Differential-Algebraic Dynamic Logic for KeYmaera X

CPS Grand Prix

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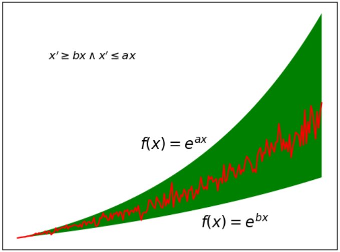

Motivation

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but how do we precisely model its semantics and prove guarantees?

Differential-Algebraic Dynamic Logic (d$\mathcal{AL}$)

$$x \le m \;\rightarrow\; \big[\{x' = v, v' = a \,\&\, v \ge 0\}\big]\, x \le m$$

Differential-Algebraic Dynamic Logic (dA$\mathcal{L}$)

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Differential-Algebraic Dynamic Logic (dA$\mathcal{L}$)

$$x \le m \rightarrow \left[\{\exists\, \delta.\, (x' = v, v' = a + \delta \,\&\, v \ge 0 \wedge \delta^2 \le \frac{|v|}{100})\} \right] x \le m$$

Differential-Algebraic Dynamic Logic (dA$\mathcal{L}$)

dA$\mathcal{L}$ = d$\mathcal{L}$ + existentially quantified ODEs

$$\{\exists\, \bar{y}.\, (x_1' = \theta_1, x_2' = \theta_2, \cdots, x_n' = \theta_n \,\&\, Q)\}$$

dA$\mathcal{L}$ Example (Perturbed Circular Motion)

$$x^2 + y^2 = 1 \rightarrow \left[\{x' = -y, y' = x\}\right] x^2 + y^2 \leq 1$$

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$$x^2 + y^2 = 1 \rightarrow \left[\{\exists e.(x' = -y + e, y' = x \,\&\, x \cdot e \leq 0)\}\right] x^2 + y^2 \leq 1$$

Uniform Substitution (Abridged)

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- Side conditions necessary...

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- Side conditions necessary...but which ones?
- Key observation: Never bind a free variable that was free!

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Instead of schema each with their own unique side conditions...

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You have substitution axioms (without side conditions)...

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and generic *admissibility rules* for each logical construct (checked recursively) preventing capture of free variables.

Uniform Substitution (Abridged)

Upshot: A significantly reduced soundness-critical core that is easier to maintain and understand

Plan of Attack

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Plan of Attack

- Modernize d$\mathsf{A}\mathcal{L}$, providing a uniform substitution calculus for it similar to that for d$\mathcal{L}$.
- Implement uniform substitution axioms into KeYmaeraX.
- Implement derived axioms and tactics into KeYmaeraX.
- Prove stuff!

d$\mathcal{L}$ Recap

DW $[c \,\&\, q(\bar{x})]p(\bar{x}) \leftrightarrow [c \,\&\, q(\bar{x})](q(\bar{x}) \rightarrow p(\bar{x}))$

DC $([c \,\&\, q(\bar{x})]p(\bar{x}) \leftrightarrow [c \,\&\, q(\bar{x}) \wedge r(\bar{x})]p(\bar{x})) \leftarrow [c \,\&\, q(\bar{x})]r(\bar{x})$

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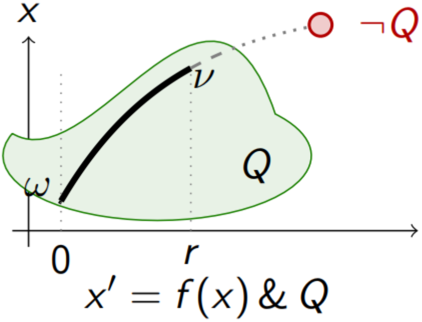

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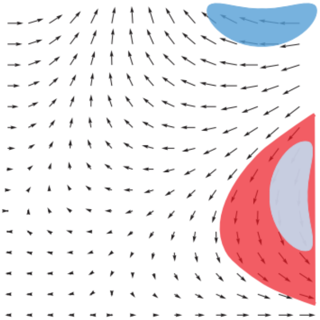

