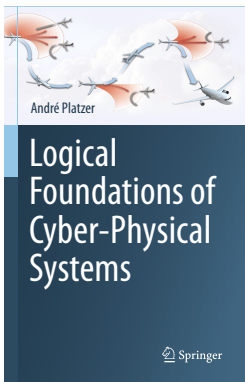
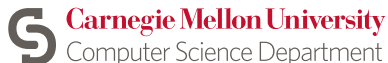


21: Virtual Substitution & Real Arithmetic

Logical Foundations of Cyber-Physical Systems



André Platzer



- 1 Learning Objectives
- 2 Real Arithmetic
 - Recap: Quadratic Equations
 - Quadratic Weak Inequalities
 - Infinity ∞ Virtual Substitution
 - Expedition: Infinities
 - Quadratic Strict Inequalities
 - Infinitesimal ϵ Virtual Substitution
- 3 Quantifier Elimination by Virtual Substitution of Quadratics
- 4 Summary



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2 Real Arithmetic

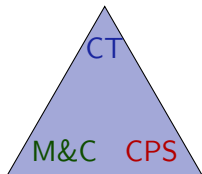
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3 Quantifier Elimination by Virtual Substitution of Quadratics

4 Summary



- rigorous arithmetical reasoning
- miracle of quantifier elimination
- logical trinity for reals
- switch between syntax & semantics at will
- virtual substitution lemma
- bridge gap between semantics and inexpressibles
- infinities & infinitesimals



analytic complexity
modeling tradeoffs

verifying CPS at scale

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Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$a \neq 0 \vee b \neq 0 \vee c \neq 0 \rightarrow$$

$$\left(\exists x (ax^2 + bx + c = 0 \wedge F) \leftrightarrow \right.$$

$$a = 0 \wedge b \neq 0 \wedge F_{\bar{x}}^{-c/b}$$

$$\vee a \neq 0 \wedge b^2 - 4ac \geq 0 \wedge \left(F_{\bar{x}}^{(-b + \sqrt{b^2 - 4ac})/(2a)} \vee F_{\bar{x}}^{(-b - \sqrt{b^2 - 4ac})/(2a)} \right)$$

Lemma (Virtual Substitution Lemma for $\sqrt{\cdot}$)

$$\text{Extended logic} \rightarrow F_x^{(a+b\sqrt{c})/d} \equiv F_{\bar{x}}^{(a+b\sqrt{c})/d} \leftarrow \text{FOL}_{\mathbb{R}}$$

$$\omega_x^r \in \llbracket F \rrbracket \text{ iff } \omega \in \llbracket F_{\bar{x}}^{(a+b\sqrt{c})/d} \rrbracket \text{ where } r = (\omega[a] + \omega[b]\sqrt{\omega[c]}) / (\omega[d]) \in \mathbb{R}$$

Theorem (Virtual Substitution: Quadratic Inequality $x \notin a, b, c$)

$$\exists x (ax^2 + bx + c \leq 0 \wedge F) \leftrightarrow$$

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$$\vee F_{\bar{x}}^{\text{small}} \dots$$

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$$\left. \vee F_{\bar{x}}^{-\infty} \right) \dots$$

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$-\infty$ the rubber band number that's smaller on any comparison

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$-\infty$ needs to satisfy the quadratic inequality (obvious for roots, not $-\infty$)

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Virtual Substitution of $-\infty$ into Comparisons

$$p = \sum_{i=0}^n a_i x^i$$

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Ultimately negative at $-\infty$

$$\lim_{x \rightarrow -\infty} p(x) < 0$$

$$p(-\infty) < 0 \stackrel{\text{def}}{\equiv} \left\{ \right.$$

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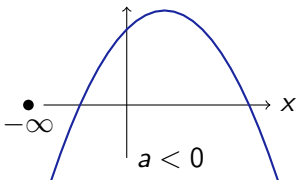
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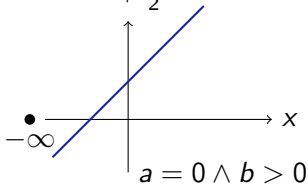
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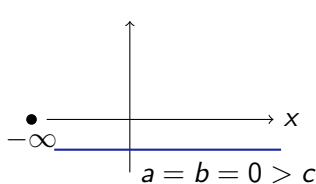
$$-x^2 + x + 1$$



