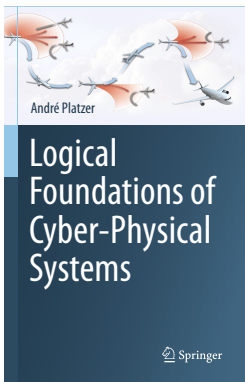
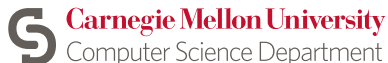


05: Dynamical Systems & Dynamic Axioms

Logical Foundations of Cyber-Physical Systems



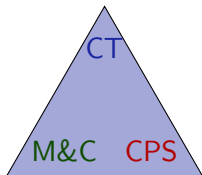
André Platzer



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 - Soundness
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- 6 Summary

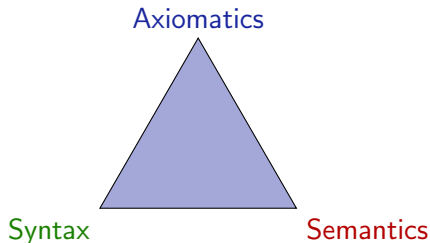
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rigorous reasoning about CPS
dL as verification language



cyber+physics interaction
relate discrete+continuous

align semantics+reasoning
operational CPS effects



Syntax defines the notation

What problems are we allowed to write down?

Semantics what carries meaning.

What real or mathematical objects does the syntax stand for?

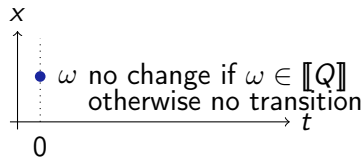
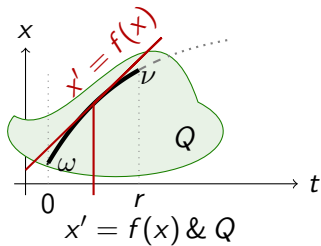
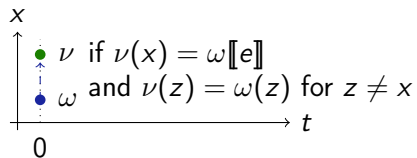
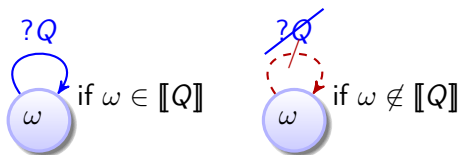
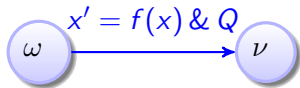
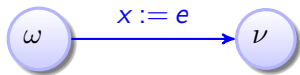
Axiomatics internalizes semantic relations into universal syntactic transformations.

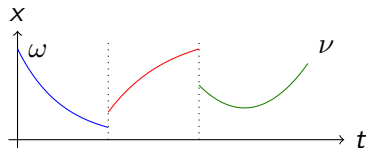
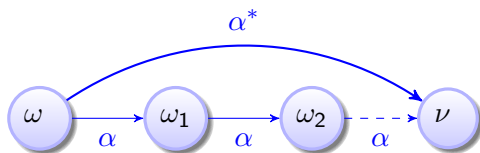
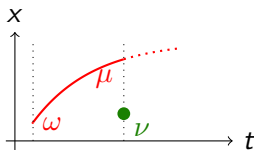
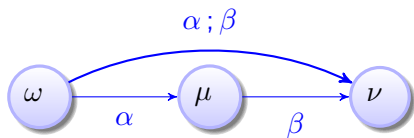
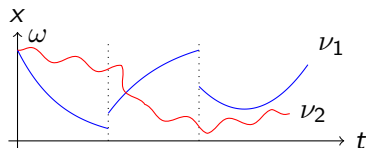
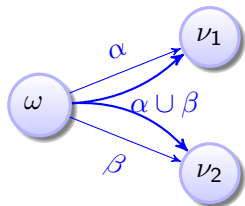
How does the semantics of A relate to semantics of $A \wedge B$, syntactically? If A is true, is $A \wedge B$ true, too? Conversely?

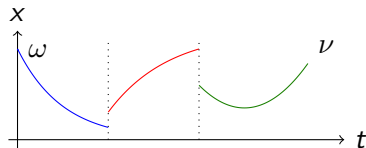
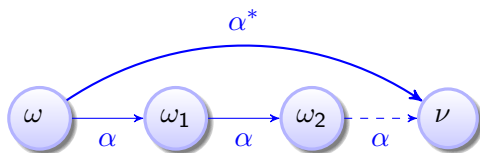
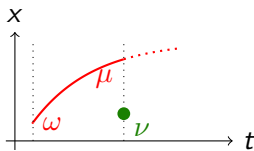
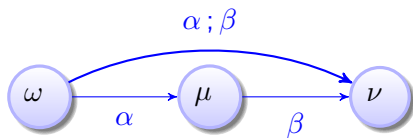
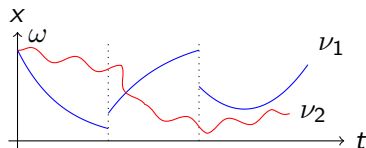
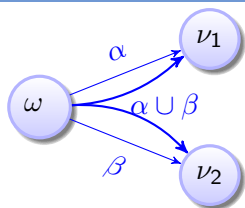
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Logical guiding principle: Compositionality

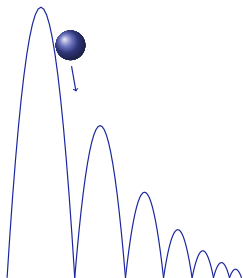
- 1 Every CPS is modeled by a hybrid program (or game ...)
- 2 All hybrid programs are combinations of simpler hybrid programs (by a program operator such as \cup and $;$ and $*$)
- 3 All CPS can be analyzed if only we identify one suitable analysis technique for each operator.
- 4 Analysis of a big CPS is an analysis chain for all individual parts.