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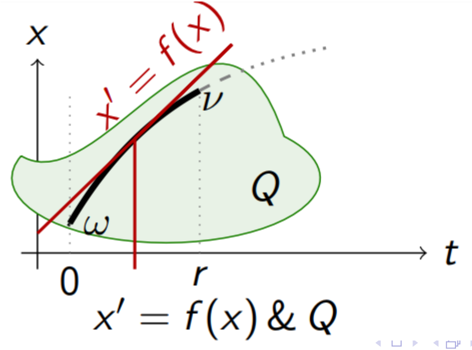

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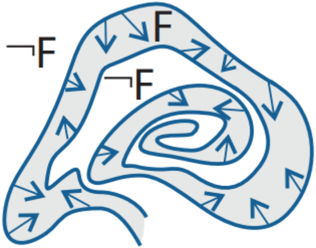

Attempts at Uniformity

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Wrong!

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Counterexample:

$$\begin{aligned}(\left[\{\exists y.(x' = y, z' = -1 \,\&\, y \geq z)\}\right] x \geq 0 \leftrightarrow \forall y.\,[?y \geq z]\, x \geq 0) \\ \leftarrow \forall y.\left[\{x' = y, z' = -1 \,\&\, y \geq z\}\right](x \geq 0)'\end{aligned}$$

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Pick a state with $x \geq 0 \wedge z \geq 0$:

- Premise 1: $\forall y.\,[\{x' = y, z' = -1\,\&\,y \geq z\}]\,(x \geq 0)'$ True!

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- Premise 2: $\forall y.\, [?y \geq z]\, x \geq 0$ True!
- Conclusion: $[\{\exists y.(x' = y, z' = -1 \,\&\, y \geq z)\}]\, x \geq 0$ False??

Uniform Substitution for dA$\mathcal{L}$

$$\text{DC} \quad ([c \,\&\, q]p \leftrightarrow [c \,\&\, q \wedge r]p) \leftarrow [c \,\&\, q]r$$

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\text{DAW} & [\{\exists \bar{y}.(c \,\&\, q)\}]\, p \leftrightarrow [\{\exists \bar{y}.(c \,\&\, q)\}]\, (q \rightarrow p) \\
\text{DAE} & \left[\{\exists \bar{y}.(x' = f, c \,\&\, q)\}\right] p \leftrightarrow \forall \bar{y}.\, \left[\{\exists \bar{y}.(x' = f, c \,\&\, q)\}\right] \left[x' := f\right] p
\end{array}
$$

Uniform Substitution for dA$\mathcal{L}$

$$\begin{aligned}
&\text{DAC} && ([\{\exists \bar{y}.(c \,\&\, q)\}]\, p \leftrightarrow [\{\exists \bar{y}.(c \,\&\, q \land r)\}]\, p) \leftarrow [\{\exists \bar{y}.(c \,\&\, q)\}]\, r \\
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DAE too weak for technical reasons, DAW actually unnecessary!

Uniform Substitution for dA$\mathcal{L}$

$$\text{DAC} \quad ([\{\exists \bar{y}.(c \,\&\, q)\}]\, p \leftrightarrow [\{\exists \bar{y}.(c \,\&\, q \land r)\}]\, p) \leftarrow [\{\exists \bar{y}.(c \,\&\, q)\}]\, r$$

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$$\text{DAS} \quad [\{\exists \bar{y}.(c \,\&\, q)\}]\, p \leftrightarrow \forall \bar{y}.\, [\{\exists \bar{y}.(c \,\&\, q)\}]\, [\{c \,\&\, q\}]\, p$$

What we actually need is a 'differential algebraic stutter' axiom (DAS)!