$$x + \frac{1}{2}$$



$$-1$$



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- Order: $\forall x (-\infty \leq x \leq \infty)$
- Complete lattice since every subset has a supremum and infimum
- Arithmetic? $\infty + 1?$

$$\infty + x =$$

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$$\infty \cdot x =$$

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$$\infty - \infty = \text{undefined} \quad \infty + (-\infty) = \infty + (-\infty + 1) = (\infty - \infty) + 1$$

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$$-\infty \cdot x = -\infty \quad \text{for all } x > 0$$

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$$\infty - \infty = \text{undefined} \quad \infty + (-\infty) = \infty + (-\infty + 1) = (\infty - \infty) + 1$$

$$0 \cdot \infty = \text{undefined}$$

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$$1/0 =$$

- Order: $\forall x (-\infty \leq x \leq \infty)$
- Complete lattice since every subset has a supremum and infimum
- Arithmetic? $\infty + 1$? $\infty \leq \infty + 1$ but $\infty + 1 \leq \infty$ by order

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Virtual Substitution

Infinities only needed virtually during virtual substitution, never explicitly.

Theorem (Virtual Substitution: Quadratic Inequality $x \notin a, b, c$)

$$\exists x (ax^2 + bx + c < 0 \wedge F) \leftrightarrow$$

Theorem (Virtual Substitution: Quadratic Inequality $x \notin a, b, c$)

$$a \neq 0 \vee b \neq 0 \vee c \neq 0 \rightarrow$$

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$$a = 0 \wedge b \neq 0 \wedge F_{\bar{x}}^{-c/b}$$

$$\vee a \neq 0 \wedge b^2 - 4ac \geq 0 \wedge \left(F_{\bar{x}}^{(-b + \sqrt{b^2 - 4ac})/(2a)} \vee F_{\bar{x}}^{(-b - \sqrt{b^2 - 4ac})/(2a)} \right)$$

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strict inequality never true at the roots but slightly off

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ϵ the rubber band number that's smaller in magnitude on any comparison

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ε is “always as small as needed”

- Positive: $\varepsilon > 0$
- Smaller: $\forall x \in \mathbb{R} (x > 0 \rightarrow \varepsilon < x)$
- Standard \mathbb{R} are Archimedean:

$$\forall x \in \mathbb{R} \setminus \{0\} \exists n \in \mathbb{N} \underbrace{|x + x + \cdots + x|}_{n \text{ times}} > 1$$

- $\mathbb{R}[\varepsilon]$ are non-Archimedean: $\underbrace{\varepsilon + \varepsilon + \cdots + \varepsilon}_{\text{any } n \in \mathbb{N} \text{ times}} < 1$
- Infinitesimals as inverses of infinities?

$$\varepsilon \cdot \infty = 1? \quad -\varepsilon \cdot -\infty = 1? \quad (\varepsilon + 1) \cdot \underbrace{(\infty + 2)}_{\infty?} = \dots$$

- How to order for $x \neq 0$?

$$\varepsilon^2 \quad \varepsilon \quad x^2 + \varepsilon \quad (x + \varepsilon)^2 \quad x^2 + 2\varepsilon x + 5\varepsilon + \varepsilon^2$$

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Virtual Substitution of $e + \varepsilon$ into Comparisons

$$p = \sum_{i=0}^n a_i x^i$$

$$(p = 0)_{\bar{x}}^{e+\varepsilon} \equiv$$

$$(p \leq 0)_{\bar{x}}^{e+\varepsilon} \equiv$$

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$$(p < 0)_{\bar{x}}^{e+\varepsilon} \equiv (p^+ < 0)_{\bar{x}}^e$$

