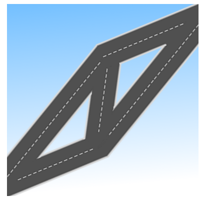

Formal Verification of Traffic Networks at Equilibrium

15-624 Final Project

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1 Introduction

The traffic network equilibrium problem, also known as the traffic assignment problem, aims to identify the traffic patterns on transportation networks which arise after a set of travellers determine the least cost path through the network between their origin and destination (Nagurney, 2009). The travel cost along each link in the network is a function of the traffic volume on the link, so each traveller must take into account the actions of other travellers when determining their own least cost route.

Solving the traffic assignment problem is a critical to transportation planning because it can compute roadway usage given travel demand and some mild assumptions on traveller behavior. Often, planning applications take the form of a transportation network design problem (NDP) in which design parameters are optimized with traffic equilibrium as a constraint. The design parameters influence the cost function of the network. As a result, different values of the design parameters induce different equilibrium traffic flow on the network. For example, Sheffi and Powell (1983) find the signal timings that minimize travel delay subject to the constraint that traffic is at equilibrium with respect to the cost induced by that signal timing.

This project identifies a dynamical system for a very simple network over both traffic flow and the design parameter that characterizes the equilibria as the value of the design parameters changes. In particular this dynamical system

1. preserves as an invariant the equilibrium state as the design parameter smoothly changes, and
2. is expressible in differential dynamic logic, the logic we have used in this course.

This formulation lays the foundation for a new approach to solving NDPs by characterizing their constraint set in a novel manner.

The remainder of this paper is organized as follows. Section 2 defines traffic equilibrium and offers a brief overview of existing formulations and applications of traffic equilibrium. Section 3 Briefly introduces the morning commute problem: the canonical motivating example for the *static traffic assignment problem.* Section 4 introduces the notation used to formalize network equilibrium. Section 5 introduces two different dynamical systems formulations for network equilibrium. Section 6 formulates the NDP problem in terms of projected dynamical systems, one of the two dynamical systems introduced in the previous section. Section 7 introduces a concrete simple network and equilibrium-preserving dynamics. It is shown that these dynamics preserve equilibrium under a changing control parameter. Finally section 8 reviews what has been done in this project as well as the limitations of this approach. Additionally, several avenues for further work are identified.

2 Background

The traffic network equilibrium problem has been extensively studied in the transportation literature. Equilibrium conditions were first characterized by Wardrop (1952) who defined the following two route choice criteria.

1. The journey times on all the paths actually used are equal, and less than those which would be experienced by a single vehicle on any unused route.

2. The average journey time is a minimum.

When traffic flow is distributed on the network according to the first criterion, no driver has a less costly alternative route. As a result it is known as the *user equilibrium* (UE) traffic distribution, because it requires no coordination between agents to achieve or maintain. When traffic flow is distributed on the network according to the second criterion, there is no less costly alternative traffic distribution, but it may be possible for a single driver to find a less costly route at the expense of the system. Because the second criterion requires the coordination of drivers is known as the *system optimum* (SO).

This project will deal exclusively with user equilibrium as it is a more realistic model of driver route-choice behavior. Smith (1979) and Dafermos (1980) then formulated the user equilibrium as a *variational inequality problem* (VIP), which has become the prevailing formulation used in the literature (Nagurney, 2009). Of additional interest is the alternative fixed-point formulation introduced by Smith (1979).

Of particular importance to this project are the dynamical systems formulations of Nagurney and Zhang (1997) and Smith (1984) who introduce two different formulations of traffic equilibrium as the solution to a differential equation. Moreover, they show that each system converges to equilibrium from a non-equilibrium point.

Both Smith (1979) and Dafermos (1980) formulate the *static traffic assignment* problem in which the demand for travel on the network is fixed, i.e. not time-dependent. Correspondingly, the flow on the network is not time-dependent. This is to be contrasted with the *dynamic traffic assignment* problem in which travel demand on the network, and necessarily link flow, is a function of time.

Network equilibrium is also utilized as a constraint in many network design problems (NDP) (Sheffi and Powell, 1983). NDPs of this nature are formulated as mathematical programs with equilibrium constraints (MPEC) (Luo et al., 1996). Because the equilibrium constraint is a variational inequality problem, it is non-convex and non-differentiable. Iterative algorithms which approximate the gradient through the VIP exist but are computationally expensive (Josefsson and Patriksson, 2007). Developing efficient (approximate) solution algorithms for MPECs remains an area of active research. A logical treatment of network design problems would aim to verify properties of the network at equilibrium *without* explicitly computing the equilibrium state.

Formal methods have been successfully applied to verify safety in several traffic control systems. Loos et al. (2011) prove collision-freedom in cars equipped with local adaptive cruise control. Loos and Platzer (2011) design and verify a smart traffic light system at an intersection. Mitsch et al. (2012) model variable speed limits applied to portions of the roadway and prove conditions on model parameters that guarantee safety. Of particular importance to this project is the safety verification of traffic networks presented by Müller et al. (2015) which enables verification of link properties (for example, that flow does not exceed capacity) of arbitrary networks by decomposition into separable components subject to contracts on their flow inputs and outputs.

Formal methods have also been developed to describe adversarial dynamics within a hybrid system. Hybrid Games (Platzer, 2013) describe hybrid systems in which choices in the hybrid system are made by one of two players each of whom aims to reach complementary and mutually exclusive “winning” conditions. Differential Hybrid Games (Platzer, 2017) extend hybrid games to include differential equations to which each players provides their own input control.

3 A motivating example

Consider the morning commute. Suppose every morning a set of drivers each leave their homes at the same time each headed to their respective workplaces. Every morning each driver decides which path they should take to work. They do not know what paths any of the other drivers will choose today, but each knows what happened yesterday. In particular, we suppose that each has full information of what happened yesterday.

Consider the following behavioral principle: drivers only change route in such a way to reduce travel time *based on yesterday's travel costs*. We suppose that drivers follow this principle when deciding which route to take every morning. Note that they need not change route from day to day, but if they do the chosen route *must* decrease travel costs. The fixed points of this decision principle are those route choices which *cannot* change from one day to the next. According to the decision principle yesterday's route choices cannot change if and only if no driver has a less costly alternative route. This coincides exactly with Wardrop's first condition so the fixed points of this decision processes are exactly the equilibria as defined by Wardrop.

We will return to this example in subsequent sections to motivate several concepts.

4 Methods

Loosely following Smith (1979) we formulate the traffic assignment problem as follows. Let $G = (N, L)$ represent a finite directed graph with N denoting the set of nodes with n members and L the set of links with m members. We denote each link as $a_i \in L$ for $i = 1, \cdots, m$ and the traffic flow across each link as x_i for $i = 1, \cdots, m$. We collect the link flows in a vector $\mathbf{x} = (x_1, \cdots, x_m) \in \mathbb{R}^m_+$. Where $\mathbb{R}^m_+$ denotes the non-negative orthant of $\mathbb{R}^m$. Similarly we define a vector of link capacities, $\mathbf{c} = (c_1, \cdots, c_m) \in \mathbb{R}^m_+$.

