Focusing for $\mathsf{d}\mathcal{L}$
15-824, Fall 2018

Klaas Pruiksma
(kpruiksm)

1 Introduction

Focusing, as first presented by Andreoli [1], is an attempt to separate out the essential parts of a proof. A focused proof in Andreoli's system is a canonical representative of a class of ordinary non-focused proofs which are equivalent in that they only differ in "non-essential" ways. The goal of a focused system for a given logic is to be sound and complete with respect to the original logic, but to restrict the set of possible proofs sufficiently that the process of proof search is much more limited. We set out to develop a focused version of differential dynamic logic, or $\mathsf{d}\mathcal{L}$ [5]. It is not immediately clear how much practical use this would have, but it does open up more possibilities in the space of proof search.

2 Related Work

While there is a decently large body of work on focusing [1, 2, 3, 4, 7], little work exists discussing focusing in the context of modal logics. The only information we were able to find is in an off-hand comment by Reed [6, Section 3]. However, this does not generalize well to the situation of $\mathsf{d}\mathcal{L}$, whose modalities behave rather differently than those described by Reed [6] — in particular, the fact that there exist multiple rules which can act to decompose a modal formula of the form $[\alpha]\phi$ or $\langle\alpha\rangle\phi$, rather than each modality $[\alpha]$ or $\langle\alpha\rangle$ being determined by introduction/elimination or left/right rules, makes $\mathsf{d}\mathcal{L}$a significantly different setting.

3 Contributions

Our contributions are somewhat lacking, due to the incomplete nature of this work. We present a system which behaves in many respects like one would expect a focused system for $\mathsf{d}\mathcal{L}$to behave (Sections 4 and 5). However, we only prove the soundness of this system with respect to $\mathsf{d}\mathcal{L}$(Section 6). It appears, however, that completeness (or focalization) does not hold for this system, and so instead we present the path that we planned to take to showing focalization, as well as the issues that arose along the way to doing so (Section 7). We believe that these issues are nonessential, but have not as yet succeeded in reworking our focused system to avoid them.

For the purposes of this discussion, we will restrict our attention to the fragment of $\mathsf{d}\mathcal{L}$with $\neg$ and $\wedge$ as its only propositional connectives, as $\vee$, $\rightarrow$, and $\leftrightarrow$ are all derivable from these.

4 Polarized $\mathsf{d}\mathcal{L}$

As is standard in approaches to focusing [1, 4, 7], we begin by discussing how to *polarize* $\mathsf{d}\mathcal{L}$— that is, we separate the connectives into *synchronous* and *asynchronous*, based on their behaviour during proof search. While Platzer [5] presents $\mathsf{d}\mathcal{L}$using a two-sided sequent calculus, the classical nature of $\mathsf{d}\mathcal{L}$means that the sequent calculus is symmetric, and so rather than referring to connectives as right- or left-synchronous, we will simply say synchronous to mean right-synchronous/left-asynchronous and vice versa. Per Andreoli [1], asynchronous connectives are those which can be eagerly decomposed (on the right) without affecting provability, while synchronous connectives cannot be eagerly decomposed, generally because their rules make some form of choice that we may not have enough information to make at the time. In the classical setting, this is not inherent to the (propositional) connectives, and so, following Liang and Miller [4], we instead split $\wedge$ and $\neg$ into two connectives each — synchronous $\wedge^+$ and $\neg^+$ and asynchronous $\wedge^-$ and $\neg^-$. The quantifiers $\exists$ and $\forall$ are synchronous and asynchronous, respectively.

It now only remains to consider the modalities $[\alpha]$ and $\langle\alpha\rangle$, for which it is less obvious how to proceed. Examining the G rules (rule schemata) (G1)-(G4) [5] (Figure 1) suggests that both $[\alpha]$ and $\langle\alpha\rangle$ should be synchronous connectives. This is more apparent when we rewrite (G1)-(G4) as the equivalent (G1')-(G4'), which make it clear that if these connectives occur on the right-hand side of a sequent, we have multiple choices of rule to apply — there is always some applicable D rule which breaks down the hybrid program α, and either (G1') or (G2') is always applicable. Moreover, applying (G1') or (G2') involves making a choice of the formula ϕ that we are weakening our postcondition to, and this choice cannot be made eagerly without affecting provability. As a result of this, we take both $[\alpha]$ and $\langle\alpha\rangle$ to be synchronous

For technical reasons, we will also introduce two new connectives: $\uparrow$ and $\downarrow$, which serve to make explicit the transition from synchronous to asynchronous. We pronounce these as *up* and *down*, respectively, and refer to them together as *shifts*. In keeping with prior work [7], $\uparrow$ is an asynchronous connective and $\downarrow$ a synchronous connective. With this addition, we can explicitly write down a grammar for synchronous and asynchronous propositions, as shown in fig. 2. The existence of these shifts will simplify the rules of focused $\mathsf{d}\mathcal{L}$, at the (minor) expense of making the statements of the focalization and defocalization theorems slightly more involved.