Uniform Substitution for dA$\mathcal{L}$

$$
\begin{array}{rl}
\text{DAC} & ([\{\exists\bar{y}.(c \,\&\, q)\}]\, p \leftrightarrow [\{\exists\bar{y}.(c \,\&\, q \wedge r)\}]\, p) \leftarrow [\{\exists\bar{y}.(c \,\&\, q)\}]\, r \\
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Theorem (Soundness of Uniform Substitution Calculus for dA$\mathcal{L}$)

The above substitution axioms for dA$\mathcal{L}$ are sound.

Uniform Substitution for dA$\mathcal{L}$

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Theorem (Soundness of Uniform Substitution Calculus for dA$\mathcal{L}$)

The above substitution axioms for dA$\mathcal{L}$ are sound.

We have created a sound and 'minimal' uniform substitution calculus for dA$\mathcal{L}$ that we can implement into KeYmaeraX!

Implementation Details

- Modified the parser and core data structures in KeYmaeraX to support dA$\mathcal{L}$
- We support singly-quantified differential systems (due to lack of vectorial support and significant compatibility changes required)
- Modified to unification, uniform substitution and other necessary algorithms supported by KeYmaeraX
- Uniform Substitution axioms added to trusted axiom base
- Derived axioms and derived tactics proven from trusted axioms
- Tested proving examples using derived axioms and tactics

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A successful extension that minimally extends the trusted core!

Derived Tactics

$$\frac{Q(x,\bar{y}) \vdash P(x)}{\Gamma \vdash \left[\{\exists \bar{y}.(x' = f(x,\bar{y}) \,\&\, Q(x,\bar{y}))\}\right] P(x), \Delta}\,\mathtt{dAW}$$

$$\frac{Q(x,\bar{y}) \vdash \left[x' := f(x,\bar{y})\right] (P(x))'}{P(x) \vdash \left[\{\exists \bar{y}.(x' = f(x,\bar{y}) \,\&\, Q(x,\bar{y}))\}\right] P(x)}\,\mathtt{dAI}$$

$$\frac{\Gamma \vdash [\{\exists \bar{y}.(c \,\&\, Q \wedge R)\}]\, P, \Delta \qquad \Gamma \vdash [\{\exists \bar{y}.(c \,\&\, Q)\}]\, R, \Delta}{\Gamma \vdash [\{\exists \bar{y}.(c \,\&\, Q)\}]\, P, \Delta}\,\mathtt{dAC}$$

Example (Perturbed Circular Motion)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{x \cdot e \leq 0 \vdash 2x(-y+e) + 2yx \leq 0}\ \mathbb{R}}{x \cdot e \leq 0 \vdash \left[x' := -y + e\right] \left[y' := x\right] 2xx' + 2yy' \leq 0}\ [':=]}{x \cdot e \leq 0 \vdash \left[x' := -y + e; y' := x\right] (x^2 + y^2 \leq 1)'}\ ['],[;]}{x^2 + y^2 = 1 \vdash \left[\{\exists e.(x' = -y + e, y' = x \,\&\, x \cdot e \leq 0)\}\right] x^2 + y^2 \leq 1}\ \mathtt{dAI}}{\vdash x^2 + y^2 = 1 \rightarrow \left[\{\exists e.(x' = -y + e, y' = x \,\&\, x \cdot e \leq 0)\}\right] x^2 + y^2 \leq 1}\ {\rightarrow}\mathrm{R}$$

$$\dfrac{Q(x,\bar{y}) \vdash \left[x' := f(x,\bar{y})\right] (P(x))'}{P(x) \vdash \left[\{\exists \bar{y}.(x' = f(x,\bar{y}) \,\&\, Q(x,\bar{y}))\}\right] P(x)}\ \mathtt{dAI}$$

Conclusion and Future Work

- Constructed a uniform substitution calculus for d$A\mathcal{L}$
- Implemented it in KeYmaeraX for a single existentially quantified variable
- Constructed derived axioms and rules for actual use
- Future extensions:
  - Support for multiple quantifiers
  - WebUI support
  - 'Desugared' syntax
  - Support for more derived rules
  - Support for hybrid games with differential-algebraic components

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Questions?