Let $Q \subseteq N \times N$ denote the set of origin-destination pairs on the network with $|Q| = d$. Denote by q_i the travel demand between each origin-destination pair for $i = 1, \cdots, d$. We collect the demand as a vector $\mathbf{q} = (q_1, \cdots, q_d) \in \mathbb{R}^d_+$. The goal of the traffic assignment problem is to allocate the volume of travel demand to links in the network such that all travellers reach their destination and Wardrop's first condition is met.

Intuitively, however, the choice faced by drivers is in terms of paths through the network, not specific links. A *path* in the network retains its graph-theoretic definition: an ordered sequence of directed links in the network which connect two nodes in the graph. We may enumerate all (directed) paths in the network as a set P and let $p = |P|$ A *path flow* is a vector $\mathbf{F} \in \mathbb{R}^p_+$ which represents the volume of traffic along each path in the network. A path flow is *demand feasible* if the total volume of traffic on all paths between each origin destination pair is equal to the travel demand between that origin destination pair. In other words, if there are 10 vehicles trying to get from point A to point B and there are two paths connecting A and B, the sum of the volumes on the two paths had better be equal to 10. Define $M \in \{0, 1\}^{d \times p}$ to be a binary matrix representing the trip-path incidence relation: $M_{ij} = 1$ if the j-th path in P connects the origin-destination pair q_i and equals 0 otherwise. With this definition, a path flow $\mathbf{F}$ is demand feasible if $M\mathbf{F} = \mathbf{q}$.

A path flow vector induces traffic volume on each link. We can make explicit this relation by defining the *link-path* incidence matrix. Let $D \in \{0, 1\}^{p \times m}$ be a binary matrix representing the link-path incidence relation: $D_{ij} = 1$ if link a_j is on the i-th path of P and equals 0 otherwise. The volume of traffic on each link is given by $\mathbf{x} \in \mathbb{R}^m_+$ and referred to as the *link flow*. A link flow $\mathbf{x}$ corresponds to a path flow $\mathbf{F}$ if $\mathbf{x} = D^T\mathbf{F}$. A link flow is said to be *demand-feasible* if is corresponds to a demand-feasible path-flow. A link flow $\mathbf{x} \in \mathbb{R}^m_+$ is said to be *supply-feasible* if its value on each link does not exceed the capacity of the link: $x_i \leq c_i$ for all $i = 1 \cdots m$.

We may now represent the set of demand-feasible and supply-feasible link flows.

$$\Omega_d = \{\mathbf{x} \in \mathbb{R}^m_+ \mid \exists \mathbf{F} \in \mathbb{R}^p_+ M(\mathbf{F} = \mathbf{q} \wedge D^T\mathbf{F} = \mathbf{x})\} \tag{1}$$

$$\Omega_s = \{\mathbf{x} \in \mathbb{R}^m_+ \mid \mathbf{x} \preceq \mathbf{c}\} \tag{2}$$

$$\Omega = \Omega_d \cap \Omega_s \tag{3}$$

Where we use $\preceq$ to denote element-wise inequality. To simplify the notation for the rest of this paper we will use Ω to refer to the supply and demand feasible set $\Omega_d \cap \Omega_s$.

Finally, we introduce the link cost function $\mathbf{t} : \Omega_s \to \mathbb{R}^m_+$ which reports the travel cost of each link as a function of the link flows. We denote the cost of a single link a_i as $t_i(\mathbf{x})$.

The user equilibrium condition introduced by Wardrop (1952) can be expressed succinctly as a *variational inequality* (Dafermos, 1980; Nagurney, 2009; Smith, 1979). We motivate the variational inequality formulation by returning to the decision process outlined in section 3. Using our notation let $\mathbf{x}^*$ denote yesterday's link flows and let $\mathbf{x}$ denote today's link flows which have yet to be decided. The link costs induced by yesterday's flow is therefore given by $\mathbf{t}(\mathbf{x}^*)$. This cost may be regarded as fixed because it was induced by link flows from yesterday. In order for yesterday's link flows to be at equilibrium, there can be no alternative feasible link flow which reduces the total cost based on yesterday's costs. Mathematically, yesterday's total cost is given by $\mathbf{t}(\mathbf{x}^*)\mathbf{x}^*$ and the cost of today's potential route choice $\mathbf{x}$ based on yesterday's costs is given by $\mathbf{t}(\mathbf{x}^*)\mathbf{x}$. So $\mathbf{x}^*$ is at equilibrium if and only if $\mathbf{t}(\mathbf{x}^*)\mathbf{x}^* \leq \mathbf{t}(\mathbf{x}^*)\mathbf{x}$.

We therefore say that $\mathbf{x}^* \in \Omega$ is an equilibrium link flow if (4) holds.

Theorem 1 (Variational Inequality Formulation)**.** *A link flow pattern $\mathbf{x}^* \in \Omega$ is a traffic network equilibrium link flow if and only if it satisfies the following variational inequality problem:*

$$\mathbf{t}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \geq 0, \ \textit{for all } \mathbf{x} \in \Omega \tag{4}$$

Theorem 2 (Existence)**.** *There exists an equilibrium link flow $\mathbf{x}^* \in \Omega_d \cap \Omega_s$ if the following conditions are met.*

1. *The link cost function $\mathbf{t} : \Omega_s \to \mathbb{R}_+^m$ is continuous.*
2. *The demand-feasible set of link flows Ω_d is closed and convex.*
3. *Every demand-feasible link flow is also supply feasible: $\Omega_d \subseteq \Omega_s$.*

The proof of theorem 2 is given in Smith (1979). Of the conditions on existence that theorem 2 requires, only one is non-trivial. Condition 1 is very mild and not unreasonable in practice. The demand-feasible set as defined in (1) meets condition 2 since it is defined by a set of linear constraints. Condition 3, as Smith notes, is rather strong. This condition arises from the use of the Brouwer fixed-point theorem, which states that every continuous mapping from a non-empty convex and compact set into itself has a fixed point. In short, the proof leverages a continuous mapping $T : \Omega_d \cap \Omega_s \to \Omega_d$ so in order to apply the theorem, Ω_d needs to be a subset of Ω_s. Besides, $\mathbf{t}$ is not defined outside of Ω_s. There are alternate existence theorems that relax this constraint but for simplicity we will use this one.

There are also uniqueness conditions which we do not consider in this project. Uniqueness is used to show that an algorithm that finds *an* equilibrium has found the right one. In this project we do not compute the equilibrium state explicitly, we do not care much if there is more than one equilibrium. For the rest of this paper we **do not** assume uniqueness of the equilibrium state.

The network design problem additionally introduces a network control system which implicitly modifies the network cost. A classical example of a network design problem is presented in Sheffi and Powell (1983). The timing of traffic signals in the network is controlled with the intent of minimizing the total delay in the network. What makes this problem difficult is that changes to the signal timing changes the travel costs of the links which changes the equilibrium traffic flow which determines the total delay.