Finally, we allow for both asynchronous and synchronous atoms n^- and p^+ to be used in polarized $\mathsf{d}\mathcal{L}$.

$$\frac{\vdash \forall^\alpha(\phi \to \psi)}{[\alpha]\phi \vdash [\alpha]\psi}\ (\text{G1}) \qquad \frac{\vdash [\alpha]\phi \quad \vdash \forall^\alpha(\phi \to \psi)}{\vdash [\alpha]\psi}\ (\text{G1}')$$

$$\frac{\vdash \forall^\alpha(\phi \to \psi)}{\langle\alpha\rangle\phi \vdash \langle\alpha\rangle\psi}\ (\text{G2}) \qquad \frac{\vdash \langle\alpha\rangle\phi \quad \vdash \forall^\alpha(\phi \to \psi)}{\vdash \langle\alpha\rangle\psi}\ (\text{G2}')$$

$$\frac{\vdash \forall^\alpha(\phi \to [\alpha]\phi)}{\phi \vdash [\alpha^*]\phi}\ (\text{G3}) \qquad \frac{\vdash \phi \quad \vdash \forall^\alpha(\phi \to [\alpha]\phi)}{\vdash [\alpha^*]\phi}\ (\text{G3}')$$

$$\frac{\vdash \forall^\alpha \forall v > 0(\varphi(v) \to \langle\alpha\rangle\phi(v-1))}{\exists v\varphi(v) \vdash \langle\alpha^*\rangle\exists v \leq 0\varphi(v)}\ (\text{G4}) \qquad \frac{\vdash \exists v\varphi(v) \quad \vdash \forall^\alpha \forall v > 0(\varphi(v) \to \langle\alpha\rangle\phi(v-1))}{\vdash \langle\alpha^*\rangle\exists v \leq 0\varphi(v)}\ (\text{G4}')$$

Figure 1: Global (G) rules of $\mathsf{d}\mathcal{L}$, in two forms

Asynchronous propositions A^-, B^-	$::=$	$n^- \mid A^- \wedge^- B^- \mid \neg^- A^+ \mid \forall x.A^- \mid\uparrow A^+$
Synchronous propositions A^+, B^+	$::=$	$p^+ \mid A^+ \wedge^+ B^+ \mid \neg^+ A^- \mid \exists x.A^+ \mid\downarrow A^- \mid [\alpha]A^+ \mid \langle\alpha\rangle A^+$

Figure 2: A grammar of synchronous and asynchronous propositions in polarized $\mathsf{d}\mathcal{L}$

4.1 Depolarization

In order to relate formulae of polarized $\mathsf{d}\mathcal{L}$to formulae of $\mathsf{d}\mathcal{L}$, we define an operation of depolarization or erasure which turns a polarized formula into an unpolarized formula. This operation, which we write $(\cdot)^\bullet$, has a straighforward inductive definition, shown in fig. 3. Intuitively, $(A^\pm)^\bullet$ is the result of removing all shifts and all superscripts $^+$ and $^-$ from the formula $A^\pm$.

$$\begin{array}{rlrl}
(n^-)^\bullet = & n & (p^+)^\bullet = & p \\
(A^- \wedge^- B^-)^\bullet = & (A^-)^\bullet \wedge (B^-)^\bullet & (A^+ \wedge^+ B^+)^\bullet = & (A^+)^\bullet \wedge (B^+)^\bullet \\
(\neg^- A^+)^\bullet = & \neg(A^+)^\bullet & (\neg^+ A^-)^\bullet = & \neg(A^-)^\bullet \\
(\forall x.A^-)^\bullet = & \forall x.(A^-)^\bullet & (\exists x.A^+)^\bullet = & \exists x.(A^+)^\bullet \\
(\uparrow A^+)^\bullet = & (A^+)^\bullet & (\downarrow A^-)^\bullet = & (A^-)^\bullet \\
([\alpha]A^+)^\bullet = & [\alpha](A^+)^\bullet & (\langle\alpha\rangle A^-)^\bullet = & \langle\alpha\rangle(A^-)^\bullet
\end{array}$$

Figure 3: The depolarization operation $(\cdot)^\bullet$

5 A focused sequent calculus for d$\mathcal{L}$

Having polarized d$\mathcal{L}$, we are now equipped to present our focused sequent calculus. We will use the notation $[\![\phi]\!]$ for formulae in focus, in order to avoid confusion with the box modality $[\alpha]$.