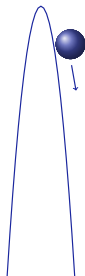
compositional semantics \Rightarrow compositional proofs!



Example (Quantum the Bouncing Ball)

$$0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \rightarrow$$

$$[(\{x' = v, v' = -g \ \& \ x \geq 0\}; (?x=0; v := -cv \cup ?x \neq 0))^*] (0 \leq x \wedge x \leq H)$$



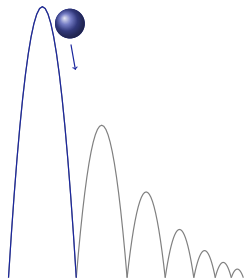
Example (Quantum the Bouncing Ball)

(Single-hop)

$$0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \rightarrow$$

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Removing the repetition grotesquely changes the behavior to a single hop



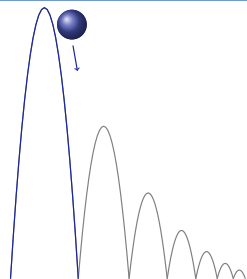
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How to prove ;

Example (Quantum the Bouncing Ball)

(Single-hop)

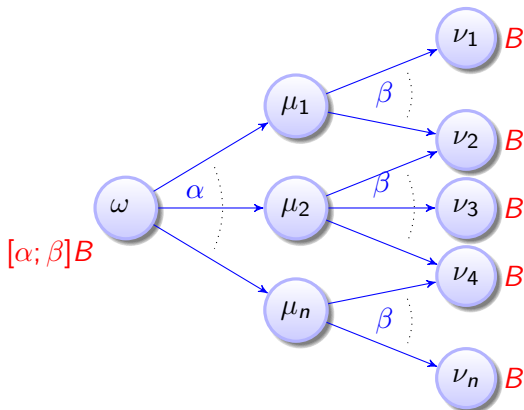
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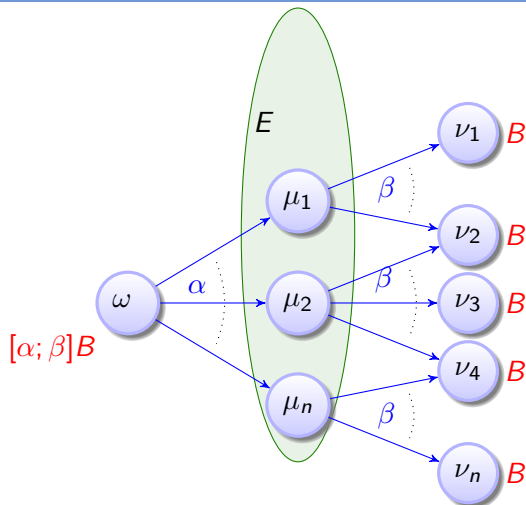
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$$H; \frac{}{A \rightarrow [\alpha; \beta]B}$$

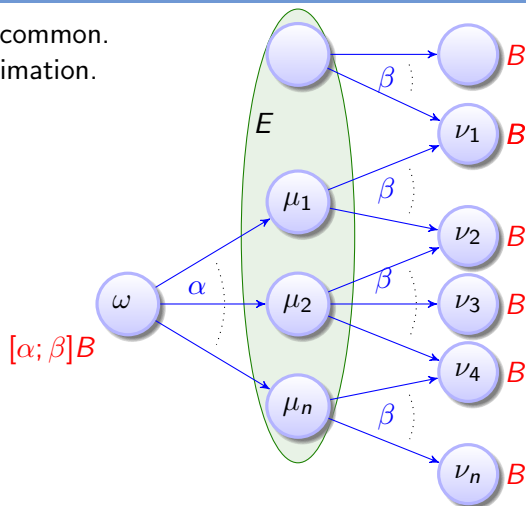


$$H; \frac{}{A \rightarrow [\alpha; \beta]B}$$



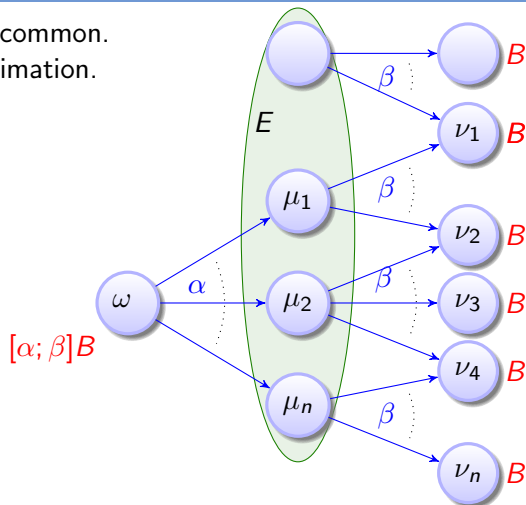
E summarizes what μ_i have in common.
 E is often imprecise overapproximation.

$$H; \frac{A \rightarrow [\alpha]E \quad E \rightarrow [\beta]B}{A \rightarrow [\alpha; \beta]B}$$



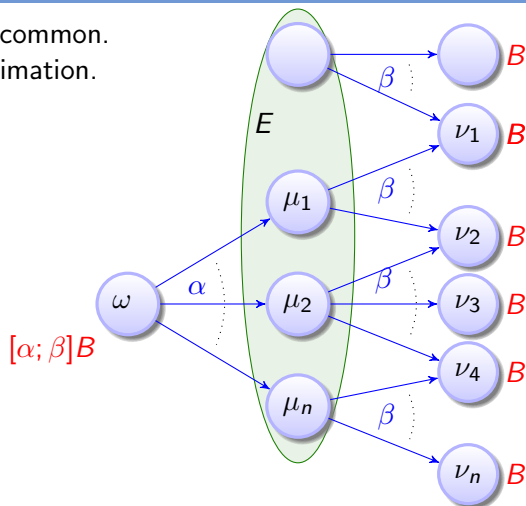
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 Just need to find this E ...