Let $Y \subseteq \mathbb{R}^k$ denote the domain of the control parameters. The link cost $\mathbf{t}$ now additionally depends on the control parameters. Let $g : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}$ be a network performance function which depends jointly on the state of the network and on the control parameters. In the example given by Sheffi and Powell (1983), this would simply be the total delay in the network. We express the network design problem as a mathematical program with equilibrium constraints in (5).

$$\min_{\mathbf{y} \in Y} \ g(\mathbf{y}, \mathbf{x}^*) \tag{5a}$$

$$\text{s.t.} \ \ \mathbf{x}^* \in \Omega \tag{5b}$$

$$\mathbf{t}(\mathbf{y}, \mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \geq 0, \ \text{for all } \mathbf{x} \in \Omega \tag{5c}$$

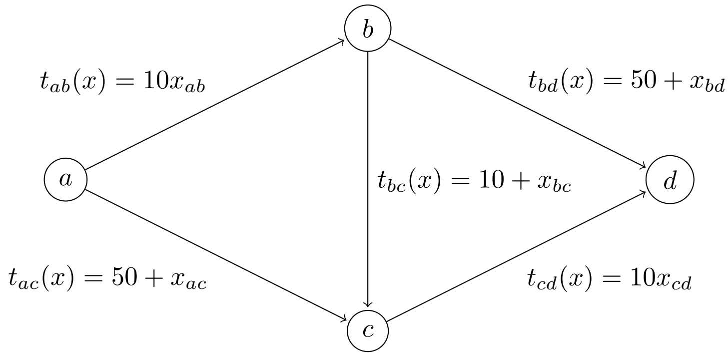

Figure 1: The Braess network. Each link is labeled with the link cost function.

4.1 The Braess Network

As a croncrete network we introduce the Braess Network as described by Murchland (1970) in fig. 1.

On this network there is only one origin-destination pair: (a, d). Let $q = 6$ denote the travel demand from a to d. Further, we assume that the road network has sufficient capacity for all vehicles. In particular, we may take capacity equal to q for each link. There are three paths between a and d on this network: (a, b, d), (a, c, d), and (a, b, c, d). We will refer to these three paths as *upper*, *lower*, and *zig-zag* paths respectively.

Murchland (1970) states the equilibrium is achieved when each of the three paths is used by two vehicles each. Immediately we see that such a path flow is demand-feasible since the path flows sum to equal the demand. To convince ourselves that such a path flow is indeed at equilibrium, we can apply Wardrop's first principle directly: all used paths should have equal travel cost. First, let us compute the corresponding link flow. We could construct the link-path incidence matrix, but for this small example it is easier to simply enumerate in table 1. The link cost is computed by applying the link cost function associated with each link in fig. 1.

Table 1: Link flow and link costs

Link	Upper	Lower	Zig-zag	Link flow	Link cost
ab	2	0	2	4	40
bd	2	0	0	2	52
ac	0	2	0	2	52
cd	0	2	2	4	40
bc	0	0	2	2	12

We can compute the path costs by simply adding up the costs on each link of the path. The upper path is composed of ab and bd, the lower by ac and cd, and the zig-zag by ab, bc, and cd. The path costs are therefore $40 + 52 = 92$ for the upper path, $52 + 40 = 92$ for the lower path, and $40 + 12 + 40 = 92$ for the zig-zag path. We observe that all paths that are used have equal cost. Therefore Wardrop's first condition is met and this link flow is an equilibrium.

Alternatively we use this example to understand why Wardrop's first condition is equivalent to the condition that no driver has a less costly alternative route. When all paths between their origin and destination cost the same, the driver had no incentive to switch routes.

Additionally, we can use the VI path flow formulation at this equilibrium and show that the following holds for all feasible path flows.

$$\begin{aligned} \mathbf{T}(\mathbf{f}^*)\mathbf{f} - \mathbf{T}(\mathbf{f}^*)\mathbf{f}^* &\geq 0 \\ 92 \cdot f_{\text{upper}} + 92 \cdot f_{\text{lower}} + 92 \cdot f_{\text{zig-zag}} - (92 \cdot 2 + 92 \cdot 2 + 92 \cdot 2) &\geq 0 \\ f_{\text{upper}} + f_{\text{lower}} + f_{\text{zig-zag}} &\geq 6 \end{aligned}$$

Recall that a path flow is feasible if the total volume over paths between and origin-destination pair is equal to the demand between that pair. Therefore, $f_{\text{upper}} + f_{\text{lower}} + f_{\text{zig-zag}} = 6$ for all feasible path flows and the inequality holds.

This particular network is notable because its equilibrium traffic flow is not its most efficient traffic assignment, giving rise to what is known as the "Braess Paradox". Consider instead the demand-feasible path flow which puts three vehicles each on the upper and lower paths and none on the zig-zag path. The link costs for this path flow are given in table 2.

Table 2: Link flow and link costs

Link	Upper	Lower	Zig-zag	Link flow	Link cost
ab	3	0	0	3	30
bd	3	0	0	3	53
ac	0	3	0	3	53
cd	0	3	0	3	30
bc	0	0	0	0	10

With this path flow, the travel costs on each of the used paths is 83. This is less than the cost of the paths at equilibrium, meaning that this is globally more efficient. However, this flow pattern is not an equilibrium because the unused zig-zag path has cost equal to 70 ($= 30 + 10 + 30$), less than the cost of the used paths. As a result, a single driver on the upper or lower path would be able to switch to the zig-zag path and save time. This fact contradicts Wardrop's first condition.

As a VI in the path flow formulation we may find a path flow which does not satisfy the inequality.

$$\begin{aligned} \mathbf{T}(\mathbf{f}^*)\mathbf{f} - \mathbf{T}(\mathbf{f}^*)\mathbf{f}^* &\geq 0 \\ 83 \cdot f_{\text{upper}} + 83 \cdot f_{\text{lower}} + 70 \cdot f_{\text{zig-zag}} - (83 \cdot 3 + 83 \cdot 3 + 70 \cdot 0) &\geq 0 \\ 83 \cdot f_{\text{upper}} + 83 \cdot f_{\text{lower}} + 70 \cdot f_{\text{zig-zag}} &\geq 83 \cdot 6 \end{aligned}$$

In particular take $f_{\text{zig-zag}} = 6$ (and no flow on either of the other two paths). Substituting this path flow into the inequality yields $70 \cdot 6 \geq 83 \cdot 6$ which is false.

5 Dynamical Systems Formulation of Traffic Equilibrium

In this section two related dynamical systems formulations of traffic equilibrium are introduced.

5.1 Projected Dynamical Systems

The variational inequality formulation of the traffic equilibrium condition is a static condition. Of particular interest in re-framing traffic equilibrium in the logic of hybrid systems is to identify a dynamical system which not only preserves traffic equilibrium as an invariant but pushes a non-equilibrium state toward equilibrium. We can develop a dynamical system from the decision process from the motivating example in section 3. Let $\mathbf{x}$ denote yesterday's link flow. When choosing today's routes, the drivers wish to pick a feasible link flow $\tilde{\mathbf{x}}$ which reduces travel costs. One way to achieve this is to simply move $\mathbf{x}$ in the direction that reduces its link costs by some step size $\alpha \geq 0$, i.e. $\mathbf{x} - \alpha\mathbf{t}(\mathbf{x})$, and then project it back onto the feasible set to ensure that the resulting link flow remains feasible.