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Immediately negative at x

$$\lim_{y \searrow x} p(y) < 0$$

$$p^+ < 0 \stackrel{\text{def}}{\equiv} \left\{ \begin{array}{l} \text{if} \\ \text{if} \end{array} \right.$$

ordinary virtual $\sqrt{\cdot}$ substitution of e into immediate negativity $p^+ < 0$

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Break ties by successive derivative p' immediately negative at root of p



Immediately negative at e

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Immediately negative at e

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$$\equiv 0 \cdot 1 < 0 \vee 0 = 0 \wedge$$

$$\underbrace{((0 < 0 \vee 4a^2 \leq 0 \wedge (0 < 0 \vee -4a^2(b^2 - 4ac) < 0))}_{(2ax + b < 0)_{\bar{x}}^{(-b + \sqrt{b^2 - 4ac})/(2a)}} \vee \underbrace{0 = 0}_{(2ax + b = 0)_{\bar{x}}^{\dots}} \wedge \underbrace{2a1 < 0}_{(2a < 0)_{\bar{x}}^{\dots}}$$

$$\equiv 4a^2 \leq 0 \wedge -4a^2(b^2 - 4ac) < 0 \vee 2a < 0$$

$$\equiv a = 0 \wedge 0(b^2 - 0) < 0 \vee 2a < 0 \equiv 2a < 0$$

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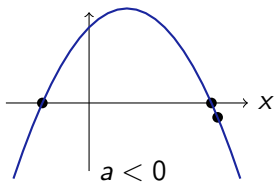
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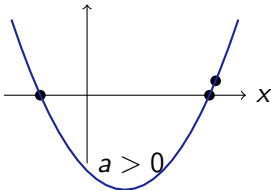
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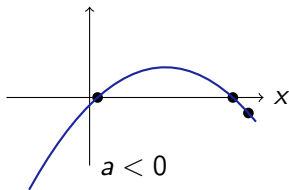
$$-x^2 + x + 1$$



$$x^2 - x - 1$$



$$-\frac{1}{2}x^2 + x - \frac{1}{10}$$



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- 1 Learning Objectives
- 2 Real Arithmetic
 - Recap: Quadratic Equations
 - Quadratic Weak Inequalities
 - Infinity ∞ Virtual Substitution
 - Expedition: Infinities
 - Quadratic Strict Inequalities
 - Infinitesimal ϵ Virtual Substitution
- 3 Quantifier Elimination by Virtual Substitution of Quadratics
- 4 Summary

Theorem (Virtual Substitution: Quadratics) (Weispfenning'97)

Let all atomic formulas in F be of the form $ax^2 + bx + c \sim 0$ with $x \notin a, b, c$ and $\sim \in \{=, \leq, <, \neq\}$ and its discriminant $d \stackrel{\text{def}}{=} b^2 - 4ac$.

$$\exists x F \leftrightarrow$$

$$F_x^{-\infty}$$

$$\bigvee_{(ax^2+bx+c\{\leq\}0) \in F} \left(a=0 \wedge b \neq 0 \wedge F_x^{-c/b} \vee a \neq 0 \wedge d \geq 0 \wedge (F_x^{(-b+\sqrt{d})/(2a)} \vee F_x^{(-b-\sqrt{d})/(2a)}) \right)$$

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Equivalence needs roots and off-roots from **all** atomic formulas in F

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$$F_x^{-\infty}$$

$$\bigvee_{(ax^2+bx+c \{ \leq \} 0) \in F} \left(a=0 \wedge b \neq 0 \wedge F_x^{-c/b} \vee a \neq 0 \wedge d \geq 0 \wedge (F_x^{(-b+\sqrt{d})/(2a)} \vee F_x^{(-b-\sqrt{d})/(2a)}) \right)$$

$$\bigvee_{(ax^2+bx+c \{ \neq, < \} 0) \in F} \left(a=0 \wedge b \neq 0 \wedge F_x^{-c/b+\varepsilon} \vee a \neq 0 \wedge d \geq 0 \wedge (F_x^{(-b+\sqrt{d})/(2a)+\varepsilon} \vee F_x^{(-b-\sqrt{d})/(2a)+\varepsilon}) \right)$$

