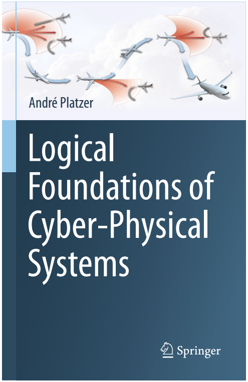
11: Differential Equations & Proofs

Logical Foundations of Cyber-Physical Systems



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Outline

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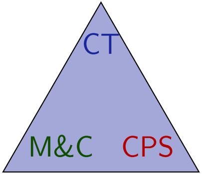
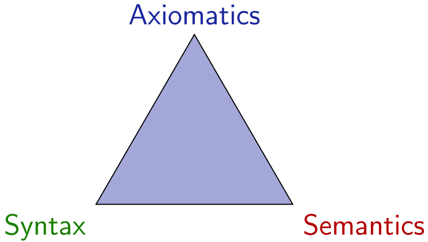
1. Learning Objectives
2. Differential Invariants
   - Recap: Ingredients for Differential Equation Proofs
   - Soundness: Derivations Lemma
   - Differential Weakening
   - Equational Differential Invariants
   - Differential Invariant Inequalities
   - Disequational Differential Invariants
   - Example Proof: Damped Oscillator
   - Conjunctive Differential Invariants
   - Disjunctive Differential Invariants
   - Assuming Invariants
3. Differential Cuts
4. Soundness
5. Summary

discrete vs. continuous analogy
rigorous reasoning about ODEs
beyond differential invariant terms
differential invariant formulas
cut principles for differential equations
axiomatization of ODEs
differential facet of logical trinity



understanding continuous dynamics
relate discrete+continuous

operational CPS effects
state changes along ODE



Syntax defines the notation
What problems are we allowed to write down?

Semantics what carries meaning.
What real or mathematical objects does the syntax stand for?

Axiomatics internalizes semantic relations into universal syntactic transformations.
How does the semantics of $e \geq \tilde{e}$ relate to semantics of $e - \tilde{e} \geq 0$, syntactically? What about derivatives?

Outline

Differentials

Syntax $e ::= x \mid x' \mid c \mid e + k \mid e \cdot k \mid (e)'$

Semantics $\omega[\![(e)']\!] = \sum_x \omega(x') \frac{\partial [\![e]\!]}{\partial x}(\omega)$

Axioms

$$(e+k)' = (e)' + (k)'$$

$$(e \cdot k)' = (e)' \cdot k + e \cdot (k)'$$

$$(c())' = 0 \qquad \text{for constants/numbers } c()$$

$$(x)' = x' \qquad \text{for variables } x \in \mathcal{V}$$

ODE

$$[\![x' = f(x) \,\&\, Q]\!] = \{(\varphi(0)|_{\{x'\}^\complement}, \varphi(r)) : \varphi \models x' = f(x) \land Q$$

$$\text{for some } \varphi : [0,r] \to \mathcal{S}, \text{ some } r \in \mathbb{R}\}$$

$$\varphi(z)(x') = \frac{\mathrm{d}\varphi(t)(x)}{\mathrm{d}t}(z) \qquad \ldots$$

Differential Substitution Lemmas

Lemma (Differential lemma) (Differential value vs. Time-derivative)

If $\varphi \models x' = f(x) \land Q$ *for duration* $r>0$, *then for all* $0 \le z \le r$, $FV(e) \subseteq \{x\}$:

$$\varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

Lemma (Differential assignment) (Effect on Differentials)

If $\varphi \models x' = f(x) \land Q$ *then* $\varphi \models P \leftrightarrow [x' := f(x)]P$

Lemma (Derivations) (Equations of Differentials)

$$(e + k)' = (e)' + (k)'$$

$$(e \cdot k)' = (e)' \cdot k + e \cdot (k)'$$

$$(c())' = 0 \qquad \textit{for constants/numbers } c()$$

$$(x)' = x' \qquad \textit{for variables } x \in \mathcal{V}$$

Lemma (Differential lemma) (Differential value vs. Time-derivative)

If $\varphi \models x' = f(x) \wedge Q$ *for duration* $r>0$*, then for all* $0 \leq z \leq r$*,* $FV(e) \subseteq \{x\}$*:*

$$\varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

Lemma (Differential assignment) (Effect on Differentials)

$DE \quad [x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q][x' := f(x)]P$

Lemma (Derivations) (Equations of Differentials)

$+' \qquad (e + k)' = (e)' + (k)'$

$\cdot' \qquad (e \cdot k)' = (e)' \cdot k + e \cdot (k)'$

$c' \qquad (c())' = 0$

$x' \qquad (x)' = x'$

Soundness: Proof of Derivations Lemma

Lemma (Derivations) (Equations of Differentials)

$+'$	$(e+k)' = (e)' + (k)'$
$\cdot'$	$(e \cdot k)' = (e)' \cdot k + e \cdot (k)'$
c'	$(c())' = 0$
x'	$(x)' = x'$

Soundness: Proof of Derivations Lemma

Lemma (Derivations) (Equations of Differentials)

$+'$ $\quad (e+k)' = (e)' + (k)'$

Proof.

$\omega[\![(e+k)']\!] =$

$\square$

Lemma (Derivations) (Equations of Differentials)

$+'$ $\quad (e+k)' = (e)' + (k)'$

Proof.

$$\omega[\![(e+k)']\!] = \sum_x \omega(x') \frac{\partial [\![e+k]\!]}{\partial x}(\omega)$$

□

Lemma (Derivations) (Equations of Differentials)

$+'$ $\quad (e+k)' = (e)' + (k)'$

Proof.

$$\omega[\![(e+k)']\!] = \sum_x \omega(x') \frac{\partial [\![e+k]\!]}{\partial x}(\omega) = \sum_x \omega(x') \frac{\partial ([\![e]\!] + [\![k]\!])}{\partial x}(\omega)$$

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Proof.

$$\omega[\![(e+k)']\!] = \sum_x \omega(x')\frac{\partial[\![e+k]\!]}{\partial x}(\omega) = \sum_x \omega(x')\frac{\partial([\![e]\!]+[\![k]\!])}{\partial x}(\omega)$$

$$= \sum_x \omega(x')\Big(\frac{\partial[\![e]\!]}{\partial x}(\omega) + \frac{\partial[\![k]\!]}{\partial x}(\omega)\Big)$$

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$$= \omega[\![(e)']\!] + \omega[\![(k)']\!]$$

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Soundness: Proof of Derivations Lemma

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$$= \omega[\![(e)']\!] + \omega[\![(k)']\!] = \omega[\![(e)' + (k)']\!] \quad \text{for all } \omega$$

□

Lemma (Differential lemma) (Differential value vs. Time-derivative)

If $\varphi \models x' = f(x) \land Q$ *for duration* $r>0$, *then for all* $0 \le z \le r$, $FV(e) \subseteq \{x\}$:

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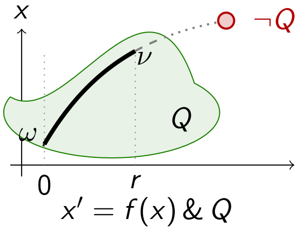
Differential Weakening



ODE

$$[\![x' = f(x) \,\&\, Q]\!] = \{(\varphi(0)|_{\{x'\}^\complement}, \varphi(r)) : \varphi \models x' = f(x) \land Q$$
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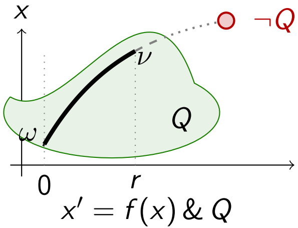


DW $[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q](Q \rightarrow P)$

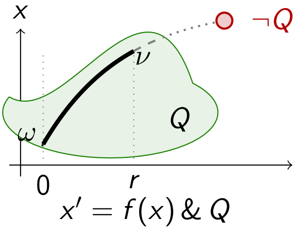
ODE

$$[\![x' = f(x) \,\&\, Q]\!] = \{(\varphi(0)|_{\{x'\}^\complement}, \varphi(r)) : \varphi \models x' = f(x) \wedge Q$$

$$\text{for some } \varphi : [0, r] \to \mathcal{S}, \text{ some } r \in \mathbb{R}\}$$

$$\varphi(z)(x') = \frac{d\varphi(t)(x)}{dt}(z)$$

Differential equations cannot leave their domains.

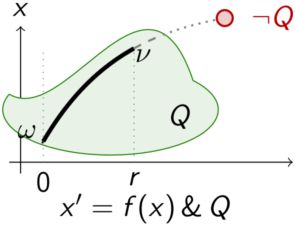


DW $[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q](Q \rightarrow P)$

Example (Bouncing ball)

$$\text{DW}\ \frac{}{\vdash [x' = v, v' = -g \,\&\, x \geq 0]\, 0 \leq x}$$

No need to solve any ODEs to prove that bouncing ball is above ground.