Nagurney and Zhang (1997) introduce a *Projected Dynamical System* formulation of traffic equilibrium given by (6) which extends the discrete decision process to a continuous one.

$$\mathbf{x}' = \Pi_{\Omega}(\mathbf{x}, -\mathbf{t}(\mathbf{x})) \tag{6}$$

with link flow $\mathbf{x} \in \Omega$ and,

$$\Pi_\Omega(x, v) = \lim_{\varepsilon \to 0} \frac{P_\Omega(x + \varepsilon v) - P_\Omega(x)}{\varepsilon} \tag{7}$$

$$= \lim_{\varepsilon \to 0} \frac{P_\Omega(x + \varepsilon v) - x}{\varepsilon} \tag{8}$$

$$P_K(u) = \arg\min_{v \in K} \|u - v\| \tag{9}$$

The operator Π_Ω can be thought of as a Gâteaux directional derivative along the negative of the link cost of the norm projection onto Ω, P_Ω for $x \in \Omega$. Under this model, link flows change in the feasible direction that offers the largest reduction in the users travel costs. The intuition behind this operator is that traffic volume should move from higher cost links to lower cost links. By moving in the direction of $-t(x)$ the traffic volume on high cost links is reduced faster than the volume on lower cost links. The net effect of the projection back onto the feasible set is then to move volume from higher cost links to lower cost links until the costs are equalized.

It's important to note that solutions of (6) are not functions of real time, but of some abstract decision space. The main result presented in Nagurney and Zhang (1997) is to show that one my discretize (6) to produce an algorithm which converges on the true solution. In this setting, we may think of this system as representing, somewhat counter-intuitively, the long term evolution of traffic flow.

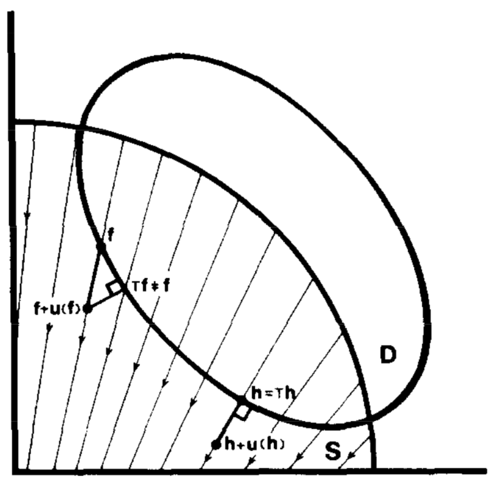

Figure 2: Visual depiction of the dynamical system in (6). Here, $D \cap S$ represents the feasible region Ω, $u(\cdot) = -t(\cdot)$ and Tf is the projection of $f + u(f)$ onto Ω. The vector field represents u. The flow h is an equilibrium. We see that the equilibrium flow h is a fixed point of T whereas the T applied to some non-equilibrium point f, is not a fixed point ($Tf \neq f$) and nudges the system toward equilibrium. The operator Π_Ω can be considered the infinitesimal version of the operator T in the diagram.

As a succinct visual aid of this process, consider fig. 2 re-printed here from Smith (1979). Lets consider how the route choice adjustment process acts on an arbitrary feasible link flow x in fig. 2. Suppose x is in the interior of $D \cap S$. In this case, x' is simply $-t$ and the link flow will evolve toward the boundary. Upon reaching the boundary, x can no longer move in the direction of $-t$ and remain in the feasible set. The dynamics ensure this through use of the projection operator. As a result, x evolves in the direction most closely aligned with $-t$ that still points in the direction of Ω. The link flow will then move along the boundary, approaching the equilibrium point h. At the equilibrium point we see that $\Pi_\omega(h, -t(h)) = 0$ since $P_\Omega(h - \varepsilon t(h)) = h$ for all $\varepsilon \geq 0$.

We repeat three important theorems proved in Nagurney and Zhang (1997).

Theorem 3. *A link flow $x^* \in \Omega$ satisfies the variational inequality problem (4) if and only if it is a stationary point for the differential equation (6), that is,*

$$0 = \Pi_\Omega(\mathbf{x}, -\mathbf{t}(\mathbf{x}))$$

Definition 1 (Stability)**.** *The route choice adjustment process given by (6) is **stable** if for every initial flow pattern $\mathbf{x}_0 \in \Omega$ and every equilibrium flow pattern $\mathbf{x}^*$, the distance $\|\mathbf{x}^* - \mathbf{x}_0(s)\|$ is monotone non-increasing in s.*

Theorem 4. *The route choice adjustment process is stable if the link cost t is monotone increasing in the link flow.*

Definition 2 (Asymptotic Stability)**.** *The route choice adjustment process given by (6) is **asymptotically stable** if it is stable and, for any initial flow pattern $\mathbf{x}_0 \in \Omega$ there exists some equilibrium flow pattern $\mathbf{x}^*$ such that $\mathbf{x}_0(s) \to \mathbf{x}^*$ as $s \to \infty$.*

Theorem 5. *The route choice adjustment process is asymptotically stable if the link cost t is strictly monotone increasing in the link flow.*

Lemma 1. *The path-flow feasibility conditions can be expressed in differential dynamic logic.*

Proof. As before, let $\mathbf{q} = (q_1, \ldots, q_d)$ denote the non-negative demand between each origin-destination pair on the network and let M and D denote the trip-path indicence matrix and link-path indicence matrix respectively. $\mathbf{q}$ is given as an input to the traffic assignment problem and M and D are derived from the network topology, so we may regard these as objects as fixed.

Let $\mathbf{F} = (F_1, \ldots, F_p)$ denote a path flow. From the definition of path demand feasibility the vector product $M\mathbf{F}$ must equal the demand vector $\mathbf{q}$. We may express this in first order logic as

$$\wedge_{i \in Q} q_i = \sum_{j \in P} M_{ij} F_j \tag{10}$$

Moreover, each path flow must be non-negative.

$$\wedge_{j \in P} 0 \leq F_j \tag{11}$$

□

Lemma 2. *The link-flow feasibility conditions can be expressed in differential dynamic logic.*

Proof.

$$\bigwedge_{i \in L} (0 \leq x_i) \tag{12}$$

$$\exists F_1 \geq 0 \exists F_2 \geq 0 \cdots \exists F_p \geq 0 ((\bigwedge_{i \in Q} q_i = \sum_{j \in P} M_{ij} F_j) \wedge (\bigwedge_{k \in L} x_k = \sum_{j \in P} D_{kj} F_j)) \tag{13}$$

□

Remark 1. *Although the constraint set is expressible in differential dynamic logic, the projected dynamic system is not. The projection operator in this setting is equivalent to projection onto a simplex. This is not generally attainable in closed form. However, as a special case, there exists a closed form for projection onto the 2-simplex which we leverage in the worked example in section 7.*

From theorem 3 we may prove that the equilibrium condition $P \equiv \forall \tilde{\mathbf{x}} \in \Omega(\mathbf{t}(\mathbf{x})(\tilde{\mathbf{x}} - \mathbf{x}) \geq 0)$ is an invariant of the route choice adjustment process (6). Moreover, theorem 5 guarantees that following the route adjustment process will get us closer to equilibrium, or at least never lead us further from equilibrium. Concretely, these two facts can be expressed in differential dynamic logic as follows.