In keeping with the sequent calculus presented by Platzer [5], we will use a two-sided sequent calculus, in which sequents will have the form $\Gamma^-; L \vdash \Delta^+; R$. Here, Γ^- and Δ^+ are sets of asynchronous and synchronous formulae, respectively. L may be either Γ^+ — a (possibly empty) set of synchronous formulae — or $[\![N]\!]$ — a negative formula in focus. Dually, R may be either a (possibly empty) set of asynchronous formulae Δ^- or a synchronous formula $[\![P]\!]$ in focus. We will also require (as is standard for two-sided focusing systems [7]) that if one of L and R is a focused proposition, then the other is empty. We will frequently omit empty L and R from sequents for simplicity.

Our sequents can then be split into four different cases, depending on what forms L and R have:

$$\begin{aligned} \text{Neutral:}\quad & \Gamma^- \vdash \Delta^+ \\ \text{Inversion:}\quad & \Gamma^-; \Gamma^+ \vdash \Delta^+; \Delta^- \\ \text{Left focus:}\quad & \Gamma^-; [\![N]\!] \vdash \Delta^+ \\ \text{Right focus:}\quad & \Gamma^- \vdash \Delta^+; [\![P]\!] \end{aligned}$$

Note that neutral sequents can be seen as the special case of inversion sequents in which the set of formulae in inversion (Γ^+ and Δ^-) is empty. However, they merit special attention and a separate name because of their importance to focusing — we are only allowed to focus on a formula in a neutral sequent, in order to ensure that at most one of L and R is in focus, as discussed above. A neutral sequent should therefore be interpreted as a sequent where any next step we take in a proof is a decision of what goal to pursue. Inversion sequents can be thought of as the phase of proof search where no decisions need to be made, and we only simplify existing goals in ways that do not affect provability. Finally, focus sequents can be thought of as the phase of proof search where essential choices are made.

We introduce one final technical device before presenting the sequent calculus. Following Simmons [7], we will find it useful to discuss *suspended propositions*, which we will write as $\langle\!|A|\!\rangle$. A suspended formula $\langle\!|\phi^+|\!\rangle$ should be thought of as a synchronous formula which we do not have the information to break down any further at the moment, and similarly for asynchronous $\langle\!|\phi^-|\!\rangle$. We therefore allow suspended synchronous propositions to live in the same context as asynchronous propositions and vice versa, so Γ^- and Δ^+ are no longer strictly asynchronoous and synchronous, but rather contain formulae which cannot be broken down immediately. In practice, the only suspended formulae we will encounter in real proofs will be atomic propositions (as we have the information to break down non-atomic propositions), but it is useful for the proof theory to allow for arbitrary suspended propositions, as this allows us to prove identity expansion for the focused system.

With this in mind, we present the sequent calculus for focused d$\mathcal{L}$in fig. 4 and **??**. We omit the D rules of [5] for brevity, but they may be included by the following schemata:

(1)

$$\text{If } \frac{\psi^+}{\phi^+}\ (Dn) \text{ is a D rule, then we may form the rule } \frac{\Gamma^- \vdash \Delta^+; [\![\psi^+]\!]}{\Gamma^- \vdash \Delta^+; [\![\phi^+]\!]}\ (Dn)R$$

(2)

$$\text{If } \frac{\psi^+}{\phi^+}\ (Dn) \text{ is a D rule, then we may form the rule } \frac{\Gamma^-; \Gamma^+, \psi^+ \vdash \Delta^+; \Delta^-}{\Gamma^-; \Gamma^+, \phi^+ \vdash \Delta^+; \Delta^-}\ (Dn)L$$

Note that we make explicit in the rules (F3) and (F6) that we only apply QE at neutral sequents. It is unclear whether this is strictly necessary, but it does not appear to cause problems thus far.

6 Defocalization

In order to state the defocalization and focalization theorems, we extend erasure $(\cdot)^\bullet$ such that $([\![\phi^\pm]\!])^\bullet = (\phi^\pm)^\bullet$ and $(\langle\!\langle\phi^\pm\rangle\!\rangle)^\bullet = (\phi^\pm)^\bullet$. We then write $(\Gamma)^\bullet$ for the natural (pointwise) lifting of erasure to contexts.

Theorem 1 (Defocalization)**.** *If* $\Gamma^-; L \vdash \Delta^+; R$ *is derivable, then* $(\Gamma^-)^\bullet \vdash (\Delta^+)^\bullet$ *is derivable.*

Proof. We prove this statement by induction over the derivation of $\Gamma^-; L \vdash \Delta^+; R$.