Equivalence needs roots and off-roots from **all** atomic formulas in F

Theorem (Virtual Substitution: Quadratics) (Weispfenning'97)

Let all atomic formulas in F be of the form $ax^2 + bx + c \sim 0$ with $x \notin a, b, c$ and $\sim \in \{=, \leq, <, \neq\}$ and its discriminant $d \stackrel{\text{def}}{=} b^2 - 4ac$.

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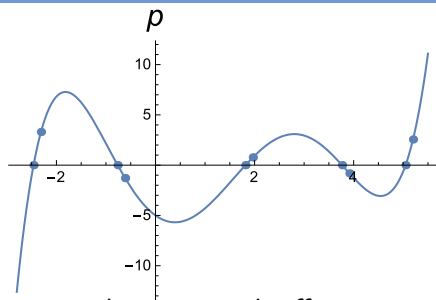
$F_{\bar{x}}^{-\infty}$

$$\bigvee_{(ax^2+bx+c \left\{ \begin{smallmatrix} \leq \\ \leq \end{smallmatrix} \right\} 0) \in F} (a=0 \wedge b \neq 0 \wedge F_{\bar{x}}^{-c/b} \vee a \neq 0 \wedge d \geq 0 \wedge (F_{\bar{x}}^{(-b+\sqrt{d})/(2a)} \vee F_{\bar{x}}^{(-b-\sqrt{d})/(2a)}))$$

$$\bigvee_{(ax^2+bx+c \left\{ \begin{smallmatrix} \neq \\ < \end{smallmatrix} \right\} 0) \in F} (a=0 \wedge b \neq 0 \wedge F_{\bar{x}}^{-c/b+\epsilon} \vee a \neq 0 \wedge d \geq 0 \wedge (F_{\bar{x}}^{(-b+\sqrt{d})/(2a)+\epsilon} \vee F_{\bar{x}}^{(-b-\sqrt{d})/(2a)+\epsilon}))$$

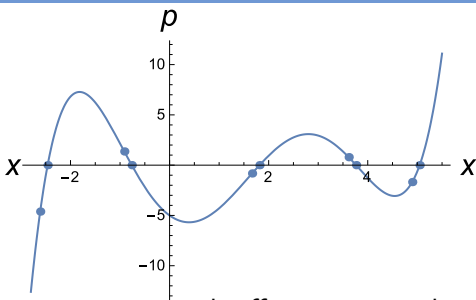
Lemma (Virtual Substitution Lemmas)

$$F_x^{(a+b\sqrt{c})/d} \equiv F_{\bar{x}}^{(a+b\sqrt{c})/d} \quad F_x^{-\infty} \equiv F_{\bar{x}}^{-\infty} \quad F_x^{e+\epsilon} \equiv F_{\bar{x}}^{e+\epsilon}$$



$-\infty$ and roots e with offsets $e + \varepsilon$

No rejection without mention



roots e with offsets $e - \varepsilon$ and ∞

Other parts of F not satisfied by the points of p have their own polynomial q that contributes different roots \tilde{e} and off-roots $\tilde{e} + \varepsilon$.

Generalizations of quantifier elimination to higher degrees also place a representative point into every region of interest, but derivatives and relationships of derivatives become crucially relevant.

Theorem (Virtual Substitution: Quadratics) (Weispfenning'97)

$$\exists x F \leftrightarrow F_{\bar{x}}^{-\infty}$$

$$\bigvee_{(ax^2+bx+c \left\{ \begin{smallmatrix} = \\ \leq \end{smallmatrix} \right\} 0) \in F} (a=0 \wedge b \neq 0 \wedge F_{\bar{x}}^{-c/b} \vee a \neq 0 \wedge d \geq 0 \wedge (F_{\bar{x}}^{(-b+\sqrt{d})/(2a)} \vee F_{\bar{x}}^{(-b-\sqrt{d})/(2a)}))$$

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“Proof” Sketch.