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Example (Quantum the Bouncing Ball)

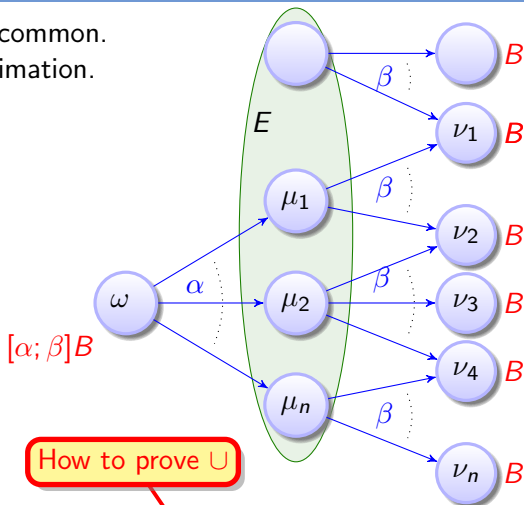
(Single-hop)

$$0 \leq x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \rightarrow [x' = v, v' = -g \ \& \ x \geq 0] E$$

$$E \rightarrow [?x=0; v := -cv \cup ?x \neq 0] (0 \leq x \wedge x \leq H)$$

E summarizes what μ_i have in common.
 E is often imprecise overapproximation.
 Just need to find this E ...

$$H; \frac{A \rightarrow [\alpha]E \quad E \rightarrow [\beta]B}{A \rightarrow [\alpha; \beta]B}$$



Example (Quantum the Bouncing Ball)

(Single-hop)

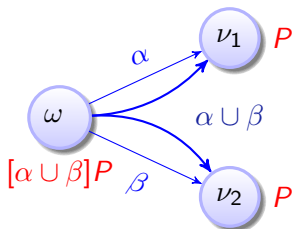
$$0 \leq x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \rightarrow [x' = v, v' = -g \ \& \ x \geq 0] E$$

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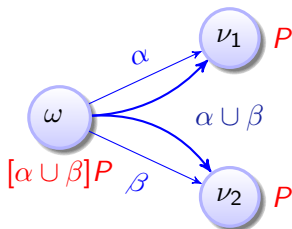
Semantics

$$\llbracket \alpha \cup \beta \rrbracket = \llbracket \alpha \rrbracket \cup \llbracket \beta \rrbracket$$



Semantics

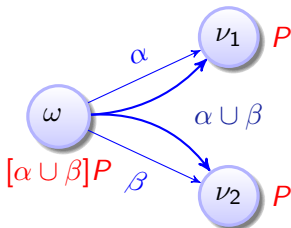
$$\llbracket \alpha \cup \beta \rrbracket = \llbracket \alpha \rrbracket \cup \llbracket \beta \rrbracket$$



- $\omega \in \llbracket [\alpha \cup \beta]P \rrbracket$ iff $\nu \in \llbracket P \rrbracket$ for all ν with $(\omega, \nu) \in \llbracket \alpha \cup \beta \rrbracket = \llbracket \alpha \rrbracket \cup \llbracket \beta \rrbracket$

Semantics

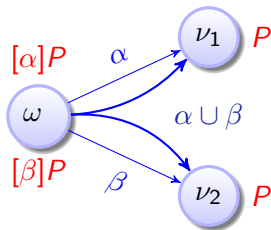
$$\llbracket \alpha \cup \beta \rrbracket = \llbracket \alpha \rrbracket \cup \llbracket \beta \rrbracket$$



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- Then $\nu \in \llbracket P \rrbracket$ for all ν with $(\omega, \nu) \in \llbracket \alpha \rrbracket$
- and $\nu \in \llbracket P \rrbracket$ for all ν with $(\omega, \nu) \in \llbracket \beta \rrbracket$

Semantics

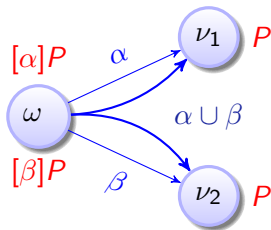
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Semantics

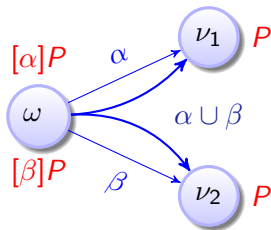
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- And vice versa.

Semantics

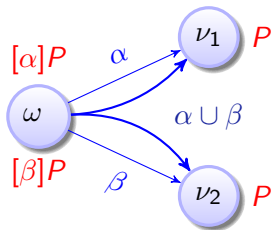
$$\llbracket \alpha \cup \beta \rrbracket = \llbracket \alpha \rrbracket \cup \llbracket \beta \rrbracket$$



- $\omega \in \llbracket [\alpha \cup \beta]P \rrbracket$ iff $\nu \in \llbracket P \rrbracket$ for all ν with $(\omega, \nu) \in \llbracket \alpha \cup \beta \rrbracket = \llbracket \alpha \rrbracket \cup \llbracket \beta \rrbracket$
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- And vice versa. So $\omega \in \llbracket [\alpha \cup \beta]P \rrbracket \leftrightarrow \llbracket [\alpha]P \rrbracket \wedge \llbracket [\beta]P \rrbracket$