In the cases of the $\mathsf{foc}^\pm$ and $\mathsf{susp}^\pm$ rules, as well as the rules for $\uparrow$ and $\downarrow$, the premiss of the rule is the same as the conclusion, once erased, and so by applying the induction hypothesis, we are immediately done.

In the cases of the $\mathsf{id}^\pm$ rules, erasing the conclusion of the rule yields a sequent which is immediately provable using the identity rule (P9) of $\mathsf{d}\mathcal{L}$.

In the cases of synchronous rules and asynchronous right rules, we apply the inductive hypothesis to the premisses of the last rule used in the derivation, and then apply the corresponding rule of $\mathsf{d}\mathcal{L}$, taking advantage of the fact that each of these rules erases exactly to a rule of $\mathsf{d}\mathcal{L}$.

A similar approach works for (G1')-(G4'), though technically each of these rules erases to a derived rule of $\mathsf{d}\mathcal{L}$formed by combining a cut (P10) with one of (G1)-(G4).

Each of the asynchronous left rules (other than those for $\wedge^-$) erases precisely to a rule of $\mathsf{d}\mathcal{L}$. The left rules for $\wedge^-$ erase to derived rules of $\mathsf{d}\mathcal{L}$formed by applying weakening to the left rule for $\wedge$, (P6).

Finally, the F rules (F3) and (F6) erase precisely to the rules (F3) and (F6) of $\mathsf{d}\mathcal{L}$, and so we are done. □

7 Focalization

We now proceed with proving focalization, following the methods of [7] and deriving focalization from cut admissibility and identity expansion for the focused logic. As such, we now need to

Non-connective rules

$$\frac{}{\Gamma^-, \llparenthesis \phi^+ \rrparenthesis \vdash \Delta^+; \llbracket \phi^+ \rrbracket}\ \mathsf{id}^+ \qquad \frac{}{\Gamma^-; \llbracket \phi^- \rrbracket \vdash \Delta^+, \llparenthesis \phi^- \rrparenthesis}\ \mathsf{id}^-$$

$$\frac{\Gamma^- \vdash \Delta^+; \llbracket \phi^+ \rrbracket}{\Gamma^- \vdash \Delta^+, \phi^+}\ \mathsf{foc}^+ \qquad \frac{\Gamma^-; \llbracket \phi^- \rrbracket \vdash \Delta^+}{\Gamma^-, \phi^- \vdash \Delta^+}\ \mathsf{foc}^-$$

$$\frac{\Gamma^-, \llparenthesis p^+ \rrparenthesis; \Gamma^+ \vdash \Delta^+; \Delta^-}{\Gamma^-; \Gamma^+, p^+ \vdash \Delta^+; \Delta^-}\ \mathsf{susp}^+ \qquad \frac{\Gamma^-; \Gamma^+ \vdash \Delta^+, \llparenthesis n^- \rrparenthesis; \Delta^-}{\Gamma^-; \Gamma^+ \vdash \Delta^+; \Delta^-, n^-}\ \mathsf{susp}^-$$

Asynchronous connectives

$$\frac{\Gamma^-; \Gamma^+ \vdash \Delta^+; \Delta^-, \phi^- \quad \Gamma^-; \Gamma^+ \vdash \Delta^+; \Delta^-, \psi^-}{\Gamma^-; \Gamma^+ \vdash \Delta^+; \Delta^-, \phi^- \wedge^- \psi^-}\ \wedge^- R$$

$$\frac{\Gamma^-; \llbracket \phi^- \rrbracket \vdash \Delta^+}{\Gamma^-; \llbracket \phi^- \wedge^- \psi^- \rrbracket \vdash \Delta^+}\ \wedge^- L1 \qquad \frac{\Gamma^-; \llbracket \psi^- \rrbracket \vdash \Delta^+}{\Gamma^-; \llbracket \phi^- \wedge^- \psi^- \rrbracket \vdash \Delta^+}\ \wedge^- L2$$

$$\frac{\Gamma^-; \Gamma^+, \phi^+ \vdash \Delta^+; \Delta^-}{\Gamma^-; \Gamma^+ \vdash \Delta^+; \Delta^-, \neg^- \phi^+}\ \neg^- R \qquad \frac{\Gamma^- \vdash \Delta^+; \llbracket \phi^+ \rrbracket}{\Gamma^-; \llbracket \neg^- \phi^+ \rrbracket \vdash \Delta^+}\ \neg^- L$$