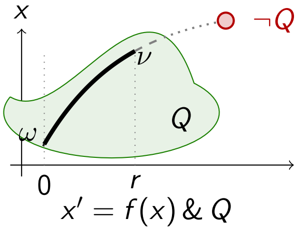
DW $[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q](Q \to P)$

Example (Bouncing ball)

$$\text{DW}\dfrac{\text{G}\dfrac{}{\vdash [x' = v, v' = -g \,\&\, x \geq 0](x \geq 0 \to 0 \leq x)}}{\vdash [x' = v, v' = -g \,\&\, x \geq 0]\, 0 \leq x}$$

No need to solve any ODEs to prove that bouncing ball is above ground.

Differential Weakening



DW $[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q](Q \rightarrow P)$

Example (Bouncing ball)

$$\dfrac{\dfrac{\dfrac{}{\vdash x \geq 0 \rightarrow 0 \leq x}\;\mathbb{R}}{\vdash [x' = v, v' = -g \,\&\, x \geq 0](x \geq 0 \rightarrow 0 \leq x)}\;\text{G}}{\vdash [x' = v, v' = -g \,\&\, x \geq 0]\, 0 \leq x}\;\text{DW}$$

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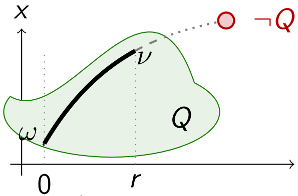

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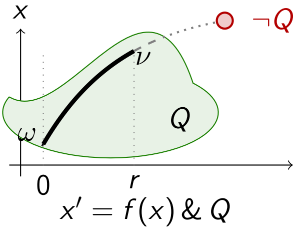
No need to solve any ODEs to prove that bouncing ball is above ground.

Differential Weakening

$$\text{dW}\quad \frac{}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta}$$



$$\text{DW}\quad [x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q](Q \rightarrow P)$$

Example (Bouncing ball)

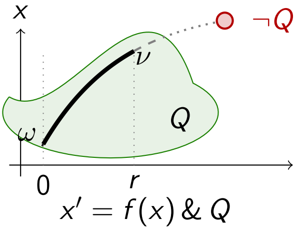
$$\dfrac{\dfrac{\dfrac{*}{\vdash x \geq 0 \rightarrow 0 \leq x}\;\mathbb{R}}{\vdash [x' = v, v' = -g \,\&\, x \geq 0](x \geq 0 \rightarrow 0 \leq x)}\;\text{G}}{\vdash [x' = v, v' = -g \,\&\, x \geq 0]\, 0 \leq x}\;\text{DW}$$

No need to solve any ODEs to prove that bouncing ball is above ground.

Differential Weakening

Differential Weakening

$$\text{dW}\quad \frac{Q \vdash P}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta}$$



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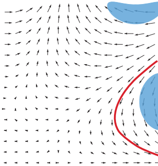
Differential Invariant Terms for Differential Equations



Differential Invariant

$$\text{dI}\ \frac{\vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x)\,\&\,Q]e = 0}$$

$$\text{DI}\ \big([x' = f(x)]\,e = 0 \leftrightarrow e = 0\big) \leftarrow [x' = f(x)]\,(e)' = 0$$

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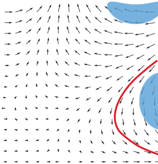
$$\text{DW}\ [x' = f(x)\,\&\,Q]P \leftrightarrow [x' = f(x)\,\&\,Q](Q \rightarrow P)$$



Differential Invariant

$$\text{dI}\quad \frac{Q \vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x) \,\&\, Q]e = 0}$$

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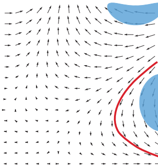

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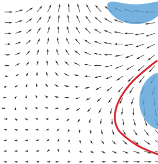
Proof (dI is a derived rule).

□

$$\text{DI}\ \frac{}{e = 0 \vdash [x' = f(x) \,\&\, Q]e = 0}$$

Differential Invariant Terms for Differential Equations



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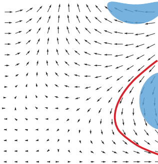

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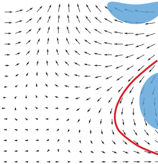
□



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Proof (dI is a derived rule).

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\vdash [x' = f(x) \,\&\, Q](Q \to [x' := f(x)](e)' = 0)}\text{G},\!\to\!\text{R}}{\vdash [x' = f(x) \,\&\, Q][x' := f(x)](e)' = 0}\text{DW}}{\vdash [x' = f(x) \,\&\, Q](e)' = 0}\text{DE}}{e = 0 \vdash [x' = f(x) \,\&\, Q]e = 0}\text{DI}$$

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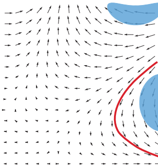

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Proof (dI is a derived rule).

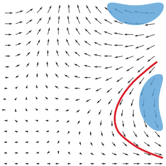
$$\begin{array}{ll}
 & Q \vdash [x' := f(x)](e)' = 0 \\ \hline
\text{G},\to\text{R} & \vdash [x' = f(x)\,\&\,Q](Q \to [x' := f(x)](e)' = 0) \\ \hline
\text{DW} & \vdash [x' = f(x)\,\&\,Q][x' := f(x)](e)' = 0 \\ \hline
\text{DE} & \vdash [x' = f(x)\,\&\,Q](e)' = 0 \\ \hline
\text{DI} & e = 0 \vdash [x' = f(x)\,\&\,Q]e = 0
\end{array}
\qquad \text{G}\ \frac{P}{[\alpha]P}$$

□

Differential Invariant Equations

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \;\; \textit{for } r > 0 \;\; \Rightarrow \;\; \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$



Differential Invariant

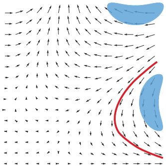
$$\text{dI} \quad \frac{}{e = k \vdash [x' = f(x)]e = k}$$

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$

Differential Invariant

$$\text{dI} \quad \frac{\vdash [x' := f(x)](e)' = (k)'}{e = k \vdash [x' = f(x)]e = k}$$



$$\text{DI} \quad \big([x' = f(x)]\, e = k \leftrightarrow e = k\big) \leftarrow [x' = f(x)]\,(e)' = (k)'$$

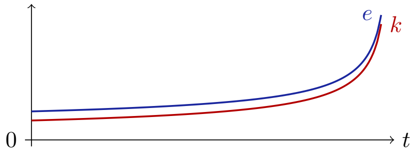
Differential Invariant Equations

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$

Differential Invariant

$$\text{dI} \quad \frac{\vdash [x' := f(x)](e)' = (k)'}{e = k \vdash [x' = f(x)]e = k}$$



$$\text{DI} \quad \big([x' = f(x)]\, e = k \leftrightarrow e = k\big) \leftarrow [x' = f(x)]\,(e)' = (k)'$$

Proof (= rate of change from = initial value. Mean-value theorem).

$$\frac{d\varphi(t)[\![e]\!]}{dt}(z) = \varphi(z)[\![(e)']\!] = \varphi(z)[\![(k)']\!] = \frac{d\varphi(t)[\![k]\!]}{dt}(z)$$

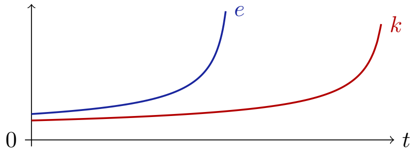
□

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$

Differential Invariant

$$\text{dI} \;\; \frac{\vdash [x' := f(x)](e)' \ge (k)'}{e \ge k \vdash [x' = f(x)]e \ge k}$$



$$\text{DI} \;\; \big([x' = f(x)]\, e \ge k \leftrightarrow e \ge k\big) \leftarrow [x' = f(x)]\,(e)' \ge (k)'$$

Proof ($\ge$ rate of change from $\ge$ initial value. Mean-value theorem).

$$\frac{d\varphi(t)[\![e]\!]}{dt}(z) = \varphi(z)[\![(e)']\!] \ge \varphi(z)[\![(k)']\!] = \frac{d\varphi(t)[\![k]\!]}{dt}(z)$$

□

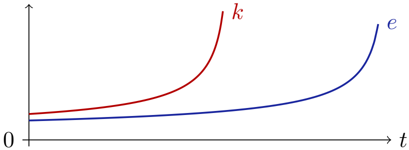
Differential Invariant Inequalities

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$

Differential Invariant

$$\text{dI} \quad \frac{\vdash [x' := f(x)](e)' \le (k)'}{e \le k \vdash [x' = f(x)]e \le k}$$



$$\text{DI} \quad \big([x' = f(x)]\, e \le k \leftrightarrow e \le k\big) \leftarrow [x' = f(x)]\,(e)' \le (k)'$$