Theorem 6. *The following is a valid formula.*

$$\forall \tilde{\mathbf{x}} \in \Omega(\mathbf{t}(\mathbf{x})(\tilde{\mathbf{x}} - \mathbf{x}) \geq 0) \to [\mathbf{x}' = \Pi_\Omega(\mathbf{x}, -\mathbf{t}(\mathbf{x})) \;\&\; \mathbf{x} \in \Omega] \forall \tilde{\mathbf{x}} \in \Omega(\mathbf{t}(\mathbf{x})(\tilde{\mathbf{x}} - \mathbf{x}) \geq 0)$$

Proof. By (4), the pre-condition implies that $\mathbf{x}$ is an equilibrium point. By theorem 3, we have that since $\mathbf{x}$ is an equilibrium point, $\Pi_\Omega(\mathbf{x}, -\mathbf{t}(\mathbf{x})) = 0$. Therefore, $\mathbf{x}' = 0$ initially. Because $\mathbf{x}'$ depends on $\mathbf{x}$ only, $\mathbf{x}$ will remain unchanged forever so the post-condition is true. □

Theorem 7. *Let Ω^* denote the set of equilibria. If $\mathbf{t} : \Omega_s \to \mathbb{R}^m_+$ is strictly monotone increasing then the following is a valid formula.*

$$\mathbf{x} \in \Omega \to \exists \mathbf{x}^* \in \Omega^* \forall \epsilon > 0 \ \langle \mathbf{x}' = \Pi_\Omega(\mathbf{x}, -\mathbf{t}(\mathbf{x})) \ \& \ \mathbf{x} \in \Omega \rangle (\|\mathbf{x} - \mathbf{x}^*\| \leq \epsilon)$$

Proof. Let $\mathbf{x}_0 \in \Omega$. Let $\mathbf{x}_0(s)$ denote the state of $\mathbf{x}_0$ after running the dynamics for duration $s \geq 0$. Note that by construction, the dynamics ensure that if $\mathbf{x}$ is in Ω initially then $\mathbf{x}$ will remain in Ω after running the dynamics for any duration, i.e. the domain constraint is always satisfied. By theorem 5, for every $\epsilon > 0$ there exists real number N such that if $s > N$ then $\|\mathbf{x}_0(s) - \mathbf{x}\| \leq \epsilon$. Concretely, we may take any $r > N$. Then $\mathbf{x}_0(r) \in \Omega$ and $\mathbf{x}_0(s) \in \Omega$ for all $0 \leq s \leq r$. Moreover, we have that there exists some $\mathbf{x}^* \in \Omega^*$ such that $\|\mathbf{x}_0(r) - \mathbf{x}^*\| \leq \epsilon$. □

5.2 Path swap dynamics

Smith (1984) introduces a dynamical system formulation of traffic equilibrium in terms of path flow. First, some notation. As above, $\mathbf{F} = (F_1, \cdots, F_p)$ is a vector of path flows. Define the relation over paths $i \sim j$ as true if and only if i and j join the same origin destination pair. Let $\mathbf{T}(\mathbf{F}) = (T_1(\mathbf{F}), \cdots, T_p(\mathbf{F}))$ denote the path cost as a function of path flow. Finally, let $(\cdot)_+ = \max\{0, \cdot\}$. We can rephrase Wardrop's equilibrium condition logically as $i \sim j \wedge T_i(\mathbf{F}) > T_j(\mathbf{F}) \to F_i = 0$ for all paths i, j. Equivalently, $\mathbf{F}$ is an equilibrium path flow if and only if whenever $i \sim j$ (14) holds.

$$\Phi_{ij}(\mathbf{F}) \equiv F_i(T_i(\mathbf{F}) - T_j(\mathbf{F}))_+ = 0 \tag{14}$$

We note that $\Phi_{ij}(\mathbf{F}) \geq 0$ for all demand feasible path flows because $F_i \geq 0$ for all i. As a result, whenever $\mathbf{F}$ is not an equilibrium path flow $\Phi_{ij}(\mathbf{F}) > 0$.

Intuitively, Φ_{ij} can be thought of as the rate at which drivers change routes from route i to route j. Concretely suppose there are only two paths in the network, i and j. Further suppose $T_i(\mathbf{F}) > T_j(\mathbf{F})$. Then drivers abandon route i a rate of Φ_{ij} and clamor to route j at a rate of Φ_{ij}. In other words, $F_i' = -\Phi_{ij}(\mathbf{F})$ and $F_j' = \Phi_{ij}(\mathbf{F})$.

To express the dynamical system in general we let $e_i \in \mathbb{R}^p$ denote the standard basis vector and define the "swap" vector $\Delta_{ij} = e_j - e_i$ if $i \sim j$ and 0 otherwise.

$$\Delta T_{ij}(\mathbf{F}) = -\mathbf{T}(\mathbf{F})\Delta_{ij} = T_i(\mathbf{F}) - T_j(\mathbf{F}) \tag{15}$$

Importantly, this implies the identity $\Delta T_{ij} = -\Delta T_{ji}$. In particular, we can express Φ_{ij} as,

$$\Phi_{ij}(\mathbf{F}) = F_i(\Delta T_{ij})_+ \tag{16}$$

Finally, the dynamics are given by (17).

$$\mathbf{F}' = \Phi(\mathbf{F}) \equiv \sum_{i,j \in P} \Phi_{ij}(\mathbf{F})\Delta_{ij} \tag{17}$$

The quantity Φ_{ij} can be thought of as the speed and Δ_{ij} the direction of F under the influence of swapping higher cost routes for lower cost ones.

We briefly repeat the main results from Smith (1984).

1. Solutions to the differential equation $F' = 0$ are equilibrium traffic flows. Related, equilibria are preserved as an invariant of the dynamics.

2. Feasible non-equilibrium traffic flows converge to equilibrium traffic flows under path swap dynamics.

3. Feasibility is preserved as an invariant under path swap dynamics.

Remark 2. *Importantly, the path swap dynamics in* (14) *are expressible in differential dynamic logic extended with the max function provided that the path flow function* T *are expressible in differential dynamic logic.*

This fact follows transparently from (17) which contains only additions, multiplications, and the max function.