- “ \leftarrow ” simple from virtual substitution lemma with (extended) term witness
- “ \rightarrow ” Valid iff true in every state, so all variables have real numeric value
 - o-minimal: solutions form finite union of disjoint intervals (univariate)
 - WLOG endpoints are the roots since all polynomials quadratic
 - All side conditions have to be met otherwise can't be solution
 - Non-point intervals contain ε offset since smaller than endpoint □

$$a \neq 0 \rightarrow (\exists x (ax^2 + bx + c = 0 \wedge x \geq 0))$$

$$\Leftrightarrow b^2 - 4ac \geq 0 \wedge (2ba \leq 0 \wedge 4ac \geq 0 \vee -2a \leq 0 \wedge 4ac \leq 0 \\ \vee 2ba \leq 0 \wedge 4ac \geq 0 \vee 2a \leq 0 \wedge 4ac \leq 0))$$

$$-(-b + \sqrt{b^2 - 4ac}) / (2a) = ((-1 + 0\sqrt{b^2 - 4ac}) / 1) \cdot ((-b + \sqrt{b^2 - 4ac}) / (2a)) \\ = (b - \sqrt{b^2 - 4ac}) / (2a)$$

$$(-x \leq 0)_{\bar{x}}^{(b - \sqrt{b^2 - 4ac}) / (2a)}$$

$$\equiv b2a \leq 0 \wedge \cancel{b^2} - (-1)^2(\cancel{b^2} - 4ac) \geq 0 \vee -1 \cdot 2a \leq 0 \wedge \cancel{b^2} - (-1)^2(\cancel{b^2} - 4ac) \leq 0$$

$$\equiv 2ba \leq 0 \wedge 4ac \geq 0 \vee -2a \leq 0 \wedge 4ac \leq 0$$

$$(-x \leq 0)_{\bar{x}}^{(b + \sqrt{b^2 - 4ac}) / (2a)}$$

$$\equiv b2a \leq 0 \wedge \cancel{b^2} - 1^2(\cancel{b^2} - 4ac) \geq 0 \vee 1 \cdot 2a \leq 0 \wedge \cancel{b^2} - 1^2(\cancel{b^2} - 4ac) \leq 0$$

$$\equiv 2ba \leq 0 \wedge 4ac \geq 0 \vee 2a \leq 0 \wedge 4ac \leq 0$$



Example: Nonnegative Roots of Quadratic Polynomials

$$\exists x (x^2 - x + c = 0 \wedge x \geq 0)$$

$$\leftrightarrow (-1)^2 - 4c \geq 0 \wedge (c \geq 0 \vee c \leq 0)$$

$$\underline{\vee c \geq 0 \vee \text{false} \wedge c \leq 0}$$



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$$\leftrightarrow \underbrace{(x^2 - x + c \leq 0 \wedge x \geq 0)_{\bar{x}}^{-\infty}}_{\text{false}} \vee 1 - 4c \geq 0 \vee \underbrace{(x^2 - x + c \leq 0 \wedge x \geq 0)_{\bar{x}}^0}_{c \leq 0 \wedge 0 \geq 0 \text{ (subsumed)}}$$

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$$\exists x (x^2 - x + c \leq 0 \wedge x \geq 0 \wedge -x + 2 = 0)$$

more roots!