Proof ($\le$ rate of change from $\le$ initial value. Mean-value theorem).

$$\frac{d\varphi(t)[\![e]\!]}{dt}(z) = \varphi(z)[\![(e)']\!] \le \varphi(z)[\![(k)']\!] = \frac{d\varphi(t)[\![k]\!]}{dt}(z) \qquad \square$$

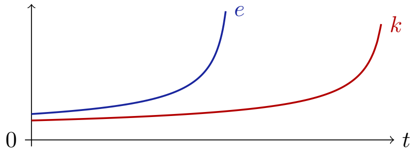
Differential Invariant Inequalities

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \textit{ for } r > 0 \;\Rightarrow\; \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

Differential Invariant

$$\text{dI}\quad \frac{\vdash [x' := f(x)](e)' > (k)'}{e > k \vdash [x' = f(x)]e > k}$$



$$\text{DI}\quad \big([x' = f(x)]\, e > k \leftrightarrow e > k\big) \leftarrow [x' = f(x)]\,(e)' > (k)'$$

Proof (> rate of change from > initial value. Mean-value theorem).

$$\frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z) = \varphi(z)[\![(e)']\!] > \varphi(z)[\![(k)']\!] = \frac{\mathrm{d}\varphi(t)[\![k]\!]}{\mathrm{d}t}(z) \qquad \square$$

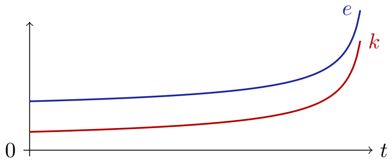
Differential Invariant Inequalities

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

Differential Invariant

$$\text{dI} \;\; \frac{\vdash [x' := f(x)](e)' \ge (k)'}{e > k \vdash [x' = f(x)]e > k}$$



$$\text{DI} \;\; \big([x' = f(x)]\, e > k \leftrightarrow e > k\big) \leftarrow [x' = f(x)]\,(e)' \ge (k)'$$

Proof ($\ge$ rate of change from $>$ initial value. Mean-value theorem).

$$\frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z) = \varphi(z)[\![(e)']\!] \ge \varphi(z)[\![(k)']\!] = \frac{\mathrm{d}\varphi(t)[\![k]\!]}{\mathrm{d}t}(z) \qquad \square$$

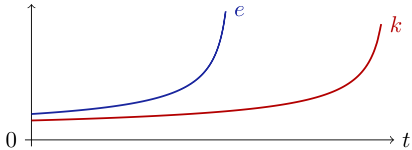
Differential Invariant Inequalities

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

Differential Invariant

$$\text{dI} \quad \frac{\vdash [x' := f(x)](e)' \neq (k)'}{e \neq k \vdash [x' = f(x)]e \neq k}$$



$$\text{DI} \quad \big([x' = f(x)]\, e \neq k \leftrightarrow e \neq k\big) \leftarrow [x' = f(x)]\,(e)' \neq (k)'$$

Proof ($\neq$ rate of change from $\neq$ initial value. Mean-value theorem).

$$\frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z) = \varphi(z)[\![(e)']\!] \neq \varphi(z)[\![(k)']\!] = \frac{\mathrm{d}\varphi(t)[\![k]\!]}{\mathrm{d}t}(z)$$

□

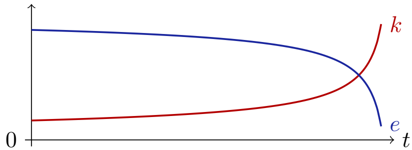
Differential Invariant Inequalities

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \wedge Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$

Differential Invariant

$$\text{dI} \quad \frac{\vdash [x' := f(x)](e)' \neq (k)'}{e \neq k \vdash [x' = f(x)]e \neq k}$$



$$\text{DI} \quad \big([x' = f(x)]\, e \neq k \leftrightarrow e \neq k\big) \leftarrow [x' = f(x)]\,(e)' \neq (k)'$$

Proof ($\neq$ rate of change from $\neq$ initial value. Mean-value theorem).

$$\frac{d\varphi(t)[\![e]\!]}{dt}(z) = \varphi(z)[\![(e)']\!] \neq \varphi(z)[\![(k)']\!] = \frac{d\varphi(t)[\![k]\!]}{dt}(z)$$ □

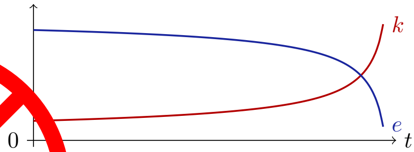
Differential Invariant Inequalities

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \text{ for } r > 0 \Rightarrow \forall 0 \le z \le r \ \ \varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$

Differential Invariant

$$\text{dI} \ \ \frac{\vdash [x' := f(x)](e)' \neq (k)'}{e \neq k \vdash [x' = f(x)]e \neq k}$$



$$\text{DI} \ \ ([x' = f(x)]\, e \neq k \leftrightarrow [?Q]\, e \neq k) \leftarrow [x' = f(x)]\, (e)' \neq (k)'$$

Proof (≠ rate of change from ≠ initial value. Mean-value theorem).

$$\frac{d\varphi(t)[\![e]\!]}{dt}(z) = \varphi(z)[\![(e)']\!] \neq \varphi(z)[\![(k)']\!] = \frac{d\varphi(t)[\![k]\!]}{dt}(z)$$

□

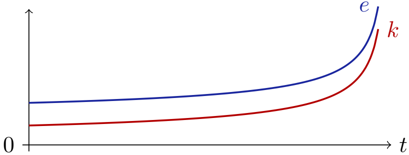
Differential Invariant Inequalities

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \text{ for } r > 0 \Rightarrow \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$

Differential Invariant

$$\text{dI} \quad \frac{\vdash [x' := f(x)](e)' = (k)'}{e \neq k \vdash [x' = f(x)]e \neq k}$$



$$\text{DI} \quad \big([x' = f(x)]\, e \neq k \leftrightarrow e \neq k\big) \leftarrow [x' = f(x)]\,(e)' = (k)'$$

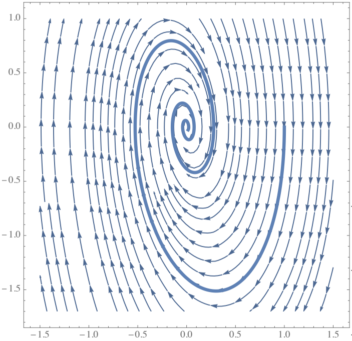
Proof (= rate of change from ≠ initial value. Mean-value theorem).

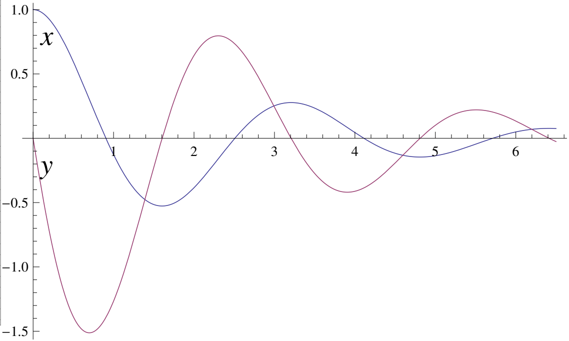
$$\frac{d\varphi(t)[\![e]\!]}{dt}(z) = \varphi(z)[\![(e)']\!] = \varphi(z)[\![(k)']\!] = \frac{d\varphi(t)[\![k]\!]}{dt}(z)$$

□

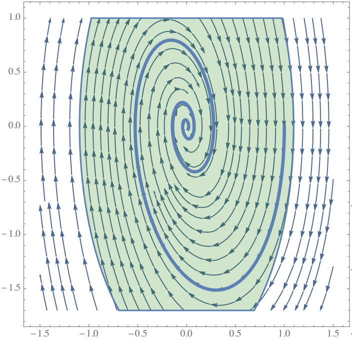
Example: Differential Invariant Inequalities

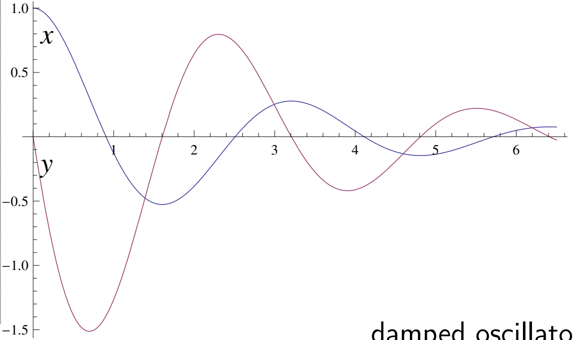
$$\frac{}{\omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega\geq 0 \land d\geq 0]\, \omega^2x^2+y^2\leq c^2}$$





$$\frac{}{\omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega\geq 0 \land d\geq 0]\, \omega^2x^2+y^2\leq c^2}$$