6 Single-Player Control

Suppose a traffic controller is able to influence the link cost function, for example, by applying a toll to one or more links. Here we are interested in examining the dynamical system governed by (18).

$$\mathbf{x}' = \Pi_\Omega(\mathbf{x}, -\mathbf{t}(\mathbf{y}, \mathbf{x})) \tag{18}$$

Where $\mathbf{y}$ is typically a function of the evolution "time" of the system. This is a trivial differential game in the sense that the dynamics do not additionally depend on a second players control. It is typically analyzed in the setting of optimal control theory. (Bardi and Capuzzo-Dolcetta, 2008)

In this section appropriate conditions on $\mathbf{t}$ and the control input $\mathbf{y}$ will be developed so that theorems 3 to 5 hold.

To aid in our discussion, we say that traffic flow $\mathbf{x}$ is at equilibrium with respect to control parameters $\mathbf{y}$ if $\Pi_\Omega(\mathbf{x}, -\mathbf{t}(\mathbf{y}, \mathbf{x})) = 0$.

6.1 Discrete case

In the discrete case, we are interested in showing that the dynamical system will converge to equilibrium after a discrete change to the control parameters.

Theorem 8 (Discrete control)**.** *Let* Y *denote domain of control parameters and* Ω^*_y *denote the set of equilibria with respect to* y*. Let* $\mathbf{t} : Y \times \Omega_s \to \mathbb{R}^m_+$ *be strictly monotone increasing in* $\mathbf{x}$ *for any fixed* $y \in Y$*. The following is a valid formula.*

$$\mathbf{x} \in \Omega \to \forall \epsilon > 0 \ \mathbf{y} := *; ?(\mathbf{y} \in Y); \langle \mathbf{x}' = \Pi_\Omega(\mathbf{x}, -\mathbf{t}(\mathbf{y}, \mathbf{x})) \ \& \ \mathbf{x} \in \Omega \rangle (\exists \mathbf{x}^* \in \Omega^*_{\mathbf{y}} \ \|\mathbf{x} - \mathbf{x}^*\| \leq \epsilon)$$

Proof. We proceed with the proof using sequent calculus.

$$\dfrac{\dfrac{\text{Theorem 7}\ \dfrac{*}{\mathbf{x} \in \Omega, \mathbf{y} \in Y \vdash \forall \epsilon > 0 \ \langle \mathbf{x}' = \Pi_\Omega(\mathbf{x}, -\mathbf{t}(\mathbf{y}, \mathbf{x})) \ \& \ \mathbf{x} \in \Omega \rangle (\exists \mathbf{x}^* \in \Omega^*_{\mathbf{y}} \ \|\mathbf{x} - \mathbf{x}^*\| \leq \epsilon)}}{\mathbf{x} \in \Omega, \epsilon > 0, \mathbf{y} \in Y \vdash \langle \mathbf{x}' = \Pi_\Omega(\mathbf{x}, -\mathbf{t}(\mathbf{y}, \mathbf{x})) \ \& \ \mathbf{x} \in \Omega \rangle (\exists \mathbf{x}^* \in \Omega^*_{\mathbf{y}} \ \|\mathbf{x} - \mathbf{x}^*\| \leq \epsilon)}}{\mathbf{x} \in \Omega \to \forall \epsilon > 0 \ \mathbf{y} := *; ?(\mathbf{y} \in Y); \langle \mathbf{x}' = \Pi_\Omega(\mathbf{x}, -\mathbf{t}(\mathbf{y}, \mathbf{x})) \ \& \ \mathbf{x} \in \Omega \rangle (\exists \mathbf{x}^* \in \Omega^*_{\mathbf{y}} \ \|\mathbf{x} - \mathbf{x}^*\| \leq \epsilon)}$$

We may apply theorem 7 in the last step because $\mathbf{y}$ is fixed in the differential equation. As a result, $\mathbf{t}(\mathbf{y}, \mathbf{x}))$ may be re-written as a function of $\mathbf{x}$ alone, say $\mathbf{t}_y(\mathbf{x})$, which meets the precondition of theorem 7. □

What if the system starts off at equilibrium? In this case we would like to show that a small change in control parameters results in a small change in equilibrium toward which the dynamics will evolve.

Theorem 9. *Let* Y *denote domain of control parameters and* Ω^*_y *denote the set of equilibria with respect to* y*. Let* $\mathbf{t} : Y \times \Omega_s \to \mathbb{R}^m_+$ *be continuous and strictly monotone increasing in* $\mathbf{x}$ *for any fixed* $y \in Y$*.*

The following are valid formulae.

$$\mathbf{x} \in \Omega^*_{\mathbf{y}} \to \forall \epsilon > 0 \ \exists \delta > 0 \ \mathbf{y} := \mathbf{y} + \delta; ?(\mathbf{y} \in Y); [\mathbf{x}' = \Pi_\Omega(\mathbf{x}, -\mathbf{t}(\mathbf{y}, \mathbf{x})) \ \& \ \mathbf{x} \in \Omega] (\exists \mathbf{x}^* \in \Omega^*_{\mathbf{y}} \ \|\mathbf{x} - \mathbf{x}^*\| \leq \epsilon)$$

$$\mathbf{x} \in \Omega^*_{\mathbf{y}} \to \forall \epsilon > 0 \ \forall \delta > 0 \ \mathbf{y} := \mathbf{y} + \delta; ?(\mathbf{y} \in Y); \langle \mathbf{x}' = \Pi_\Omega(\mathbf{x}, -\mathbf{t}(\mathbf{y}, \mathbf{x})) \ \& \ \mathbf{x} \in \Omega \rangle (\exists \mathbf{x}^* \in \Omega^*_{\mathbf{y}} \ \|\mathbf{x} - \mathbf{x}^*\| \leq \epsilon)$$

6.2 Continuous control

Consider the case in which $\mathbf{t}$ is continuous in both the traffic flow $\mathbf{x}$ and control parameter $\mathbf{y}$. It would be nice if the continuity of $\mathbf{t}$ extended to some notion of continuity of the equilibrium state. In particular, suppose $\mathbf{x}$ is an equilibrium traffic flow with respect to $\mathbf{y}$. We then suppose that we follow the system of differential equations given by (19) for some $r \geq 0$.

$$\mathbf{x}' = \Pi_\Omega(\mathbf{x}, -\mathbf{t}(\mathbf{y}, \mathbf{x})) \tag{19a}$$

$$\mathbf{y}' = f(\mathbf{y}) \tag{19b}$$

Where f is some function definable in first order arithmetic. Denote the value of $\mathbf{x}$ and $\mathbf{y}$ after running the system for duration r by $\mathbf{x}^{(r)}$ and $\mathbf{y}^{(r)}$. One reasonable question arises: if $\mathbf{x}$ was an equilibrium with respect to $\mathbf{y}$ is $\mathbf{x}^{(r)}$ an equilibrium with respect to $\mathbf{y}^{(r)}$?

Intuitively, we ought to be able to find conditions under which $\mathbf{x}^{(r)}$ is "close" to the equilibrium with respect to $\mathbf{y}^{(r)}$. Informally, we can think of this as a continuous version of the model in theorem 9, again starting from an equilibrium traffic flow x.