Semantics

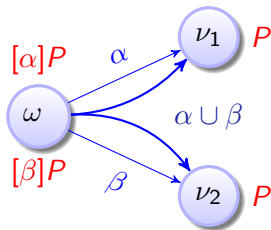
$$\llbracket \alpha \cup \beta \rrbracket = \llbracket \alpha \rrbracket \cup \llbracket \beta \rrbracket$$



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- And vice versa. So $\omega \in \llbracket [\alpha \cup \beta]P \rrbracket \leftrightarrow \llbracket [\alpha]P \rrbracket \wedge \llbracket [\beta]P \rrbracket$ for all states ω

Semantics

$$\llbracket \alpha \cup \beta \rrbracket = \llbracket \alpha \rrbracket \cup \llbracket \beta \rrbracket$$

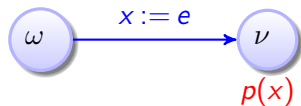


- $\omega \in \llbracket [\alpha \cup \beta]P \rrbracket$ iff $\nu \in \llbracket P \rrbracket$ for all ν with $(\omega, \nu) \in \llbracket \alpha \cup \beta \rrbracket = \llbracket \alpha \rrbracket \cup \llbracket \beta \rrbracket$
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- and $\nu \in \llbracket P \rrbracket$ for all ν with $(\omega, \nu) \in \llbracket \beta \rrbracket$ i.e., $\omega \in \llbracket [\beta]P \rrbracket$
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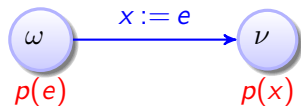
Lemma

$\llbracket \cup \rrbracket \llbracket [\alpha \cup \beta]P \rrbracket \leftrightarrow \llbracket [\alpha]P \rrbracket \wedge \llbracket [\beta]P \rrbracket$ is a sound axiom, i.e., all its instances valid.

$[\text{:=}] \quad [x := e]p(x) \leftrightarrow$

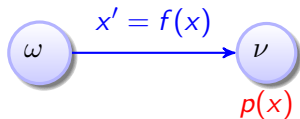
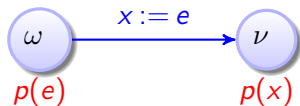


$$[:=] \quad [x := e]p(x) \leftrightarrow p(e)$$



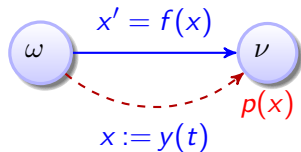
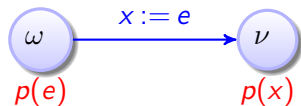
$$[:=] \quad [x := e]p(x) \leftrightarrow p(e)$$

$$['] \quad [x' = f(x)]p(x) \leftrightarrow$$



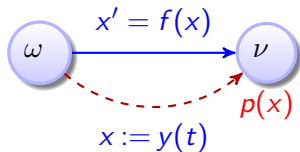
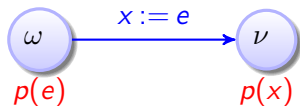
$$[:=] \quad [x := e]p(x) \leftrightarrow p(e)$$

$$['] \quad [x' = f(x)]p(x) \leftrightarrow [x := y(t)]p(x)$$

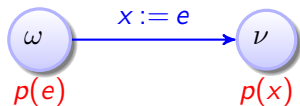


$$[:=] \quad [x := e]p(x) \leftrightarrow p(e)$$

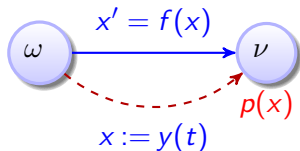
$$['] \quad [x' = f(x)]p(x) \leftrightarrow \forall t \geq 0 [x := y(t)]p(x)$$



$$[:=] [x := e]p(x) \leftrightarrow p(e)$$

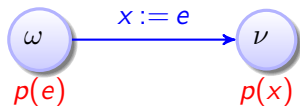


$$['] [x' = f(x)]p(x) \leftrightarrow \forall t \geq 0 [x := y(t)]p(x)$$

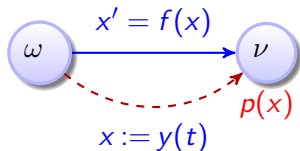


$$['] [x' = f(x) \& q(x)]p(x) \leftrightarrow \forall t \geq 0 ([x := y(t)]p(x))$$

$$[:=] [x := e]p(x) \leftrightarrow p(e)$$

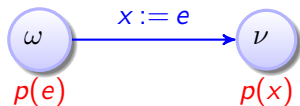


$$['] [x' = f(x)]p(x) \leftrightarrow \forall t \geq 0 [x := y(t)]p(x)$$

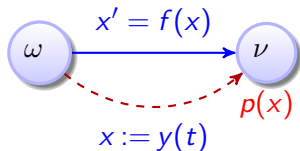


$$['] [x' = f(x) \& q(x)]p(x) \leftrightarrow \forall t \geq 0 (\forall 0 \leq s \leq t q(y(s)) \rightarrow [x := y(t)]p(x))$$

$$[:=] [x := e]p(x) \leftrightarrow p(e)$$



$$['] [x' = f(x)]p(x) \leftrightarrow \forall t \geq 0 [x := y(t)]p(x)$$



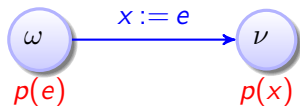
$$['] [x' = f(x) \& q(x)]p(x) \leftrightarrow \forall t \geq 0 (\forall 0 \leq s \leq t q(y(s)) \rightarrow [x := y(t)]p(x))$$

$$[?] [?Q]P \leftrightarrow$$

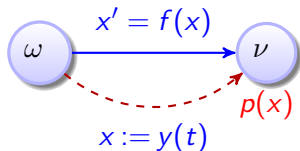


if $\omega \in [Q]$

$$[:=] [x := e]p(x) \leftrightarrow p(e)$$



$$['] [x' = f(x)]p(x) \leftrightarrow \forall t \geq 0 [x := y(t)]p(x)$$



$$['] [x' = f(x) \& q(x)]p(x) \leftrightarrow \forall t \geq 0 (\forall 0 \leq s \leq t q(y(s)) \rightarrow [x := y(t)]p(x))$$