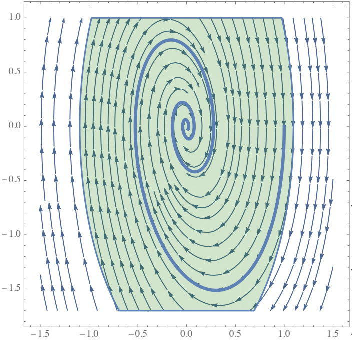


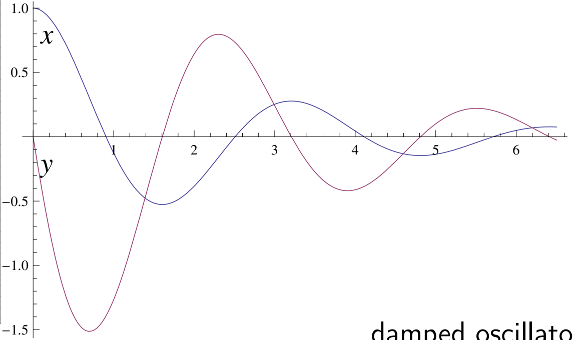


damped oscillator

Example: Differential Invariant Inequalities: Oscillator

$$\frac{\omega\geq 0 \land d\geq 0 \vdash [x':=y][y':=-\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2 y y' \leq 0}{\omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega\geq 0 \land d\geq 0]\, \omega^2 x^2 + y^2 \leq c^2}$$

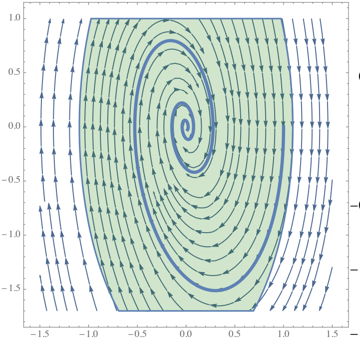


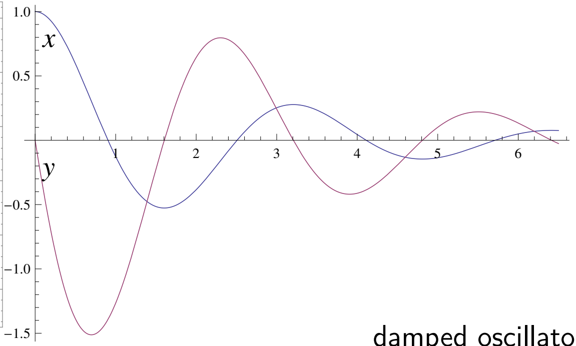


damped oscillator

Example: Differential Invariant Inequalities: Oscillator

$$\dfrac{\dfrac{\omega\geq 0 \land d\geq 0 \vdash 2\omega^2 x y + 2y(-\omega^2 x - 2d\omega y) \leq 0}{\omega\geq 0 \land d\geq 0 \vdash [x':=y][y':=-\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2 y y' \leq 0}}{\omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega\geq 0 \land d\geq 0]\, \omega^2x^2+y^2\leq c^2}$$

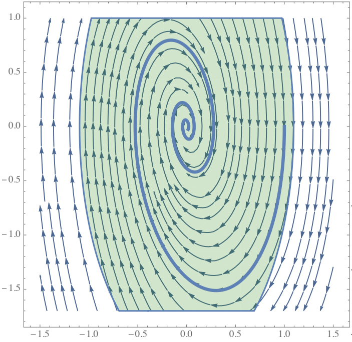


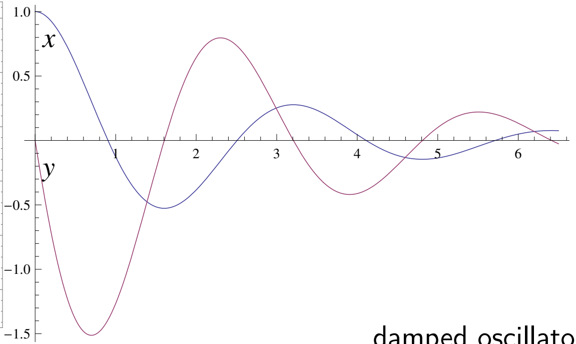


damped oscillator

Example: Differential Invariant Inequalities: Oscillator

$$\frac{\dfrac{\dfrac{*}{\omega{\geq}0 \land d{\geq}0 \vdash 2\omega^2 xy + 2y(-\omega^2 x - 2d\omega y) \leq 0}}{\omega{\geq}0 \land d{\geq}0 \vdash [x'{:=}y][y'{:=}{-}\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2yy' \leq 0}}{\omega^2x^2{+}y^2{\leq}c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega{\geq}0 \land d{\geq}0]\, \omega^2x^2{+}y^2{\leq}c^2}$$

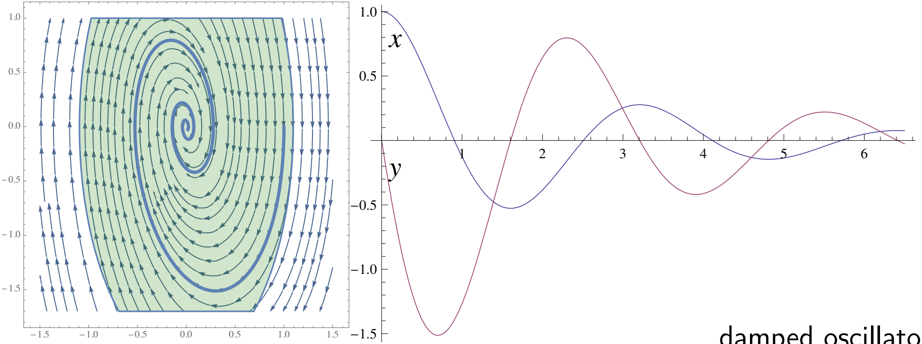




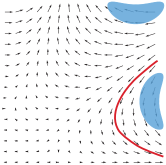
damped oscillator

Example: Differential Invariant Inequalities: Oscillator

$$\dfrac{\dfrac{\dfrac{*}{\omega{\ge}0 \land {\color{red}d{\ge}0} \vdash 2\omega^2xy + 2y(-\omega^2x - 2d\omega y) \le 0}}{\omega{\ge}0 \land {\color{red}d{\ge}0} \vdash [x'{:=}y][y'{:=}-\omega^2x - 2d\omega y]\, 2\omega^2xx' + 2yy' \le 0}}{\omega^2x^2{+}y^2{\le}c^2 \vdash [x' = y, y' = -\omega^2x - 2d\omega y \,\&\, \omega{\ge}0 \land {\color{red}d{\ge}0}]\, \omega^2x^2{+}y^2{\le}c^2}$$



damped oscillator



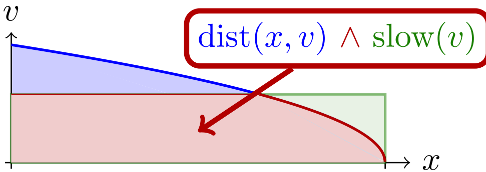
Differential Invariant

$$\text{dI}\ \frac{}{A \land B \vdash [x' = f(x)](A \land B)}$$

Differential Invariant Conjunctions

Differential Invariant

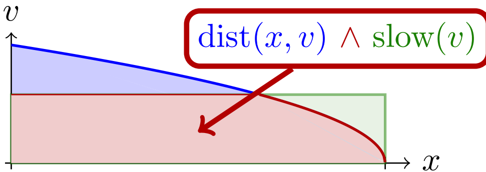
$$\text{dI}\ \frac{\vdash [x' := f(x)]((A)' \land (B)')}{A \land B \vdash [x' = f(x)](A \land B)}$$



$$\text{DI}\quad \big([x' = f(x)](A \land B) \leftrightarrow (A \land B)\big) \leftarrow [x' = f(x))]((A)' \land (B)')$$

Differential Invariant

$$\text{dI}\ \frac{\vdash [x' := f(x)]((A)' \land (B)')}{A \land B \vdash [x' = f(x)](A \land B)}$$



$$\text{DI}\ \ ([x' = f(x)](A \land B) \leftrightarrow (A \land B)) \leftarrow [x' = f(x))]((A)' \land (B)')$$

Proof (separately).