$$\frac{\Gamma^-; \Gamma^+ \vdash \Delta^+; \Delta^-, \phi^-(s(X_1, \ldots, X_n))}{\Gamma^-; \Gamma^+ \vdash \Delta^+; \Delta^-, \forall x.\phi^-(x)}\ \forall R \qquad \frac{\Gamma^-; \llbracket \phi^-(X) \rrbracket \vdash \Delta^+}{\Gamma^-; \llbracket \forall x.\phi^-(x) \rrbracket \vdash \Delta^+}\ \forall L$$

$$\frac{\Gamma^-; \Gamma^+ \vdash \Delta^+, \phi^+; \Delta^-}{\Gamma^-; \Gamma^+ \vdash \Delta^+; \Delta^-, \uparrow \phi^+}\ \uparrow R \qquad \frac{\Gamma^-; \phi^+ \vdash \Delta^+; \cdot}{\Gamma^-; \llbracket \uparrow \phi^+ \rrbracket \vdash \Delta^+}\ \uparrow L$$

Synchronous connectives

$$\frac{\Gamma^- \vdash \Delta^+; \llbracket \phi^+ \rrbracket \quad \Gamma^- \vdash \Delta^+; \llbracket \psi^+ \rrbracket}{\Gamma^- \vdash \Delta^+; \llbracket \phi^+ \wedge^+ \psi^+ \rrbracket}\ \wedge^+ R \qquad \frac{\Gamma^-; \Gamma^+, \phi^+, \psi^+ \vdash \Delta^+; \Delta^-}{\Gamma^-; \Gamma^+, \phi^+ \wedge^+ \psi^+ \vdash \Delta^+; \Delta^-}\ \wedge^+ L$$

$$\frac{\Gamma^-; \llbracket \phi^- \rrbracket \vdash \Delta^+}{\Gamma^- \vdash \Delta^+; \llbracket \neg^+ \phi^- \rrbracket}\ \neg^+ R \qquad \frac{\Gamma^-; \Gamma^+ \vdash \Delta^+; \Delta^-, \phi^-}{\Gamma^-; \Gamma^+, \neg^+ \phi^- \vdash \Delta^+; \Delta^-}\ \neg^+ L$$

$$\frac{\Gamma^- \vdash \Delta^+; \llbracket \phi^+(X) \rrbracket}{\Gamma^- \vdash \Delta^+; \llbracket \exists x.\phi^+(x) \rrbracket}\ \exists R \qquad \frac{\Gamma^-; \Gamma^+, \phi^+(s(X_1, \ldots, X_n)) \vdash \Delta^+; \Delta^-}{\Gamma^-; \Gamma^+, \exists x.\phi^+(x) \vdash \Delta^+; \Delta^-}\ \exists L$$

$$\frac{\Gamma^-; \cdot \vdash \Delta^+; \phi^-}{\Gamma^- \vdash \Delta^+; \llbracket \downarrow \phi^- \rrbracket}\ \downarrow R \qquad \frac{\Gamma^-, \phi^-; \Gamma^+ \vdash \Delta^+; \Delta^-}{\Gamma^-; \Gamma^+, \downarrow \phi^- \vdash \Delta^+; \Delta^-}\ \downarrow L$$

Figure 4: Focused $\mathsf{d}\mathcal{L}$(part 1)

Other rules

$$\frac{\vdash \text{QE}(\forall X(\Phi(X) \vdash \Psi(X)))}{\Phi(s(X_1,\ldots,X_n)); \cdot \vdash \Psi(s(X_1,\ldots,X_n)); \cdot} \ (F3)$$

$$\frac{\vdash \text{QE}(\exists X \bigwedge_i \Phi_i \vdash \Psi_i)}{\Phi_1; \cdot \vdash \Psi_1; \cdot, \ldots, \Phi_n; \cdot \vdash \Psi_n; \cdot} \ (F6)$$

$$\frac{\Gamma^- \vdash \Delta^+; [\![[\alpha]\phi^+]\!] \quad \Gamma^- \vdash \Delta^+; \forall^\alpha \uparrow (\phi^+ \to \psi^+)}{\Gamma^- \vdash \Delta^+; [\![[\alpha]\psi^+]\!]} \ (G1')$$

$$\frac{\Gamma^- \vdash \Delta^+; [\![\langle\alpha\rangle\phi^+]\!] \quad \Gamma^- \vdash \Delta^+; \forall^\alpha \uparrow (\phi^+ \to \psi^+)}{\Gamma^- \vdash \Delta^+; [\![\langle\alpha\rangle\psi^+]\!]} \ (G2')$$