1. the control parameter y is changed by some small about δ

2. the traffic flow is no longer at equilibrium with respect to y, but by following the dynamics the link flow approaches the equilibrium with respect to (the updated) control parameter.

In a loop, these two steps can be thought of as a chase of some kind: a discrete change is made to the control and then the traffic flow "catches up" to the new equilibrium. We would rightly worry that over many iterations the traffic state would stray arbitrarily far from the equilibrium unless the dynamics were allowed to run long enough. However, this scenario has a natural expression as a hybrid game in which one player (the traffic controller) sets y and the number of iterations, while the other (the traffic) determines how long to run the differential.

In the continuous control system (19) both the control parameters *and* the link flows are changing smoothly and simultaneously. We would like to modify this system slightly in order to ensure that as y changes, the traffic state is always moving toward equilibrium. In particular we would like to show that equilibrium with respect to y is preserved as an invariant under the modified dynamics.

7 Worked example of single player dynamics

In this section we propose a dynamic system for a very simple network and show that it preserves as an invariant equilibrium traffic flow with respect to a (changing) control parameter y.

7.1 Notation

We will consider the following network.

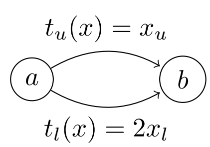

Figure 3: An example network. Each link is labeled with the link cost function.

In this example network there are two nodes, a and b, and two links, u and l. Because of this special structure, the links are also the two paths, so link and path will be used interchangeably. The only origin destination pair is (a, b). The demand from a to b is given by $q = 6$. The volume on link l is given by x_l

and on link u by x_u. The vector of link flow is given by $x = [x_u, x_l]$. The vector-valued link cost function is given by $t(x) = [t_u(x), t_l(x)]$.

The feasibility constraints are given by (20).

$$x_u \geq 0 \tag{20a}$$
$$x_l \geq 0 \tag{20b}$$
$$x_l + x_u = q \tag{20c}$$

The feasible link flows are given by the set $\Omega = \{(x_u, x_l) \mid x_u, x_l \geq 0, x_u + x_l = q\}$.

7.2 Equilibrium as a dynamical system

The dynamical system we are interested in is given by

$$x' = \Pi_\Omega(x, -t(x)) \tag{21}$$

Remark 3 (Expressible in dL)**.** *The system given by* (21) *is expressible in dL extended with* min *and* max *as in KeYmeara X.*

In general we should not expect projection onto a simplex to be expressible in first order arithmetic. However, the feasible region Ω is the straight line connecting $(0, q)$ and $(q, 0)$. The normal to this line is given by $(1, 1)$, so we simply follow this vector until the constraint is met. Explicitly, this distance is given by $\alpha = \frac{q - x_1 - x_2}{2}$. This will ensure that the demand constraint is met, but not necessarily the non-zero constraints. In full, the projected values $\hat{x}_u = P(x_u) = \min(\max(0, x_u + \alpha), q)$ and $\hat{x}_l = P(x_l) = \min(\max(0, x_l + \alpha), q)$.

Remark 4 (Feasibility invariant)**.** *The system given by* (21) *preserves feasibility as an invariant.*

The projection operator implicitly enforces the feasibility of link flows.

7.3 Equilibrium with respect to a control variable

Suppose a traffic controller is able to influence the link cost function, for example, by applying a toll to one or more links. Concretely consider the following network, modified slightly from fig. 4. The equilibrium condition remains the same:

$$0 = \Pi_\Omega(x, -t(y, x)) \tag{22}$$

However, because the link cost now depends on y, link flows that satisfy the condition are called equilibria with respect to y.

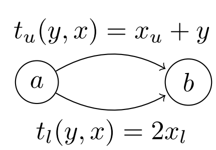

Figure 4: An example network. Each link is labeled with the link cost function.

The link cost functions are now functions of the link flow x and the control variable. We assume that y is constrained to be non-negative.

In the case where y is fixed, the problem reduces to the previous section, with a slightly different link cost function. We are interested in the case where y is changing. Concretely, we would like to define dynamics for x such that equilibrium with respect to the control variable y is an invariant of the dynamics—even as the value of y is continuously changing.

To achieve this, we modify the dynamics slightly from the previous section.

$$x' = \Pi_\Omega\left(x, -\left(t(x) + \gamma h\right)\right) \tag{23a}$$
$$y' = h \tag{23b}$$
$$\&y \geq 0 \tag{23c}$$

Where we take h a positive constant and γ will be specified below.

We can generate an intuition for this system from inspecting the time derivative of the link cost.

$$t' = \frac{\partial t}{\partial x}x' + \frac{\partial t}{\partial y}y' \tag{24}$$

In this system the link cost is changing not only in response to how the link flow is changing but how the control parameter is changing. The additional term in x' adjusts the direction of flow change to cancel the direction in which the link cost is changing as a result of the control. In effect, this ensures that the flow is always changing in the direction of equilibrium which is now moving as y changes and that the invariant remains the equilibrium condition with respect to y: $0 = \Pi_\Omega(x, -t(y, x))$.

Theorem 10 (Equilibrium Invariant)**.** *The equilibrium condition with respect to y given by* (22) *is invariant under the system given by* (23)*.*

Proof. To prove (22) invariant under the dynamics given by (23) it suffices to show that

$$\left(\Pi_\Omega(x, -t(y, x))^T \Pi_\Omega(x, -t(y, x))\right)' \leq 0 \tag{25}$$

This is equivalent to the sum of the square of each component. From remark 3, the projection can be written using first order arithmetic. Here we denote $\min(q, \max(0, \cdot))$ by $|_0^q$.

$$\Pi_\Omega(x, -t(y, x)) = -(\min(q, \max(0, \Phi_l)), \min(q, \max(0, \Phi_u))) \tag{26}$$

where Φ_l, Φ_u are defined,

$$\Phi_l = x_u - {}^1\!/\!_2(x_u - t_u - x_l - t_l) - {}^1\!/\!_2 q \tag{27}$$
$$= {}^1\!/\!_2(t_u - t_l) \tag{28}$$
$$\Phi_u = x_l + {}^1\!/\!_2(x_u - t_u - x_l - t_l) - {}^1\!/\!_2 q \tag{29}$$
$$= -{}^1\!/\!_2(t_u - t_l) \tag{30}$$

To simply we use the fact that $x_l + x_u = q$ is an invariant by construction. Expressing (25) in terms of Φ,

$$((|_0^q \Phi_l)^2 + (|_0^q \Phi_u)^2)' \leq 0 \tag{31}$$

Although the $|_0^q$ operator is not differentiable at 0 or q, it does have well-defined sub- and super-differentials. Define $\mathbb{I}_P(x)$ as an indicator function for term x and predicate P which is 1 whenever $P(x)$ is true and 0 otherwise. Denote the set of sub-differentials of a function f by $D\ f$. In particular,

$$\mathbb{I}_{0 \leq f \leq q} \in D\ |_0^q \tag{32}$$

So it follows (and analogously for Φ_u),

$$-2(|_0^q \Phi_l)\mathbb{I}_{0 \leq \Phi_l \leq q}(\Phi_l)\Phi_l' \in D\ (|_0^q \Phi_l)^2 \tag{33}$$