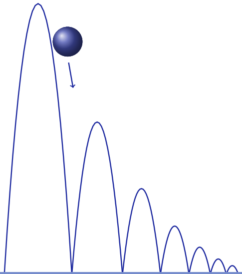
$$\text{[]}\land\text{,WL}\ \frac{\text{DI}\ \dfrac{\vdash [x' = f(x)](A)'}{A \vdash [x' = f(x)]A} \qquad \text{DI}\ \dfrac{\vdash [x' = f(x)](B)'}{B \vdash [x' = f(x)]B}}{A \land B \vdash [x' = f(x)](A \land B)}$$

□

$$\text{[]}\land\ \ [\alpha](P \land Q) \leftrightarrow [\alpha]P \land [\alpha]Q$$

$$\frac{}{2gx=2gH-v^2 \vdash [x'' = -g \,\&\, x\geq 0](2gx=2gH-v^2 \land x\geq 0)}$$

No solutions but still a proof.
Simple proof with simple arithmetic.
Independent proofs for independent questions.

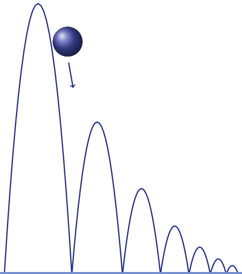


Quantum's Back for a Differential Invariant Proof

$[]\wedge \quad [\alpha](P \wedge Q) \leftrightarrow [\alpha]P \wedge [\alpha]Q$

$$[]\wedge \frac{\dfrac{}{2gx{=}2gH{-}v^2 \vdash [x''{=}{-}g \,\&\, x{\geq}0]2gx{=}2gH{-}v^2} \qquad \dfrac{}{\vdash [x''{=}{-}g \,\&\, x{\geq}0]x{\geq}0}}{2gx{=}2gH{-}v^2 \vdash [x'' = -g \,\&\, x{\geq}0](2gx{=}2gH{-}v^2 \wedge x{\geq}0)}$$

No solutions but still a proof.
Simple proof with simple arithmetic.
Independent proofs for independent questions.



Quantum's Back for a Differential Invariant Proof

$$
\dfrac{\dfrac{\dfrac{}{x\geq 0 \vdash [{\color{red}x'}{:=}v][{\color{red}v'}{:=}-g]2g{\color{red}x'} = -2v{\color{red}v'}}}{2gx{=}2gH-v^2 \vdash [x''{=}-g \;\&\; x\geq 0]2gx{=}2gH-v^2}\,\mathrm{dI} \qquad \dfrac{}{\vdash [x''{=}-g \;\&\; x\geq 0]x\geq 0}}{2gx{=}2gH-v^2 \vdash [x'' = -g \;\&\; x\geq 0](2gx{=}2gH-v^2 \wedge x\geq 0)}\,[]\wedge
$$

No solutions but still a proof.

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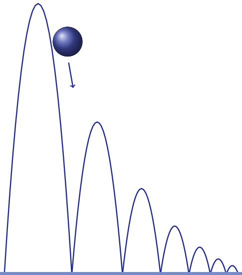
Independent proofs for independent questions.



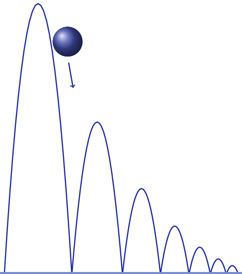
Quantum's Back for a Differential Invariant Proof

$$\dfrac{\dfrac{\dfrac{\overline{x\geq 0 \vdash 2g\color{red}{v} = -2v(\color{red}{-g})}}{x\geq 0 \vdash [x':=\color{red}{v}][v':=\color{red}{-g}]2gx' = -2vv'}[:=]}{2gx=2gH-v^2 \vdash [x''=-g\ \&\ x\geq 0]2gx=2gH-v^2}\text{dI} \qquad \overline{\vdash [x''=-g\ \&\ x\geq 0]x\geq 0}}{2gx=2gH-v^2 \vdash [x''=-g\ \&\ x\geq 0](2gx=2gH-v^2 \land x\geq 0)}[]\land$$

No solutions but still a proof.

Simple proof with simple arithmetic.

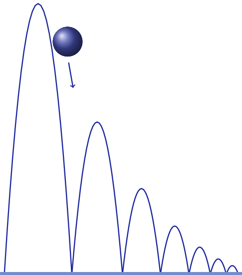
Independent proofs for independent questions.



Quantum's Back for a Differential Invariant Proof

$$\text{[]}\wedge\dfrac{\text{dI}\dfrac{[:=]\dfrac{\mathbb{R}\dfrac{*}{x\geq0 \vdash 2gv = -2v(-g)}}{x\geq0 \vdash [x':=v][v':=-g]2gx' = -2vv'}}{2gx=2gH-v^2 \vdash [x''=-g \,\&\, x\geq0]2gx=2gH-v^2} \qquad \dfrac{}{\vdash [x''=-g \,\&\, x\geq0]x\geq0}}{2gx=2gH-v^2 \vdash [x'' = -g \,\&\, x\geq0](2gx=2gH-v^2 \wedge x\geq0)}$$

No solutions but still a proof.
Simple proof with simple arithmetic.
Independent proofs for independent questions.



Quantum's Back for a Differential Invariant Proof

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{*}{x{\geq}0 \vdash 2gv = -2v(-g)}\,\mathbb{R}
    }{x{\geq}0 \vdash [x' {:=} v][v' {:=} -g]2gx' = -2vv'}\,[{:=}]
  }{2gx{=}2gH{-}v^2 \vdash [x''{=}{-}g \,\&\, x{\geq}0]2gx{=}2gH{-}v^2}\,\text{dI}
  \quad
  \dfrac{
    \dfrac{}{\color{red}{x{\geq}0} \vdash x{\geq}0}
  }{\vdash [x''{=}{-}g \,\&\, \color{red}{x{\geq}0}]x{\geq}0}\,\text{dW}
}{2gx{=}2gH{-}v^2 \vdash [x'' = -g \,\&\, x{\geq}0](2gx{=}2gH{-}v^2 \land x{\geq}0)}\,[]\land
$$

No solutions but still a proof.

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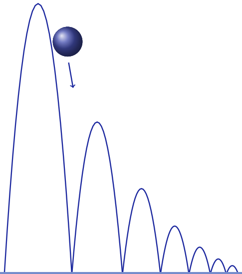

$$\dfrac{\dfrac{\dfrac{\dfrac{*}{x\geq0 \vdash 2gv = -2v(-g)}\ \mathbb{R}}{x\geq0 \vdash [x':=v][v':=-g]2gx' = -2vv'}\ [:=]}{2gx{=}2gH{-}v^2 \vdash [x''{=}{-}g \,\&\, x\geq0]2gx{=}2gH{-}v^2}\ \mathrm{dI} \qquad \dfrac{\dfrac{*}{x\geq0 \vdash x\geq0}\ \mathrm{id}}{\vdash [x''{=}{-}g \,\&\, x\geq0]x\geq0}\ \mathrm{dW}}{2gx{=}2gH{-}v^2 \vdash [x'' = -g \,\&\, x\geq0](2gx{=}2gH{-}v^2 \wedge x\geq0)}\ []\wedge$$

No solutions but still a proof.
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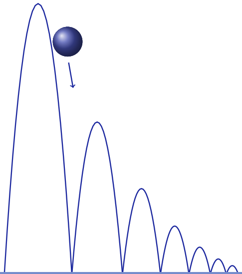
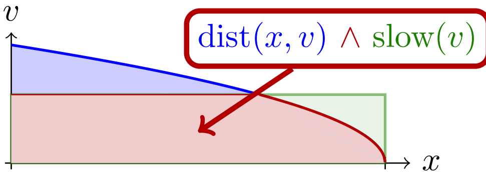


Differential Invariant

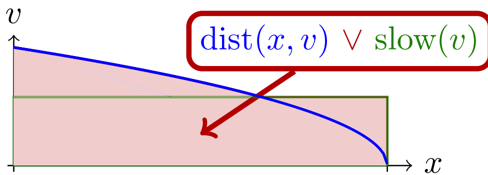
$$\text{dI}\quad \frac{\vdash [x' := f(x)]((A)' \land (B)')}{A \land B \vdash [x' = f(x)](A \land B)}$$



$$\text{DI}\quad \big([x' = f(x)](A \land B) \leftrightarrow (A \land B)\big) \leftarrow [x' = f(x))]((A)' \land (B)')$$

Differential Invariant

$$\text{dI}\quad \frac{\vdash [x' := f(x)]((A)' \lor (B)')}{A \lor B \vdash [x' = f(x)](A \lor B)}$$

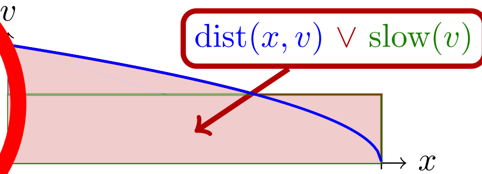


$$\text{DI}\quad \big([x' = f(x)](A \lor B) \leftrightarrow (A \lor B)\big) \leftarrow [x' = f(x))]((A)' \lor (B)')$$