$$\frac{\Gamma^- \vdash \Delta^+; [\![\phi^+]\!] \quad \Gamma^- \vdash \Delta^+; \forall^\alpha \uparrow (\phi^+ \to [\alpha]\phi^+)}{\Gamma^- \vdash \Delta^+; [\![[\alpha^*]\phi^+]\!]} \ (G3')$$

$$\frac{\Gamma^- \vdash \Delta^+; [\![\exists v \varphi(v)]\!] \quad \Gamma^- \vdash \Delta^+; \forall^\alpha \forall v > 0 \uparrow (\varphi^+(v) \to \langle\alpha\rangle\varphi^+(v-1))}{\Gamma^- \vdash \Delta^+; [\![\langle\alpha^*\rangle \exists v \le 0 \varphi^+(v)]\!]} \ (G4')$$

Figure 5: Focused d$\mathcal{L}$(part 2)

introduce the concept of *suspension-normality* [7], which is used in giving a statement of cut admissibility.. A sequent $\Gamma^-; L \vdash \Delta^+; R$ is said to be *suspension-normal* if the only suspended formulae which occur in Γ^- and Δ^+ are atomic propositions.

Unfortunately, we have not yet successfully reached a full cut admissibility result for this focused system (and as a result, we fail to show focalization). The obstacle to cut admissibility comes from the G rules (G1')-(G4'). Principal cuts involving a G rule on the left, with the principal formula of the cut being the formula in focus in that G rule, are difficult to see how to eliminate, as even if we know that the last rule used in the derivation on the right-hand side of the cut is a D rule decomposing the principal formula of the cut, the premiss of that D rule has little to do with the premisses of the G rule, and so it is unclear how to proceed. Despite the lack of cut admissibility, we do, however, have what Simmons [7] refers to as *focal substitution*, which specifies admissibility of a pair of cut-like rules.

Theorem 2 (Focal substitution)**.** *The following two rules are admissible:*

$$\frac{\Gamma^- \vdash \Delta^+; [\![\phi^+]\!] \quad \Gamma^-, \langle\!|\phi^+|\!\rangle; L \vdash \Delta^+; R}{\Gamma^-; L \vdash \Delta^+; R} \ \mathsf{subst}^+ \qquad \frac{\Gamma^-; L \vdash \Delta^+, \langle\!|\phi^-|\!\rangle; R \quad \Gamma^-; [\![\phi^-]\!] \vdash \Delta^+}{\Gamma^-; L \vdash \Delta^+; R} \ \mathsf{subst}^-$$

Proof. This theorem follows immediately from induction over the derivation of $\Gamma^-, \langle\!|\phi^+|\!\rangle; L \vdash \Delta^+; R$ (for the first substitution rule) or the derivation of $\Gamma^-; L \vdash \Delta^+, \langle\!|\phi^-|\!\rangle; R$ (for the second substitution rule).

In base cases, where the last rule used is an id rule, we simply return the other derivation (the one that we are not inducting over). In all other cases, we simply apply the inductive hypothesis to the premiss(es) of the last rule R to be used, which cannot break down $\langle\!|\phi^{\pm}|\!\rangle$ due to the suspension. We may then apply R to the result of these applications of the inductive hypothesis to get the desired result. $\square$

It appears that cut admissibility (1) holds for the fragment of focused d$\mathcal{L}$without the G rules, but we do not as yet have a completed proof.

Conjecture 1 (Cut admissibility)**.** *Suppose that all sequents under consideration in the following are suspension-normal. Then, we have all of the following:*

(1) If $\Gamma^- \vdash \Delta^+; [\![\phi^+]\!]$ *and* $\Gamma^-; \Gamma^+, \phi^+ \vdash \Delta^+; \Delta^-$*, then* $\Gamma^-; \Gamma^+ \vdash \Delta^+; \Delta^-$*.*

(2) If $\Gamma^-; \Gamma^+ \vdash \Delta^+; \Delta^-, \phi^-$ *and* $\Gamma^-; [\![\phi^-]\!] \vdash \Delta^+$*, then* $\Gamma^-; \Gamma^+ \vdash \Delta^+; \Delta^-$*.*

(3) If $\Gamma^- \vdash \Delta^+; \Delta^-, \phi^-$ *and* $\Gamma^-, \phi^-; \Gamma^+ \vdash \Delta^+; \Delta^-$*, then* $\Gamma^-; \Gamma^+ \vdash \Delta^+; \Delta^-$*.*

(4) If $\Gamma^- \vdash \Delta^+; \phi^-$ *and* $\Gamma^-, \phi^-; L \vdash \Delta^+; R$*, then* $\Gamma^-; L \vdash \Delta^+; R$*.*