$$\leftrightarrow \underbrace{(x^2 - x + c \leq 0 \wedge \dots)^{-\infty}_{\bar{x}}}_{\text{false}}$$

$$\vee 1 - 4c \geq 0 \wedge \underbrace{(x^2 - x + c \leq 0 \wedge x \geq 0 \wedge -x + 2 = 0)^{-1 \pm \sqrt{1-4c}/2}_{\bar{x}}}_{8-4c=0}$$

$$\vee -1 \neq 0 \wedge \underbrace{(x^2 - x + c \leq 0 \wedge x \geq 0 \wedge -x + 2 = 0)^0_x}_{c \leq 0 \wedge 0 \geq 0 \wedge 2=0}$$

$$\vee -1 \neq 0 \wedge \underbrace{(x^2 - x + c \leq 0 \wedge x \geq 0 \wedge -x + 2 = 0)^2_x}_{2+c \leq 0} \equiv 2 + c \leq 0$$



- 1 Learning Objectives
- 2 Real Arithmetic
 - Recap: Quadratic Equations
 - Quadratic Weak Inequalities
 - Infinity ∞ Virtual Substitution
 - Expedition: Infinities
 - Quadratic Strict Inequalities
 - Infinitesimal ϵ Virtual Substitution
- 3 Quantifier Elimination by Virtual Substitution of Quadratics
- 4 Summary

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$a \neq 0 \vee b \neq 0 \vee c \neq 0 \rightarrow$$

$$\left(\exists x (ax^2 + bx + c = 0 \wedge F) \leftrightarrow \right.$$

$$a = 0 \wedge b \neq 0 \wedge F_{\bar{x}}^{-c/b}$$

$$\vee a \neq 0 \wedge b^2 - 4ac \geq 0 \wedge \left(F_{\bar{x}}^{(-b + \sqrt{b^2 - 4ac})/(2a)} \vee F_{\bar{x}}^{(-b - \sqrt{b^2 - 4ac})/(2a)} \right)$$

Lemma (Virtual Substitution Lemma for $\sqrt{\cdot}$)

Extended logic $\rightarrow F_x^{(a+b\sqrt{c})/d} \equiv F_{\bar{x}}^{(a+b\sqrt{c})/d} \leftarrow \text{FOL}_{\mathbb{R}}$

$$\omega_x^r \in \llbracket F \rrbracket \text{ iff } \omega \in \llbracket F_{\bar{x}}^{(a+b\sqrt{c})/d} \rrbracket \text{ where } r = (\omega[a] + \omega[b]\sqrt{\omega[c]}) / (\omega[d]) \in \mathbb{R}$$

Theorem (Virtual Substitution: Quadratics) (Weispfenning'97)

Let all atomic formulas in F be of the form $ax^2 + bx + c \sim 0$ with $x \notin a, b, c$ and $\sim \in \{=, \leq, <, \neq\}$ and its discriminant $d \stackrel{\text{def}}{=} b^2 - 4ac$.

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$$\bigvee_{(ax^2+bx+c \left\{ \begin{smallmatrix} < \\ < \end{smallmatrix} \right\} 0) \in F} (a=0 \wedge b \neq 0 \wedge F_{\bar{x}}^{-c/b+\epsilon} \vee a \neq 0 \wedge d \geq 0 \wedge (F_{\bar{x}}^{(-b+\sqrt{d})/(2a)+\epsilon} \vee F_{\bar{x}}^{(-b-\sqrt{d})/(2a)+\epsilon}))$$

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- Miracle: $\text{FOL}_{\mathbb{R}}$ is decidable: Tarski'31
Algorithm decides whether (closed) formula valid or not
- Quantifier elimination computes quantifier-free equivalent
- Successive quantifier elimination decides $\text{FOL}_{\mathbb{R}}$ (after universal closure)
- QE accepts free variables, giving equivalent that identifies the requirements for truth (synthesis)
- Virtual substitution does QE for degree ≤ 3 by equivalent syntactic rephrasing of semantics Weispfenning'97
- QE proceeds inside out, so degree ≤ 3 needed on *each* iteration
- Important fragments permit many optimizations your research?
- Universally quantified weak inequalities / existentially quantified strict inequalities are easier since infinitesimals/infinities don't satisfy =
- Cylindrical algebraic decomposition (CAD) any degree Collins'75
- Simplify arithmetic to relevant parts, transform to fit together



André Platzer.

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