Differential Invariant Disjunctions

Differential Invariant

$$\text{dI}\quad \frac{\vdash [x' := f(x)]((A)' \vee (B)')}{A \vee B \vdash [x' = f(x)](A \vee B)}$$

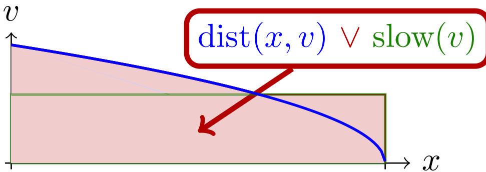


$$\text{DI}\quad \big([x' = f(x)](A \vee B) \leftrightarrow (A \vee B)\big) \leftarrow [x' = f(x))]((A)' \vee (B)')$$

Differential Invariant Disjunctions

Differential Invariant

$$\text{dI}\ \frac{\vdash [x' := f(x)]((A)' \land (B)')}{A \lor B \vdash [x' = f(x)](A \lor B)}$$

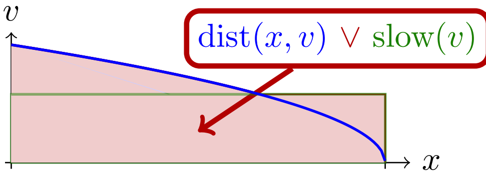


$$\text{DI}\quad \big([x' = f(x)](A \lor B) \leftrightarrow (A \lor B)\big) \leftarrow [x' = f(x))]((A)' \land (B)')$$

Differential Invariant Disjunctions

Differential Invariant

$$\text{dI}\ \frac{\vdash [x' := f(x)]((A)' \land (B)')}{A \lor B \vdash [x' = f(x)](A \lor B)}$$



$$\text{DI}\quad ([x' = f(x)](A \lor B) \leftrightarrow (A \lor B)) \leftarrow [x' = f(x))]((A)' \land (B)')$$

Proof (separately).

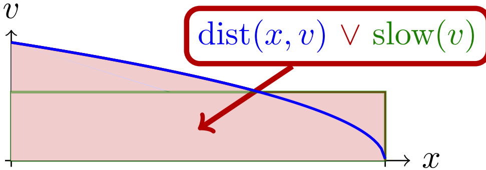
$$\text{∨L}\ \dfrac{\text{MR}\ \dfrac{\dfrac{*}{A \vdash A \lor B} \quad \text{DI}\ \dfrac{\vdash [x'=f(x)](A)'}{A \vdash [x'=f(x)]A}}{A \vdash [x' = f(x)](A \lor B)} \quad \text{MR}\ \dfrac{\dfrac{*}{B \vdash A \lor B} \quad \text{DI}\ \dfrac{\vdash [x'=f(x)](B)'}{B \vdash [x'=f(x)]B}}{B \vdash [x' = f(x)](A \lor B)}}{A \lor B \vdash [x' = f(x)](A \lor B)}$$

□

Differential Invariant Disjunctions

Differential Invariant

$$\text{dI}\ \frac{\vdash [x' := f(x)]((A)' \land (B)')}{A \lor B \vdash [x' = f(x)](A \lor B)}$$



$$\text{DI}\quad ([x' = f(x)](A \lor B) \leftrightarrow (A \lor B)) \leftarrow [x' = f(x))]((A)' \land (B)')$$

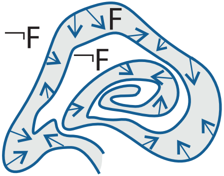
Proof (separately).

$$\text{∨L}\ \dfrac{\text{MR}\ \dfrac{\dfrac{*}{A \vdash A \lor B} \quad \text{DI}\ \dfrac{\vdash [x'=f(x)](A)'}{A \vdash [x'=f(x)]A}}{A \vdash [x' = f(x)](A \lor B)} \quad \text{MR}\ \dfrac{\dfrac{*}{B \vdash A \lor B} \quad \text{DI}\ \dfrac{\vdash [x'=f(x)](B)'}{B \vdash [x'=f(x)]B}}{B \vdash [x' = f(x)](A \lor B)}}{A \lor B \vdash [x' = f(x)](A \lor B)}$$

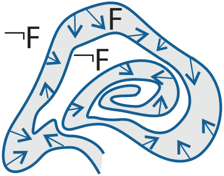
□

$$[]\land\quad [\alpha](P \land Q) \leftrightarrow [\alpha]P \land [\alpha]Q$$

Assuming Invariants

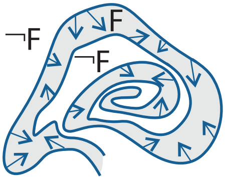


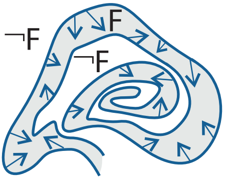
$$\frac{Q \rightarrow [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$



$$\frac{F \land Q \rightarrow [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

$$\text{loop} \quad \frac{F \vdash [\alpha]F}{F \vdash [\alpha^*]F}$$



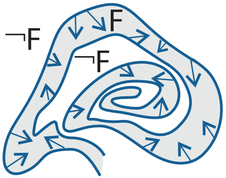


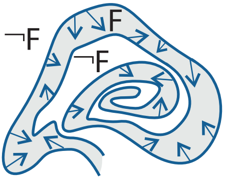
$$\frac{Q \rightarrow [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

$$\frac{\color{red}{F} \land Q \rightarrow [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Example (Restrictions)

$$\frac{}{v^2 - 2v + 1 = 0 \vdash [v' = w, w' = -v]\, v^2 - 2v + 1 = 0}$$



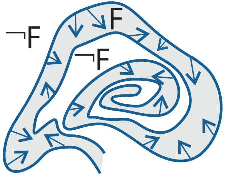


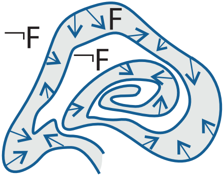
$$\frac{Q \rightarrow [x' := f(x)](F)'}{F \vdash [x' = f(x)\,\&\,Q]F}$$

$$\frac{{\color{red}F} \wedge Q \rightarrow [x' := f(x)](F)'}{F \vdash [x' = f(x)\,\&\,Q]F}$$

Example (Restrictions)

$$\frac{\overline{v^2 - 2v + 1 = 0 \vdash [v' := w][w' := -v]\,2vv' - 2v' = 0}}{v^2 - 2v + 1 = 0 \vdash [v' = w, w' = -v]\,v^2 - 2v + 1 = 0}$$



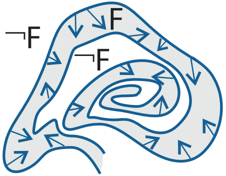


$$\frac{Q \rightarrow [x' := f(x)](F)'}{F \vdash [x' = f(x)\,\&\, Q]F} \qquad \frac{F \wedge Q \rightarrow [x' := f(x)](F)'}{F \vdash [x' = f(x)\,\&\, Q]F}$$

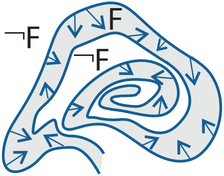
Example (Restrictions)

$$\cfrac{\cfrac{\cfrac{}{v^2 - 2v + 1 = 0 \vdash 2vw - 2w = 0}}{v^2 - 2v + 1 = 0 \vdash [v':=w][w':=-v]\, 2vv' - 2v' = 0}}{v^2 - 2v + 1 = 0 \vdash [v' = w, w' = -v]\, v^2 - 2v + 1 = 0}$$

Assuming Invariants



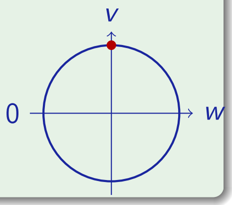
$$\frac{Q \rightarrow [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$



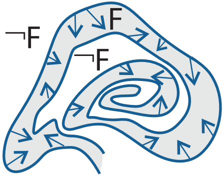
$$\frac{F \land Q \rightarrow [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Example (Restrictions)

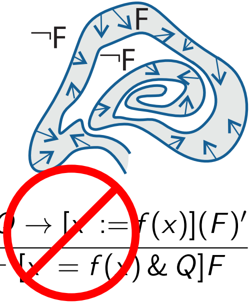
$$\frac{\frac{\frac{}{v^2 - 2v + 1 = 0 \vdash 2vw - 2w = 0}}{v^2 - 2v + 1 = 0 \vdash [v' := w][w' := -v]\, 2vv' - 2v' = 0}}{v^2 - 2v + 1 = 0 \vdash [v' = w, w' = -v]\, v^2 - 2v + 1 = 0}$$