(5) If $\Gamma^-; \Gamma^+ \vdash \Delta^+, \phi^+; \Delta^-$ *and* $\Gamma^-; \phi^+ \vdash \Delta^+; \Delta^-$*, then* $\Gamma^-; \Gamma^+ \vdash \Delta^+; \Delta^-$*.*

(6) If $\Gamma^-; L \vdash \Delta^+, \phi^+; R$ *and* $\Gamma^-; \phi^+ \vdash \Delta^+$*, then* $\Gamma^-; L \vdash \Delta^+; R$*.*

Informally, (1) and (2) specify that principal cuts (i.e. those where the principal formula of the cut is being decomposed on either side of the cut) are admissible (Technically, to enforce this, we need to order the inversion contexts Γ^+ *and* Δ^- *to force a particular formula to be inverted at a given time. Without this, (1) and (2) include some commutative cuts in addition to the principal cuts)*

(3) and (4) specify admissibility of right-commutative cuts — that is, those where the right-hand derivation of the cut decomposes a formula other than the principal formula of the cut.

(5) and (6) specify admissibility of left-commutative cuts, where the left-hand derivation of the cut decomposes a formula other than the principal formula of the cut.

Proof sketch. This proof proceeds by induction over the tuple $(\phi^{\pm}, i, \mathcal{D}, \mathcal{E})$, ordered lexicographically, where $\phi^{\pm}$ is the principal formula of the cut, i is the number of the case of the theorem statement that we are in, $\mathcal{D}$ is the left-hand derivation of the cut, and $\mathcal{E}$ is the right-hand derivation of the cut. In each principal case, we push the cut up along both sides of the proof, replacing it with a cut at a smaller formula (or, in the case where the principal formula is atomic, using the fact that one of the derivations of the cut must end in an identity rule, removing the cut altogether).

In each commutative case, left- and right-, we show that we can push the cut upwards through any rule which does not decompose the principal formula of the cut.

This is essentially standard, although there are many cases, due to the large number of D rules. $\square$

As in [7], we work to prove identity expansion in its more standard form by first proving admissibility of generalized susp rules, which can then be combined with the id rules that we have to get the desired result (that $\Gamma^-; \phi^+ \vdash \Delta^+, \phi^+$ and $\Gamma^-, \phi^- \vdash \Delta^+; \phi^-$). This result, much like cut elimination, breaks down as a result of repetition. Here, however, rather than succeeding if we restrict our set of rules, we get a result by restricting which formulae we consider.

Theorem 3 (Admissibility of susp)**.** *For any $\Gamma^-, \Gamma^+, \Delta^+, \Delta^-, \phi^+, \phi^-$ (with $\phi\pm$ not containing any repetitions $[\alpha^*]$ or $\langle\alpha^*\rangle$), the following hold:*

(1) If $\Gamma^-, \langle\!|\phi^+|\!\rangle; \Gamma^+ \vdash \Delta^+; \Delta^-$, then $\Gamma^-; \Gamma^+, \phi^+ \vdash \Delta^+; \Delta^-$.

(2) If $\Gamma^-; \Gamma^+ \vdash \Delta^+, \langle\!|\phi^-|\!\rangle; \Delta^-$, then $\Gamma^-; \Gamma^+ \vdash \Delta^+; \Delta^-, \phi^-$.

Proof. This proof proceeds by induction over the structure of $\phi^\pm$.

For atomic ϕ, the result is immediate from the susp rules.

For first-order ϕ, we first construct a proof either of $[\![\phi^+]\!]$ or using $[\![\phi^-]\!]$ from suspended copies of the subformulae of ϕ using the id rules and a rule for the top-level connective of ϕ (a right rule if ϕ is synchronous, and a left rule if ϕ is asynchronous). This proof may then be combined with the given proof (with suitable weakenings) via focal substitution to get a proof of the desired result using suspended copies of the subformulae of ϕ in place of ϕ. We then apply the inductive hypothesis to each of these to remove the suspension, and apply the not-yet-used rule for the top-level connective of ϕ (a left rule if ϕ is synchronous, and a right rule if ϕ is asynchronous).