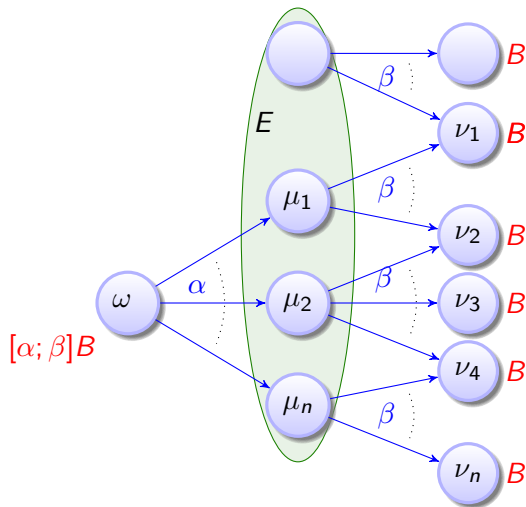
$$[?] [?Q]P \leftrightarrow (Q \rightarrow P)$$



if $\omega \in [Q]$

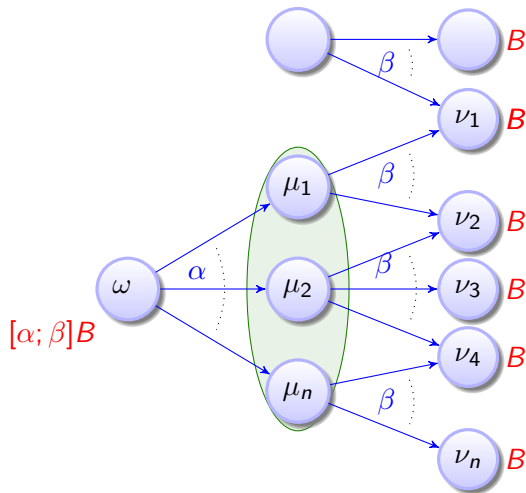
What is the most precise E summary?

$$H; \frac{A \rightarrow [\alpha]E \quad E \rightarrow [\beta]B}{A \rightarrow [\alpha; \beta]B}$$



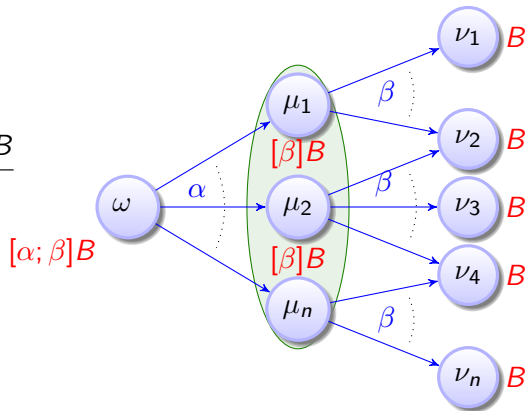
What is the most precise E summary?

$$H; \frac{A \rightarrow [\alpha]E \quad E \rightarrow [\beta]B}{A \rightarrow [\alpha; \beta]B}$$



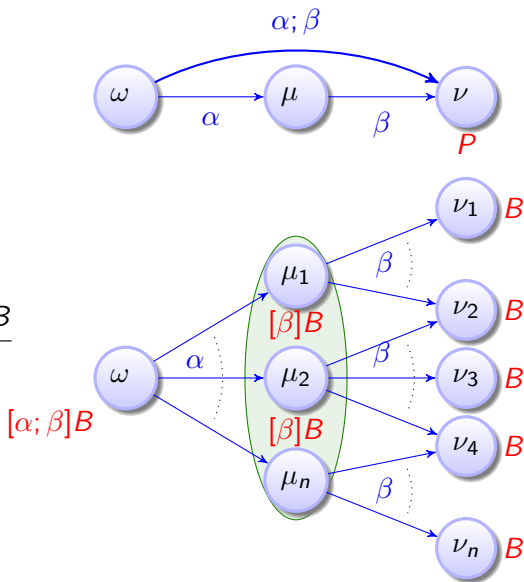
What is the most precise E summary? $[\beta]B$

$$H; \frac{A \rightarrow [\alpha][\beta]B \quad [\beta]B \rightarrow [\beta]B}{A \rightarrow [\alpha; \beta]B}$$



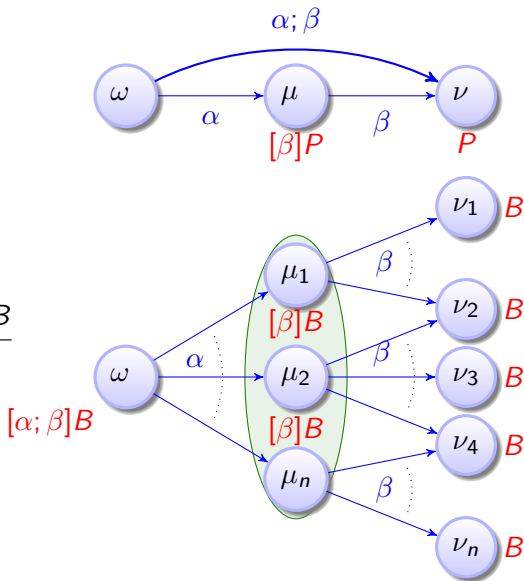
[:] $[\alpha; \beta]P \leftrightarrow$

$$H; \frac{A \rightarrow [\alpha][\beta]B \quad [\beta]B \rightarrow [\beta]B}{A \rightarrow [\alpha; \beta]B}$$