Assuming Invariants



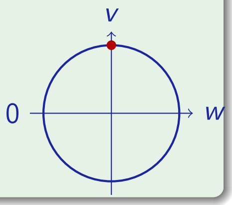
$$\frac{Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$



$$\frac{F \land Q \to [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Example (Restrictions are unsound!)

$$\cfrac{\cfrac{\cfrac{(\text{unsound})}{v^2 - 2v + 1 = 0 \vdash 2vw - 2w = 0}}{v^2 - 2v + 1 = 0 \vdash [v':=w][w':=-v]\,2vv' - 2v' = 0}}{v^2 - 2v + 1 = 0 \vdash [v' = w, w' = -v]\,v^2 - 2v + 1 = 0}$$

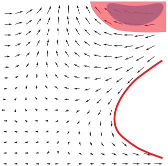


Outline

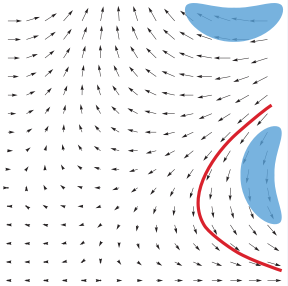
Differential Cuts

Differential Cut

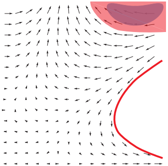
$$\frac{}{F \vdash [x' = f(x)]F}$$



Differential Cut



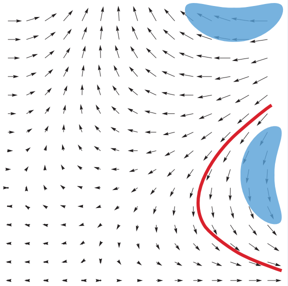
Differential Cuts



Differential Cut

$$\frac{F \vdash [x' = f(x)]C}{F \vdash [x' = f(x)]F}$$

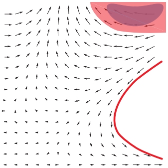
Differential Cut



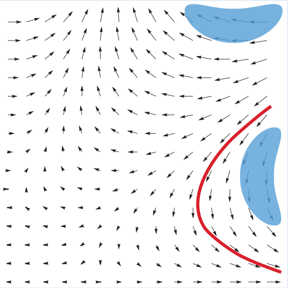
Differential Cuts

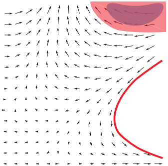
Differential Cut

$$\frac{F \vdash [x' = f(x)]C \qquad F \vdash [x' = f(x) \,\&\, C]F}{F \vdash [x' = f(x)]F}$$



Differential Cut

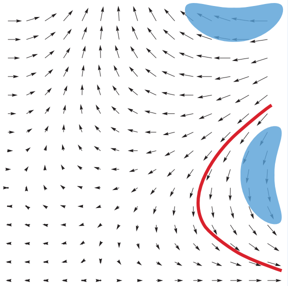


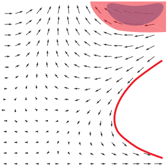


Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \land C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Differential Cut

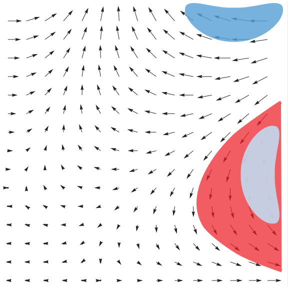


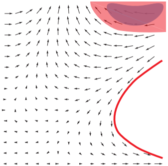


Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \land C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Differential Cut

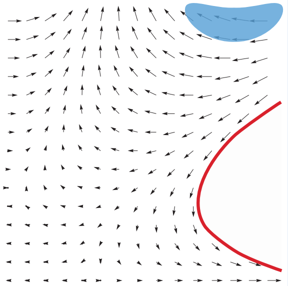




Differential Cut

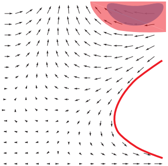
$$\frac{F \vdash [x' = f(x)\,\&\,Q]C \quad F \vdash [x' = f(x)\,\&\,Q \land C]F}{F \vdash [x' = f(x)\,\&\,Q]F}$$

Differential Cut

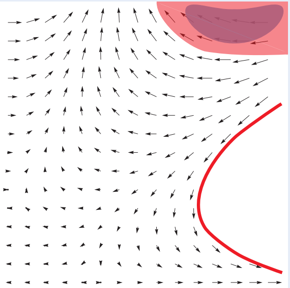


Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \land C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$

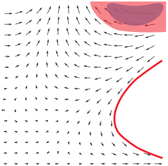


Differential Cut

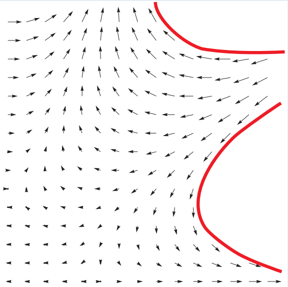


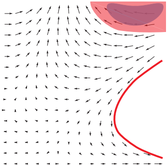
Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \land C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$



Differential Cut

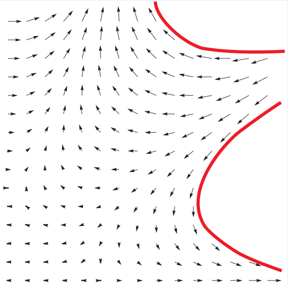




Differential Cut

$$\frac{F \vdash [x' = f(x)\,\&\,Q]C \quad F \vdash [x' = f(x)\,\&\,Q \land C]F}{F \vdash [x' = f(x)\,\&\,Q]F}$$

Differential Cut



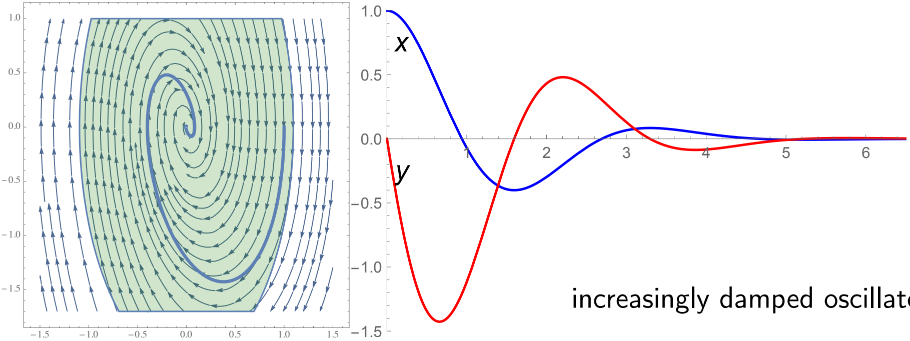
Proof (Soundness).

Let $\varphi \models x' = f(x) \land Q$ starting in $\omega \in [\![F]\!]$.
$\omega \in [\![[x' = f(x)\,\&\,Q]C]\!]$ by left premise.
Thus, $\varphi \models x' = f(x) \land Q \land C$.
Thus, $\varphi(r) \in [\![F]\!]$ by second premise. □

$$\text{dC}\ \frac{}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\geq 0]\,\omega^2x^2+y^2\leq c^2}$$

Differential Cut Example: Increasingly Damped Oscillator

$$\text{dC}\ \frac{}{\omega^2x^2+y^2\le c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\ge 0]\,\omega^2x^2+y^2\le c^2}$$



increasingly damped oscillator

Differential Cut Example: Increasingly Damped Oscillator

$$\text{dI}\ \dfrac{}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\geq 0\land d\geq 0]\,\omega^2x^2+y^2\leq c^2}$$

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$$\text{dI}\ \dfrac{}{d\ge0 \vdash [x' = y, y' = -\omega^2x - 2d\omega y, d'=7\ \&\ \omega\ge0]\ d\ge0}$$

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$$\text{[:=]}\ \dfrac{}{\text{dI}\ \dfrac{\omega\ge0 \vdash [d':=7]\, d'\ge0}{d\ge0 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\ge0]\, d\ge0}}$$

increasingly damped oscillator

$$\text{dI}\ \dfrac{}{\text{dC}\ \dfrac{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq 0 \land d\geq 0]\, \omega^2x^2+y^2\leq c^2}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq 0]\, \omega^2x^2+y^2\leq c^2}}$$

$$\text{dI}\ \dfrac{[:=]\ \dfrac{\mathbb{R}\ \dfrac{}{\omega\geq 0 \vdash 7\geq 0}}{\omega\geq 0 \vdash [d':=7]\, d'\geq 0}}{d\geq 0 \vdash [x' = y, y' = -\omega^2x - 2d\omega y, d'=7 \,\&\, \omega\geq 0]\, d\geq 0}$$