Because of the presence of the indicator, the only remaining $|_0^q$ may be dropped. This is so because $|_0^q \Phi_l$ evaluates to Φ_l whenever $\Phi_l \in [0, q]$. Outside this interval $|_0^q \Phi_l \neq \Phi_l$. However, the indicator function is 0 everywhere *except* where $\Phi_l \in [0, q]$. Therefore all terms on which the indicator is 0 drop and we are left with only those for which the indicator evaluates to 1. Concretely it now suffices to show,

$$-2\Phi_l\Phi_l' - 2\Phi_u\Phi_u' \leq 0 \tag{34}$$

Before moving forward a few useful simplifications. First, applying the same tricks to x_l', x_u' in (23), the adjustment can be extracted from the expression,

$$x_u' = \Phi_1 - {}^1\!/\!{}_2(\gamma_u - \gamma_l)h \tag{35}$$
$$x_l' = \Phi_2 + {}^1\!/\!{}_2(\gamma_u - \gamma_l)h \tag{36}$$

Additionally, t_u' and t_l' are readily computed,

$$t_u' = x_u' + y' \tag{37}$$
$$t_l' = 2x_l' \tag{38}$$

It follows that,

$$\Phi_l' = {}^1\!/\!{}_2(t_u' - t_l') \tag{39}$$
$$-2\Phi_l\Phi_l' = -2({}^1\!/\!{}_2)(t_u - t_l)({}^1\!/\!{}_2)({}^3\!/\!{}_2(t_u - t_l) - ({}^3\!/\!{}_2)(\gamma_u - \gamma_l)h + h) \tag{40}$$

Taking $\gamma = (\frac{2}{3}, 0)$ cancels the h term yielding,

$$-2\Phi_l\Phi_l' = -({}^3\!/\!{}_4)(t_u - t_l)^2 \leq 0 \tag{41}$$

Additionally, since $\Phi_u = -\Phi_l$,

$$-2\Phi_l\Phi_l' - 2\Phi_u\Phi_u' = -2\Phi_l\Phi_l' - 2(-\Phi_l)(-\Phi_l)' \tag{42}$$
$$= -4\Phi_l\Phi_l' \leq 0 \tag{43}$$
$$\tag{44}$$

The theorem is proved. □

8 Discussion and future work

This project explores the applicability of dL and methods inspired by some of its techniques to the problem of traffic network equilibrium and the network design problem with a single control variable. In addition to a translation of previous work in this area into dL, a new system was analyzed in which the link cost function not only depends on the traffic flow, but also on an external control parameter that may change over time. This control parameter, as an example, might represent a toll or the timing of traffic signals. A new dynamical system was developed which if started in an equilibrium state with respect to the initial value of the control parameter remains in an equilibrium state after running the dynamics with respect to the *current value* of y. One exciting application of this technique is a re-framing of network design problems from an mathematical program with equilibrium constraints (MPEC) to a mathematical program with PDE constraints (as in Biegler et al. (2003) for example).

Despite the work done in this project there is a significant ways to go before any meaningful application. Here we list some limitations and caveats of this work done in this project.

8.1 The general case

The most conspicuous absence is the lack of a general framework for the "adjusted" dynamics presented in section 7. In particular, it is not immediately clear how to select γ or otherwise encode the dependence on y into the dynamics of x. Additionally, it is not clear how this formulation scales with network size, or if it is possible for it to accommodate multiple origin destination pairs on the network.

Most of all, the example leverages the very special case where the projection could be specified in closed form. As a concrete next step it would be valuable to re-work section 7 using the path swap dynamical system which is expressible in differential dynamic logic.

8.2 Application to network design problems

As previously noted, this formulation enables the constraint set of many network design problem to be phrased as the solution to a system of differential equations. From a conceptual perspective there is an intuitive appeal to this formulation. Suppose we can set control parameters in such a way that the equilibrium traffic flow is trivial, for example, by ensuring that all paths but one are prohibitively expensive. By following the adjusted link/path flow dynamics from this equilibrium, all non-trivial equilibria will found as y moves in the parameter space. From the perspective of differential games, we might additionally consider an objective function as a value function which is measured as y traverses the parameter space. Although it is not immediately clear that this is a computational advantage over existing methods, it bears resemblance to sensitivity analysis based solution algorithms as described in Josefsson and Patriksson (2007). This approach may offer a more principled methodology.

8.3 Interpretation of time in the dynamics

As noted previously, the "time" derivatives explored in this paper are not derivatives with respect to real time. Physically a car cannot decide in the middle of its journey to have taken a different route. Rather time in this context represents some abstract decision space on which agents are allowed to move smoothly. The canonical example of static traffic assignment, the morning commute, however, is inherently a discrete process: drivers decide *today* based on what happened *yesterday*. So there it is not immediately clear how to interpret the states through which the dynamics pass on their way from a non-equilibrium state to an equilibrium.

8.4 Dynamic traffic assignment

This project dealt exclusively with static traffic assignment in which the demand and the flows were not time-dependent. Dynamic traffic assignment, as the name suggests, deals with demand that changes over time. As a result, traffic flows are time-dependent. Dynamic traffic assignment is a significantly harder problem but a much more practical one. As a result, extending this formulation to handle dynamic traffic assignment would be quite useful.

Wrapped up in this extension is the representation of more complicated cost functions. In this project we specify the costs analytically. In fact, if we hope to represent them in differential dynamic logic, they will have to be polynomial. Many complex features of traffic costs are naturally represented as differential equations themselves. Dynamic traffic assignment solvers rely on simulation to evaluate the travel cost of a particular path flow. Although hybrid systems are adept at handling differential equations, this formulation uses system evolution time to move through an abstract decision space. As a result it is not immediately obvious how to incorporate more complex traffic phenomena.

8.5 Two-player games

Although the user equilibrium has been extensively formulated in in game theoretic terms (c.f. Jin (2015)), I am not aware of it being phrased in the language of differential games.

Although in this project the control parameter y varies deterministically as the solution to a differential equation, it is also possible to formulate the interaction between the control and the traffic as a game. In particular, consider (23) as a differential game. In this game, y, instead of being determined by differential equation, is controlled by one player, and the adjustment term, now called z, is controlled by the other. The z player wishes to preserve equilibrium with respect to y even as the first player seeks to change it.

Another interesting extension of this formulation is the introduction of a third variable to the link cost function representing an unknown disruption on the network. Such a system could model, for example, a felled tree, a malfunctioning traffic signal, or emergency vehicles. Here we are interested in examining the differential game given by (45).

$$\mathbf{x}' = \Pi_{\Omega}(\mathbf{x}, -\mathbf{t}(\mathbf{y}, \mathbf{z}, \mathbf{x})) \tag{45}$$

This problem is an example of optimal control in the presence of an unknown disturbance, a well-established class of problems that are formulated as differential games (Bardi and Capuzzo-Dolcetta, 2008).

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