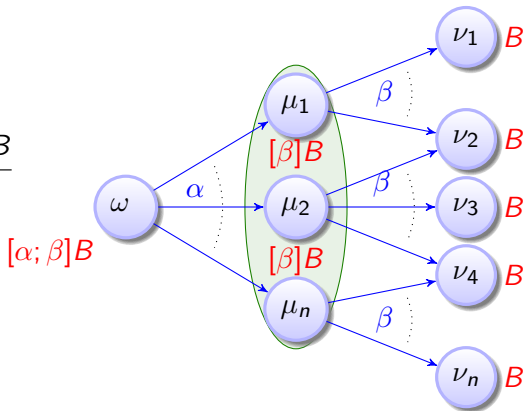
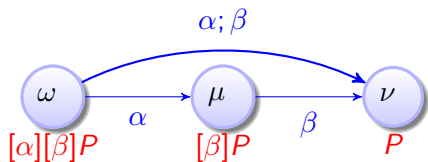
[:] $[\alpha; \beta]P \leftrightarrow$

$$H; \frac{A \rightarrow [\alpha][\beta]B \quad [\beta]B \rightarrow [\beta]B}{A \rightarrow [\alpha; \beta]B}$$



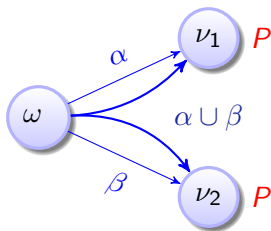
$$[;] \quad [\alpha; \beta]P \leftrightarrow [\alpha][\beta]P$$

$$H; \quad \frac{A \rightarrow [\alpha][\beta]B \quad [\beta]B \rightarrow [\beta]B}{A \rightarrow [\alpha; \beta]B}$$

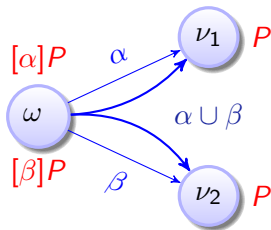


compositional semantics \Rightarrow compositional axioms!

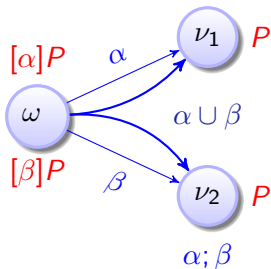
$[U] [\alpha \cup \beta] P \leftrightarrow$



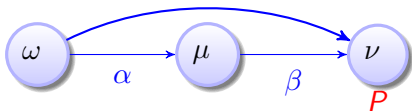
$$[U] \quad [\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$$



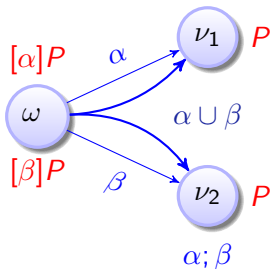
$$[\cup] \quad [\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$$



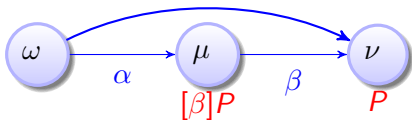
$$[;] \quad [\alpha; \beta]P \leftrightarrow$$



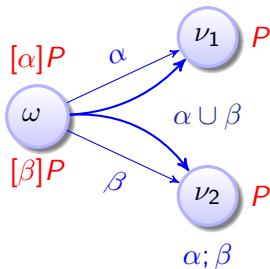
$$[U] \quad [\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$$



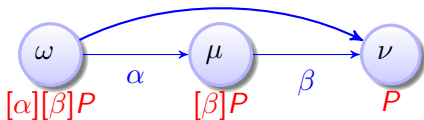
$$[:] \quad [\alpha; \beta]P \leftrightarrow$$



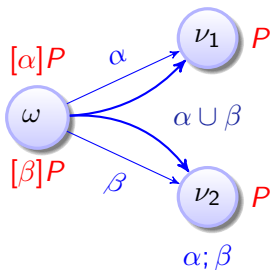
$$[U] \quad [\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$$



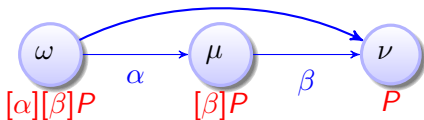
$$[;] \quad [\alpha; \beta]P \leftrightarrow [\alpha][\beta]P$$



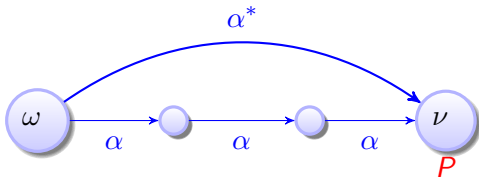
$$[U] \quad [\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$$



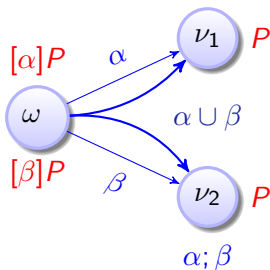
$$[;] \quad [\alpha; \beta]P \leftrightarrow [\alpha][\beta]P$$



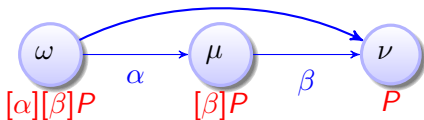
$$[*] \quad [\alpha^*]P \leftrightarrow$$



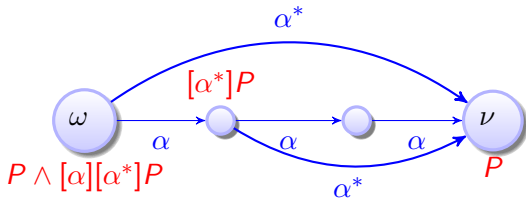
$$[U] \quad [\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$$



$$[;] \quad [\alpha; \beta]P \leftrightarrow [\alpha][\beta]P$$



$$[*] \quad [\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$$



Lemma

$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$ is sound.

