \langle \alpha \rangle \neg \Psi_2]] \\
NE(0, 1) &= [[\langle \alpha \rangle \Psi_2 \land [\alpha] \neg \Psi_1]]
\end{align*}
\]
Proof – NE(1,1)

\[ \text{NE}(1, 1) = \{ s \in S \mid \exists \text{ a Nash Equilibrium } (x^*, y^*) \text{ s.t. } p_1(s, x^*, y^*, \Psi_1) = 1 \]
\[ \text{and } p_2(s, x^*, y^*, \Psi_2) = 1 \} \]
\[ = \{ s \in S \mid \exists \text{ a Nash Equilibrium } (x^*, y^*) \text{ s.t. } \pi(s, x^*, y^*) \models \Psi_1 \]
\[ \text{and } \pi(s, x^*, y^*) \models \Psi_2 \} \quad (\ast) \]
\[ = \{ s \in S \mid \exists \text{ some transition } (s, t) \in [\alpha] \text{ s.t. } t \models \Psi_1 \text{ and } t \models \Psi_2 \} \]
\[ = \lbrack \langle \alpha \rangle (\Psi_1 \land \Psi_2) \rbrack \]
Proof— NE(1,0)

\[ NE(1, 0) = \{ s \in S \mid \exists \text{ a Nash Equilibrium } (x^*, y^*), \text{ s.t. } p_1(s, x^*, y^*, \Psi_1) = 1 \]
\[ \quad \text{and } p_2(s, x^*, y^*, \Psi_2) = 0 \} \]
\[ = \{ s \in S \mid \exists \text{ a Nash Equilibrium } (x^*, y^*), \text{ s.t. } \pi(s, x^*, y^*) \models \Psi_1 \]
\[ \quad \text{and } \pi(s, x^*, y^*) \models \neg \Psi_2 \} \]
\[ = \{ s \in S \mid \exists \text{ a pair of strategies for the two players}(x^*, y^*), \text{ s.t.} \]
\[ \quad \pi(s, x^*, y^*) \models \Psi_1 \text{ and } \forall \text{ strategy } y \text{ of Player 2, } \pi(s, x^*, y) \models \neg \Psi_2 \} \]
\[ = \llangle \alpha \rrangle \Psi_1 \land \llangle \alpha \rrangle \neg \Psi_2 \] (†)
Proof – NE(0,0)

\[ NE(0, 0) = \{ s \in S \mid \exists \text{ a Nash Equilibrium } (x^*, y^*), \text{s.t. } p_1(s, x^*, y^*, \Psi_1) = 0 \]
\[ \text{and } p_2(s, x^*, y^*, \Psi_2) = 0 \} \]
\[ = \{ s \in S \mid \exists \text{ a Nash Equilibrium } (x^*, y^*), \text{s.t. } \pi(s, x^*, y^*) \models \neg \Psi_1 \text{ and} \]
\[ \pi(s, x^*, y^*) \models \neg \Psi_2 \} \]
\[ = \{ s \in S \mid \exists \text{ a strategy pair } (x^*, y^*), \text{s.t. } \forall y, \pi(s, x^*, y) \models \neg \Psi_2 \text{ and} \]
\[ \forall x, \pi(s, x, y^*) \models \neg \Psi_1 \} \]
\[ = [\langle \alpha \rangle \neg \Psi_2 \land [\alpha] \neg \Psi_1] \]
Winning Secure Equilibrium

\[ SE(1, 1) = \llbracket \langle \alpha \rangle (\Psi_1 \land \Psi_2) \land \langle \alpha \rangle (\Psi_2 \Rightarrow \Psi_1) \land [\alpha](\Psi_1 \Rightarrow \Psi_2) \rrbracket \]

\[ SE(1, 0) = \llbracket \langle \alpha \rangle (\Psi_1 \land \neg \Psi_2) \rrbracket \]

\[ SE(0, 1) = \llbracket [\alpha](\Psi_2 \land \neg \Psi_1) \rrbracket \]

\[ SE(0, 0) = U \setminus \{SE(1, 1) \cup SE(1, 0) \cup SE(0, 1)\} \]
Cooperative Equilibrium

\[ CE(1, 1) = \llbracket (\alpha) (\Psi_1 \land \Psi_2) \land [\alpha] (\Psi_2 \Rightarrow \Psi_1) \land (\alpha) (\Psi_1 \Rightarrow \Psi_2) \rrbracket \]
\[ CE(1, 0) = \llbracket (\alpha) \Psi_1 \land (\alpha) (\Psi_1 \Rightarrow \neg \Psi_2) \rrbracket \]
\[ CE(0, 1) = \llbracket (\alpha) \Psi_2 \land \neg [\alpha] (\Psi_2 \Rightarrow \neg \Psi_1) \rrbracket \]
\[ CE(0, 0) = U \setminus \{ CE(1, 1) \cup CE(1, 0) \cup CE(0, 1) \} \]
KeYmaera X Model and Future Directions

• We have proved the existence of various secure equilibria in a box pushing game in KeYmaera X.

• Our future steps would be considering more complicated games that involves stochasticity, for example, GANs.
Reference


