Assignment 1: Introduction to Hybrid Programs
15-424/15-624/15-824 Logical Foundations of Cyber-Physical Systems
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Due Date: Thursday, September 12th, 11:59PM (no late days), worth 60 points

1. **Terms, formulas, hybrid programs, oh my!** For each of the following, determine if the expression is a (syntactically) well-formed dL term, a well-formed dL formula, a well-formed hybrid program, or none of the above (i.e., it is not well-formed). In the case that the expression is none of the above, give a short explanation why.

   (a) $z := x^5 - 1$

   (b) $?(x \cdot y \cdot z > \frac{3}{4})$

   (c) x

   (d) $42 + 6 \cdot 9$

   (e) $[g := 42]$

   (f) $z + 1 := x^5$

   (g) $x := y + 1 \cup;\ x = y'$

2. **Operator precedence.** Adopting a set of operator precedence rules helps reduce the number of parentheses (or braces) needed when writing down an expression. However, it is *essential* that you are familiar with these rules to avoid hours of debugging/ misunderstanding in your labs/theory assignments!

   For convenience, here is a cheatsheet for the operator precedence rules in dL. Refer back here whenever you are not sure about how to parse a given expression.

   - In the theory assignments and the textbook, parentheses $(\cdot)$ are used to disambiguate terms, formulas and programs. For clarity, we will always write braces around differential equations, like this: $\{x' = v, v' = a\}$ and $\{t' = 1 \,\&\, t \leq T\}$.

     In KeYmaera X (and in your lab assignments), parentheses $(\cdot)$ are used for terms and formulas, but braces $\{\cdot\}$ are used to group programs.

   - Unary operators always bind stronger than binary operators. This **includes the first-order and modal quantifiers**. Examples:
     - $\forall x\, P \wedge Q \equiv (\forall x\, P) \wedge Q$ (similarly for $\exists x$),
     - $[\alpha]P \wedge Q \equiv ([\alpha]P) \wedge Q$ (similarly for $\langle\alpha\rangle$),
     - $\neg P \wedge Q \equiv (\neg P) \wedge Q$,
     - $\alpha; \beta^* \equiv \alpha; (\beta^*)$.

   - The arithmetic operators have their usual precedence from mathematics.

- The binary logical connective $\wedge$ binds stronger than $\vee$, which in turn binds stronger than $\rightarrow, \leftrightarrow$. To avoid confusion, there is no default binding precedence between $\rightarrow$ and $\leftrightarrow$. Explicit disambiguating parentheses are required when these appear in sequence. Examples:
  - $P \wedge Q \vee R \equiv (P \wedge Q) \vee R$
  - $P \rightarrow Q \leftrightarrow R$ is considered illegal, and must be disambiguated either as $(P \rightarrow Q) \leftrightarrow R$ or $P \rightarrow (Q \leftrightarrow R)$.
- Hybrid program operator ; binds tighter than $\cup$. Example:
  - $\alpha; \beta \cup \gamma \equiv \{\alpha; \beta\} \cup \gamma$
- All arithmetic operators $+, -, \cdot$ associate to the left. All logical and program operators associate to the right. In particular, implication ($\rightarrow$) associates to the right. Examples:
  - $a - b - c \equiv (a - b) - c$
  - $P \rightarrow Q \rightarrow R \equiv P \rightarrow (Q \rightarrow R)$.
  - $\alpha; \beta; \gamma \equiv \alpha; (\beta; \gamma)$.

  Although many of these operators satisfy an associativity law (e.g., $a + (b + c) = (a + b) + c$), it is important to know their default associativity because that is also how KeYmaera X parses expressions.

For this question, you will practice applying the above precedence rules. For each formula/program below, add parentheses/braces indicating the correct binding for the connectives.

(a) $[y := 5]x = 3 \vee x = 5 \rightarrow x + 1 = 6$

(b) $\exists x\, x = 5 \rightarrow x + 1 = 6 \rightarrow x = 1$

(c) $[x := 5; y := y + x \cup \{x' = v, v' = a \,\&\, v = -1 \vee v = 1 \wedge v = 2\}]x > 0$

3. **Evolve nondeterministically!** This question will test your understanding of nondeterministic evolution.

$$\beta \stackrel{\text{def}}{\equiv} x := x_0; v := v_0; t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0\}; ?v = 0$$

Intuitively, hybrid program β first sets the initial values of x, v to x_0, v_0, and the initial value of the clock variable t to 0. It then runs the differential equations (where a is a constant acceleration) subject to the evolution domain constraint $v \geq 0$. Finally, it tests that $v = 0$ at the end of the run.

(a) Assume that $a < 0 \wedge v_0 \geq 0$. At the end of a run of hybrid program β, what is the value of t as a function of x_0, v_0, and a?

Let us modify our program a little by removing the test:

$$\gamma \stackrel{\text{def}}{\equiv} x := x_0; v := v_0; t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0\}$$

(b) Again assuming that $a < 0 \wedge v_0 \geq 0$, what are the possible values of v at the end of a run of γ? What about the possible values of t?

(c) Suppose we assume instead that $a < 0 \wedge v_0 \leq 0$ (v_0 is **less than** or equal to zero). What are the possible values of v and t at the end of a run of β?

(d) Let us consider some dL formulas that use the above programs β and γ. For each of the following formulas, state whether the formula is valid and give a brief explanation why. (The antecedents correspond to the various sign assumptions on a and v_0 from the previous parts of this question.)

i. $a < 0 \wedge v_0 \geq 0 \rightarrow [\beta]v = 0$

ii. $a < 0 \wedge v_0 < 0 \rightarrow [\beta]v = 0$

iii. $a < 0 \wedge v_0 < 0 \rightarrow \langle\beta\rangle v = 0$

iv. $a < 0 \wedge v_0 \geq 0 \rightarrow [\gamma]v = 0$

v. $a < 0 \wedge v_0 \geq 0 \rightarrow \langle\gamma\rangle v = 0$

Hint: Carefully review the semantics of differential equations with evolution domain constraints $\{x' = f(x) \,\&\, Q\}$.

4. **Search for the truth!** Determine whether each of the following formulas is valid/ satisfiable/unsatisfiable. If the formula is satisfiable, describe the set of states in which it is satisfiable. If it is unsatisfiable, briefly explain why.

(a) $\forall x\, \langle\{x' = c\}\rangle x > 0$

(b) $[?x \geq 0; x := -x]x < 0$

(c) $\langle\{z' = -c \,\&\, z > 0\}; \{z' = c \,\&\, z < 0\}\rangle z = k$

5. **Find a program!**

(a) Write down a program α that makes the following formula satisfiable, but not valid: $[\alpha]z > 5$

(b) Write down a program α that makes the formula $\forall x\, \forall y\, \langle\alpha\rangle x = y$ valid. The program may mention x **but not** y.

6. **Define an operator!** As we've seen in class, the primitive operators of hybrid programs can be used to define more complex operators. Define an n-ary nondeterministic switch statement with a fallback program β. This statement should run program α_i if formula P_i is true, for $1 \leq i \leq n$. If multiple P_i's are true, $1 \leq i \leq n$, then it chooses

nondeterministically between the corresponding α_i's. If none of the P_i's is true, then it runs β. In pseudocode, this could be written as:

switch {

  case $P_1 : \alpha_1$

  case $P_2 : \alpha_2$

  $\vdots$

  case $P_n : \alpha_n$

  default : β

}