Lemma

$[\cdot]$ $[\alpha; \beta]P \leftrightarrow [\alpha][\beta]P$ is sound.

Lemma

$[U]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$ is sound.

Proof

using $\llbracket \alpha \cup \beta \rrbracket = \llbracket \alpha \rrbracket \cup \llbracket \beta \rrbracket$.

$(\omega, \nu) \in \llbracket \alpha \cup \beta \rrbracket$ iff $(\omega, \nu) \in \llbracket \alpha \rrbracket$ or $(\omega, \nu) \in \llbracket \beta \rrbracket$.

Thus, $\omega \in \llbracket [\alpha \cup \beta]P \rrbracket$ iff both $\omega \in \llbracket [\alpha]P \rrbracket$ and $\omega \in \llbracket [\beta]P \rrbracket$. □

Lemma

$[;]$ $[\alpha; \beta]P \leftrightarrow [\alpha][\beta]P$ is sound.

Lemma

$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$ is sound.

Proof

using $\llbracket \alpha \cup \beta \rrbracket = \llbracket \alpha \rrbracket \cup \llbracket \beta \rrbracket$.

$(\omega, \nu) \in \llbracket \alpha \cup \beta \rrbracket$ iff $(\omega, \nu) \in \llbracket \alpha \rrbracket$ or $(\omega, \nu) \in \llbracket \beta \rrbracket$.

Thus, $\omega \in \llbracket [\alpha \cup \beta]P \rrbracket$ iff both $\omega \in \llbracket [\alpha]P \rrbracket$ and $\omega \in \llbracket [\beta]P \rrbracket$. □

Lemma

$[\cdot];$ $[\alpha; \beta]P \leftrightarrow [\alpha][\beta]P$ is sound.

Proof

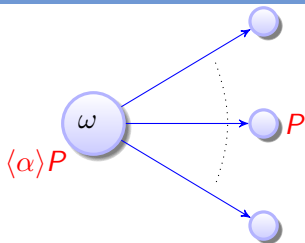
using $\llbracket \alpha; \beta \rrbracket = \llbracket \alpha \rrbracket \circ \llbracket \beta \rrbracket$.

$(\omega, \nu) \in \llbracket \alpha; \beta \rrbracket$ iff $(\omega, \mu) \in \llbracket \alpha \rrbracket$ and $(\mu, \nu) \in \llbracket \beta \rrbracket$ for some state μ .

Thus, $\omega \in \llbracket [\alpha; \beta]P \rrbracket$ iff $\mu \in \llbracket [\beta]P \rrbracket$ for all μ with $(\omega, \mu) \in \llbracket \alpha \rrbracket$.

That is, $\omega \in \llbracket [\alpha; \beta]P \rrbracket$ iff $\omega \in \llbracket [\alpha][\beta]P \rrbracket$. □

$\langle \cdot \rangle \langle \alpha \rangle P \leftrightarrow$

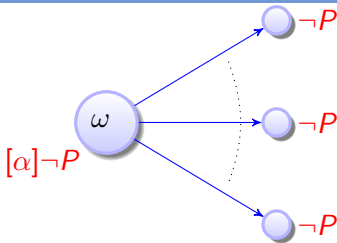


Semantics

$$\llbracket \langle \alpha \rangle P \rrbracket = \{ \omega : \nu \in \llbracket P \rrbracket \text{ for some } \nu : (\omega, \nu) \in \llbracket \alpha \rrbracket \}$$

$$\llbracket [\alpha] P \rrbracket = \{ \omega : \nu \in \llbracket P \rrbracket \text{ for all } \nu : (\omega, \nu) \in \llbracket \alpha \rrbracket \}$$

$\langle \cdot \rangle \langle \alpha \rangle P \leftrightarrow$

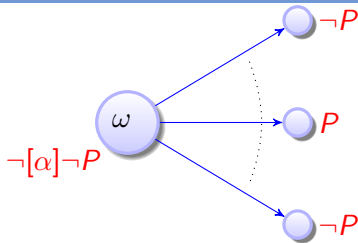


Semantics

$\llbracket \langle \alpha \rangle P \rrbracket = \{ \omega : \nu \in \llbracket P \rrbracket \text{ for some } \nu : (\omega, \nu) \in \llbracket \alpha \rrbracket \}$

$\llbracket [\alpha] P \rrbracket = \{ \omega : \nu \in \llbracket P \rrbracket \text{ for all } \nu : (\omega, \nu) \in \llbracket \alpha \rrbracket \}$

$\langle \cdot \rangle \langle \alpha \rangle P \leftrightarrow$

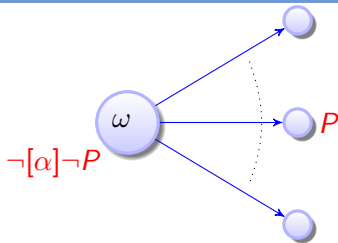


Semantics

$$\llbracket \langle \alpha \rangle P \rrbracket = \{ \omega : \nu \in \llbracket P \rrbracket \text{ for some } \nu : (\omega, \nu) \in \llbracket \alpha \rrbracket \}$$

$$\llbracket [\alpha] P \rrbracket = \{ \omega : \nu \in \llbracket P \rrbracket \text{ for all } \nu : (\omega, \nu) \in \llbracket \alpha \rrbracket \}$$

$$\langle \cdot \rangle \quad \langle \alpha \rangle P \leftrightarrow \neg[\alpha]\neg P$$