As an example, consider the case of $\wedge^-$. In this case, $\phi^- = \psi^- \wedge^- \varphi^-$, and we construct the following proof as a first step:

$$\mathcal{D}_1 = \cfrac{\cfrac{}{\Gamma^-; [\![\psi-]\!] \vdash \Delta^+, \langle\!|\psi^-|\!\rangle}\ \mathsf{id}^-}{\Gamma^-; [\![\psi - \wedge^- \varphi^-]\!] \vdash \Delta^+, \langle\!|\psi^-|\!\rangle}\ \wedge^- L1$$

We then apply focal substitution (plus some weakening) to our given proof that $\Gamma^-; \Gamma^+ \vdash \Delta^+, \langle\!|\psi^- \wedge^- \varphi^-|\!\rangle; \Delta^-$ and $\mathcal{D}_1$ to get a proof $\mathcal{E}_1$ that $\Gamma^-; \Gamma^+ \vdash \Delta^+, \langle\!|\psi^-|\!\rangle; \Delta^-$. Now, applying the inductive hypothesis twice, we get a proof $\mathcal{F}_1$ that $\Gamma^-; \Gamma^+ \vdash \Delta^+; \psi^-$. Similarly, we construct $\mathcal{F}_2$ a proof that $\Gamma^-; \Gamma^+ \vdash \Delta^+; \varphi^-$. Now, applying the $\wedge^- R$ rule to $\mathcal{F}_1$ and $\mathcal{F}_2$ yields the desired result.

It remains only to consider the cases of modal formulae $[\alpha]\phi^+$ and $\langle\alpha\rangle\phi^+$. When α is not a repetition, much the same process can be used, breaking down the formula using the suitable D rule.

For $\alpha = \beta^*$, however, this approach is insufficient — the rules (D5) and (D6) for repetition do not break down the principal formula of the rule into a smaller formula by any (reasonable) metric, and nor do the G rules. It is for this reason that we omit repetitions from this theorem. □

As a result of the shortcomings of the cut admissibility and identity expansion theorems, we are unable to develop a focalization result. The particularities of the problems with both cut admissibility and identity expansion suggest that a focalization theorem is likely to hold if we restrict both our focused system and $\mathsf{d}\mathcal{L}$to the repetition-free fragment, as this removes repetitions (allowing identity expansion to hold) and the rules (G3) and (G4). However, it is not fully obvious how to handle the rules (G1) and (G2) in cut admissibility, and this is likely to be the largest obstacle to such a result — given cut admissibility and identity expansion, the focalization theorem should follow fairly quickly.

8 Conclusion and Future Work

While this work did not yield as successful results as desired, we believe that the non-results highlight a few interesting (if not necessarily surprising) points. In particular, while it is entirely unsurprising that focalization is more difficult to achieve than defocalization (as it is very easy to restrict the set of allowable proofs in a sound way, but much harder to do so in a meaningfully complete way), and also relatively unsurprising that iteration is the obstacle to focalization, it is notable that it appears to be the only obstacle — in particular, ODEs appear to not present much difficulty for focused $\mathsf{d}\mathcal{L}$. It is similarly interesting that the rules (F3) and (F6), despite their unusual nature, appear to not pose much difficulty.

While not directly related to the failure of focalization (at least as far as we are aware), we find it interesting that the connectives $[\alpha]$ and $\langle\alpha\rangle$ appear to most naturally be considered synchronous. In particular, the identity $\langle\alpha\rangle\phi \leftrightarrow \neg[\alpha]\neg\phi$ suggests that $[\alpha]$ and $\langle\alpha\rangle$ should have opposite synchronization behaviour. We believe that further research into the G rules that prompt the choice to call both synchronous may serve both to clarify this point of confusion and to yield a focusing system which is both sound and complete with respect to $\mathsf{d}\mathcal{L}$. This is the primary direction that we see for future work in this area, although we expect that once a fully-fledged focusing system for $\mathsf{d}\mathcal{L}$is developed, other questions of interest will emerge.

References

[1] Jean-Marc Andreoli. Logic programming with focusing proofs in linear logic. *Journal of Logic and Computation*, 2(3):297–347, 1992.

[2] Kaustuv Chaudhuri and Frank Pfenning. Focusing the inverse method for linear logic. In *International Workshop on Computer Science Logic*, pages 200–215. Springer, 2005.

[3] Chuck Liang and Dale Miller. Focusing and polarization in intuitionistic logic. In *International Workshop on Computer Science Logic*, pages 451–465. Springer, 2007.

[4] Chuck Liang and Dale Miller. Focusing and polarization in linear, intuitionistic, and classical logics. *Theoretical Computer Science*, 410(46):4747–4768, 2009.

[5] André Platzer. Differential dynamic logic for hybrid systems. *J. Autom. Reas.*, 41(2):143–189, 2008. ISSN 0168-7433. doi: 10.1007/s10817-008-9103-8.

[6] Jason Reed. A judgmental deconstruction of modal logic. *Unpublished manuscript, January*, 2009.

[7] Robert J Simmons. Structural focalization. *ACM Transactions on Computational Logic (TOCL)*, 15(3):21, 2014.