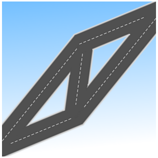
Formal Verification of Traffic Networks at Equilibrium



Matt Battifarano
mbattifa@andrew.cmu.edu

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Traffic equilibrium is the most behaviorally relevant traffic state.

- Traffic equilibrium represents the network state that arises from **selfish routing decisions** (Dafermos, 1980; Smith, 1979).
- Two equivalent statements:
  - Traffic is at **equilibrium** when no individual driver has a lower cost alternative path.
  - Traffic is at **equilibrium** the travel cost of all used paths are equal and less than the travel cost of any unused path.

Traffic equilibrium is a central component of transportation planning.

- Planning questions are **behavioral questions**.
- **How will individuals utilize** a network?
- If the network cost is changed, how will route choice change?
- What traffic control settings are "optimal"?
- Network performance need only be considered under **behaviorally relevant** traffic states.

Dynamical Systems Formulation of Traffic Equilibrium

- Dynamical systems formulations represent a **continuous decision process** over path flows x under path cost $t(x)$.
- Projected Dynamical System (Nagurney and Zhang, 1997).

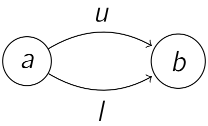
$$x' = \Pi_{\Omega}(x, -t(x))$$

- "Path Swap" Dynamical System (Smith, 1984).

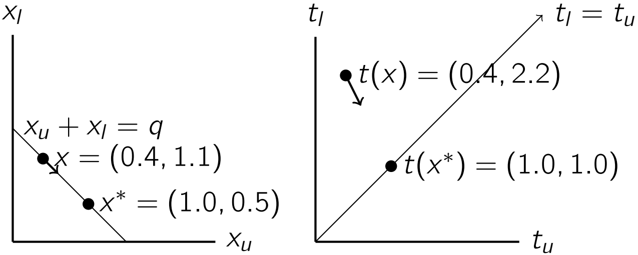
$$x' = \sum_{r,s} x_r(t_r(x) - t_s(x))_+ \Delta_{rs}$$

- In both systems:
  - $x' = 0$ if and only if x is at equilibrium.
  - Non-equilibrium states **converge** to equilibrium states.

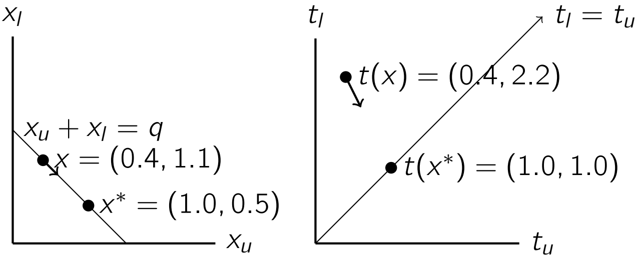
Visualizing Equilibrium Dynamics



- feasibility constraints: $x_u, x_l \geq 0$, $x_u + x_l = q = 1.5$
- path cost function: $t(x) = (t_u(x), t_l(x)) = (x_u, 2x_l)$



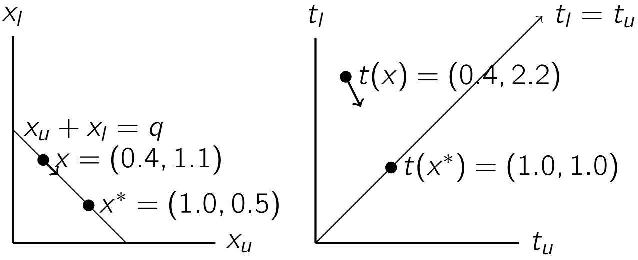
Visualizing Equilibrium Dynamics



What do the path swap dynamics look like?

$$x_u' = 1.98 = -x_u(t_u(x) - t_l(s))_+ + x_l(t_l(x) - t_u(x))_+$$
$$x_l' = -1.98 = -x_l(t_l(x) - t_u(s))_+ + x_u(t_u(x) - t_l(x))_+$$
$$t_u' = 1.98 = \frac{\partial t_u}{\partial x_l} x_l' + \frac{\partial t_u}{\partial x_u} x_u'$$
$$t_l' = -2 \cdot 1.98 = \frac{\partial t_l}{\partial x_l} x_l' + \frac{\partial t_l}{\partial x_u} x_u'$$

Visualizing Equilibrium Dynamics



What do the projected dynamics look like?

$$x'_u = 0.9 = \Pi_\Omega(x, -t(x))_u$$

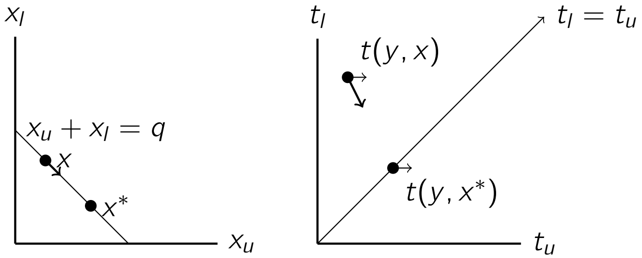
$$x'_l = -0.9 = \Pi_\Omega(x, -t(x))_l$$

$$t'_u = 0.9 = \frac{\partial t_u}{\partial x_l} x'_l + \frac{\partial t_u}{\partial x_u} x'_u$$

$$t'_l = -2 \cdot 0.9 = \frac{\partial t_l}{\partial x_l} x'_l + \frac{\partial t_l}{\partial x_u} x'_u$$

Modeling a traffic controller

- A traffic controller is considering applying a toll of the amount y on link u.
- Need to consider a **new cost function** $t(y, x) = (x_u + y, 2x_l)$.
- The traffic controller wants to know: **What happens to equilibrium as y increases from 0?**
- The path flow dynamics as formulated is **not sufficient**.



Adjusting the Path Flow Dynamics

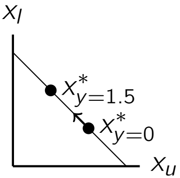
Want: $\Pi_\Omega(x, -t(y, x)) = 0$ invariant as before, but subject to the new dynamics.
Idea: adjust the path dynamics to compensate for y'.

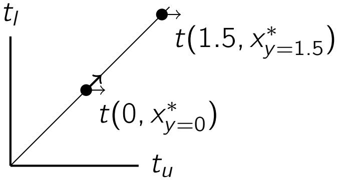
$$x' = \Pi_\Omega(x, -t(y, x) + \gamma h)$$
$$y' = h$$

What is γ?

Adjusting the Path Flow Dynamics

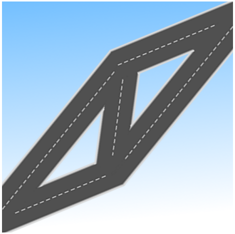
$$x' = \Pi_\Omega(x, -t(y,x) - [2/3, 0])$$
$$y' = 1$$
$$x^*_{y=0} = (1.0, 0.5) \rightarrow x^*_{y=1.5} = (0.5, 1.0)$$





$$x'_u = -1/3 = \Pi_\Omega(x, -t(x) - \gamma)_u$$
$$x'_l = 1/3 = \Pi_\Omega(x, -t(x) - \gamma)_l$$
$$t'_u = 2/3 = x'_u \partial t_u / x_u + x'_l \partial t_u / \partial x_l + y' \partial t_u / \partial y$$
$$t'_l = 2/3 = x'_u \partial t_l / x_u + x'_l \partial t_l / \partial x_l + y' \partial t_l / \partial y$$

Thank you!



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