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$$\cfrac{\mathbb{R}\;\cfrac{\cfrac{*}{\omega\geq 0 \vdash 7\geq 0}}{[:=]\;\omega\geq 0 \vdash [d':=7]\, d'\geq 0}}{\text{dI}\;d\geq 0 \vdash [x' = y, y' = -\omega^2x - 2d\omega y, d'=7 \,\&\, \omega\geq 0]\, d\geq 0}$$

ask

increasingly damped oscillator

Differential Cut Example: Increasingly Damped Oscillator

$$\dfrac{\dfrac{\dfrac{}{\omega\geq0\land d\geq0 \vdash [x':=y][y':=-\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2yy' \leq 0}\,[:=]}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq0\land d\geq0]\,\omega^2x^2+y^2\leq c^2}\,\text{dI}}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq0]\,\omega^2x^2+y^2\leq c^2}\,\text{dC}$$

$$\dfrac{\dfrac{\dfrac{*}{\omega\geq0 \vdash 7\geq0}\,\mathbb{R}}{\omega\geq0 \vdash [d':=7]\, d'\geq0}\,[:=]}{d\geq0 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7\,\&\,\omega\geq0]\, d\geq0}\,\text{dI}$$

increasingly damped oscillator

Differential Cut Example: Increasingly Damped Oscillator

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\omega\geq0\land d\geq0 \vdash 2\omega^2xy + 2y(-\omega^2x - 2d\omega y) \leq 0}\ \mathbb{R}}{\omega\geq0\land d\geq0 \vdash [x':=y][y':=-\omega^2x - 2d\omega y]\, 2\omega^2xx' + 2yy' \leq 0}\ [:=]}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq0\land d\geq0]\,\omega^2x^2+y^2\leq c^2}\ \mathrm{dI}}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq0]\,\omega^2x^2+y^2\leq c^2}\ \mathrm{dC}
$$

$$
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increasingly damped oscillator

$$\dfrac{\dfrac{\dfrac{*}{\omega\geq0\land d\geq0 \vdash 2\omega^2xy + 2y(-\omega^2x - 2d\omega y) \leq 0}\ \mathbb{R}}{\omega\geq0\land d\geq0 \vdash [x':=y][y':=-\omega^2x - 2d\omega y]\,2\omega^2xx' + 2yy' \leq 0}\ [:=]}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq0\land d\geq0]\,\omega^2x^2+y^2\leq c^2}\ \mathrm{dI}$$

$$\mathrm{dC}\quad \omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq0]\,\omega^2x^2+y^2\leq c^2$$

DC

$$\dfrac{\dfrac{\dfrac{*}{\omega\geq0 \vdash 7\geq0}\ \mathbb{R}}{\omega\geq0 \vdash [d':=7]\,d'\geq0}\ [:=]}{d\geq0 \vdash [x' = y, y' = -\omega^2x - 2d\omega y, d'=7\,\&\,\omega\geq0]\,d\geq0}\ \mathrm{dI}$$

increasingly damped oscillator

$$\dfrac{\dfrac{\dfrac{\dfrac{*}{\omega\geq0\land d\geq0 \vdash 2\omega^2xy+2y(-\omega^2x-2d\omega y)\leq 0}\ \mathbb{R}}{\omega\geq0\land d\geq0 \vdash [x':=y][y':=-\omega^2x-2d\omega y]\,2\omega^2xx'+2yy'\leq 0}\ [:=]}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y,y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\geq0\land d\geq0]\,\omega^2x^2+y^2\leq c^2}\ \text{dI}}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y,y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\geq0]\,\omega^2x^2+y^2\leq c^2}\ \text{dC}$$

init

$$\dfrac{\dfrac{\dfrac{*}{\omega\geq0 \vdash 7\geq0}\ \mathbb{R}}{\omega\geq0 \vdash [d':=7]\,d'\geq0}\ [:=]}{d\geq0 \vdash [x'=y,y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\geq0]\,d\geq0}\ \text{dI}$$

$$*$$

$$\mathbb{R}\quad \omega\geq0\land d\geq0 \vdash 2\omega^2xy + 2y(-\omega^2x - 2d\omega y) \leq 0$$

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$$\text{dI}\quad \omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq0\land d\geq0]\, \omega^2x^2+y^2\leq c^2$$

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$$[:=]\quad \omega\geq0 \vdash [d':=7]\, d'\geq0$$

$$\text{dI}\quad d\geq0 \vdash [x' = y, y' = -\omega^2x - 2d\omega y, d'=7 \,\&\, \omega\geq0]\, d\geq0$$

Could repeatedly diffcut in formulas to help the proof

$$\text{dC}\ \frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

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$$\dfrac{\text{dI}\quad \dfrac{}{x^3 \geq -1 \vdash [x' = (x-2)^4 + y^5, y' = y^2 \,\&\, y^5 \geq 0]x^3 \geq -1} \;\triangleright}{\text{dC}\quad x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

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$$\dfrac{\dfrac{\dfrac{}{y^5 \geq 0 \vdash [x':=(x-2)^4+y^5][y':=y^2]3x^2x' \geq 0}[:=]}{x^3 \geq -1 \vdash [x'=(x-2)^4+y^5, y'=y^2 \,\&\, y^5 \geq 0]x^3 \geq -1 \quad \triangleright}\text{dI}}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x'=(x-2)^4+y^5, y'=y^2]x^3 \geq -1}\text{dC}$$

$$\dfrac{\dfrac{\dfrac{*}{\vdash 5y^4y^2 \geq 0}\mathbb{R}}{\vdash [x':=(x-2)^4+y^5][y':=y^2]5y^4y' \geq 0}[:=]}{y^5 \geq 0 \vdash [x'=(x-2)^4+y^5, y'=y^2]y^5 \geq 0}\text{dI}$$

$$\mathbb{R}\ \dfrac{}{y^5 \geq 0 \vdash 3x^2((x-2)^4 + y^5) \geq 0}$$

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Outline

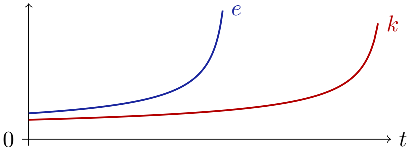
Soundness Proof: Differential Invariants

Lemma (Differential lemma) (Differential value vs. Time-derivative)

$$\varphi \models x' = f(x) \land Q \text{ for } r > 0 \;\Rightarrow\; \forall 0 \le z \le r \;\; \varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$

Differential Invariant

$$\text{DI} \quad \begin{array}{l}([x' = f(x)]e \ge 0 \leftrightarrow e \ge 0) \\ \leftarrow [x' = f(x)](e)' \ge 0\end{array}$$



Proof ($\ge$ rate of change from $\ge$ initial value. Case $r = 0$ is easier.)

$h(t) \overset{\text{def}}{=} \varphi(t)[\![e]\!]$ is differentiable on $[0, r]$ if $r > 0$ by diff. lemma.

$\frac{dh(t)}{dt}(z) = \frac{d\varphi(t)[\![e]\!]}{dt}(z) = \varphi(z)[\![(e)']\!] \ge 0$ by lemma + assume for all z.

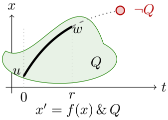
$h(r) - \underbrace{h(0)}_{\ge 0} = \underbrace{(r - 0)}_{>0} \underbrace{\frac{dh(t)}{dt}(\xi)}_{\ge 0} \ge 0$ by mean-value theorem for some ξ. □

Outline

Summary: Differential Invariants for Differential Equations

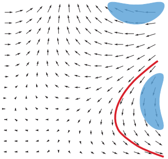
Differential Weakening

$$\frac{Q \vdash F}{\Gamma \vdash [x' = f(x) \,\&\, Q]F}$$



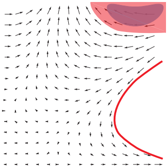
Differential Invariant

$$\frac{Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$



Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \land C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$



Summary: Differential Invariants for Differential Equations

Differential Weakening

$$\frac{Q \vdash F}{\Gamma \vdash [x' = f(x) \,\&\, Q]F}$$

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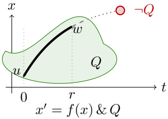
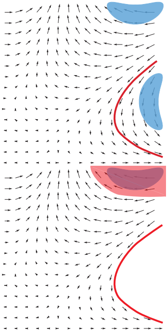
$$\frac{Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \land C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$





DW $[x' = f(x) \,\&\, Q]F \leftrightarrow [x' = f(x) \,\&\, Q](Q \to F)$

DI $\big([x' = f(x) \,\&\, Q]F \leftrightarrow [?Q]F\big) \leftarrow \big(Q \to [x' = f(x) \,\&\, Q](F)'\big)$

DC $\big([x' = f(x) \,\&\, Q]F \leftrightarrow [x' = f(x) \,\&\, Q \land C]F\big) \leftarrow [x' = f(x) \,\&\, Q]C$

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