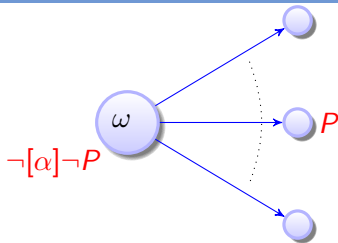


Semantics

$$\llbracket \langle \alpha \rangle P \rrbracket = \{ \omega : \nu \in \llbracket P \rrbracket \text{ for some } \nu : (\omega, \nu) \in \llbracket \alpha \rrbracket \}$$

$$\llbracket [\alpha] P \rrbracket = \{ \omega : \nu \in \llbracket P \rrbracket \text{ for all } \nu : (\omega, \nu) \in \llbracket \alpha \rrbracket \}$$

$$\langle \cdot \rangle \quad \langle \alpha \rangle P \leftrightarrow \neg[\alpha]\neg P$$



Duality axiom $\langle \cdot \rangle$ relates $\langle \alpha \rangle$ to $[\alpha]$ for arbitrary HP α

Semantics

$$\llbracket \langle \alpha \rangle P \rrbracket = \{ \omega : \nu \in \llbracket P \rrbracket \text{ for some } \nu : (\omega, \nu) \in \llbracket \alpha \rrbracket \}$$

$$\llbracket [\alpha] P \rrbracket = \{ \omega : \nu \in \llbracket P \rrbracket \text{ for all } \nu : (\omega, \nu) \in \llbracket \alpha \rrbracket \}$$



- 1 Learning Objectives
- 2 Approach & Reminder
- 3 Intermediate Conditions for CPS
- 4 Dynamic Axioms for Dynamical Systems
 - Nondeterministic Choices
 - Assignments
 - Differential Equations
 - Tests
 - Sequential Compositions
 - Loops
 - Soundness
 - Diamonds
- 5 **First Bouncing Ball Proof**
- 6 Summary

$$[\text{?}] \frac{}{A \rightarrow [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)}$$

$$A \stackrel{\text{def}}{\equiv} 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \stackrel{\text{def}}{\equiv} 0 \leq x \wedge x \leq H$$

$$\{x'' = -g\} \stackrel{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

$$[;] \quad [\alpha; \beta]P \leftrightarrow [\alpha][\beta]P$$

$$\frac{[U]}{A \rightarrow [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v)}$$

$$[;] \quad A \rightarrow [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)$$

$$A \stackrel{\text{def}}{\equiv} 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \stackrel{\text{def}}{\equiv} 0 \leq x \wedge x \leq H$$

$$\{x'' = -g\} \stackrel{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

$$[\cup] \quad [\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$$

$$\frac{[i] \quad A \rightarrow [x'' = -g]([\text{?}x = 0; v := -cv]B(x,v) \wedge [\text{?}x \geq 0]B(x,v))}{[i] \quad A \rightarrow [x'' = -g][\text{?}x = 0; v := -cv \cup \text{?}x \geq 0]B(x,v)}$$

$$\frac{[i] \quad A \rightarrow [x'' = -g][\text{?}x = 0; v := -cv \cup \text{?}x \geq 0]B(x,v)}{[i] \quad A \rightarrow [x'' = -g; (\text{?}x = 0; v := -cv \cup \text{?}x \geq 0)]B(x,v)}$$

$$[i] \quad A \rightarrow [x'' = -g; (\text{?}x = 0; v := -cv \cup \text{?}x \geq 0)]B(x,v)$$

$$A \stackrel{\text{def}}{\equiv} 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \stackrel{\text{def}}{\equiv} 0 \leq x \wedge x \leq H$$

$$\{x'' = -g\} \stackrel{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

$$[;] [\alpha; \beta]P \leftrightarrow [\alpha][\beta]P$$

$$\frac{[?],[?]}{A \rightarrow [x'' = -g]([?x = 0][v := -cv]B(x,v) \wedge [?x \geq 0]B(x,v))}$$

$$\frac{[;]}{A \rightarrow [x'' = -g]([?x = 0; v := -cv]B(x,v) \wedge [?x \geq 0]B(x,v))}$$

$$\frac{[U]}{A \rightarrow [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v)}$$

$$\frac{[;]}{A \rightarrow [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)}$$

$$A \stackrel{\text{def}}{=} 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \stackrel{\text{def}}{=} 0 \leq x \wedge x \leq H$$

$$\{x'' = -g\} \stackrel{\text{def}}{=} \{x' = v, v' = -g\}$$

$$[?] \quad [?Q]P \leftrightarrow (Q \rightarrow P)$$

$$[:=] \frac{}{A \rightarrow [x'' = -g]((x = 0 \rightarrow [v := -cv]B(x,v)) \wedge (x \geq 0 \rightarrow B(x,v)))}$$

$$[?],[?] \frac{}{A \rightarrow [x'' = -g]([?x = 0][v := -cv]B(x,v) \wedge [?x \geq 0]B(x,v))}$$

$$[:] \frac{}{A \rightarrow [x'' = -g]([?x = 0; v := -cv]B(x,v) \wedge [?x \geq 0]B(x,v))}$$

$$[\cup] \frac{}{A \rightarrow [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v)}$$

$$[:] \frac{}{A \rightarrow [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)}$$

$$A \stackrel{\text{def}}{=} 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \stackrel{\text{def}}{=} 0 \leq x \wedge x \leq H$$

$$\{x'' = -g\} \stackrel{\text{def}}{=} \{x' = v, v' = -g\}$$

$$[:=] \quad [x := e]p(x) \leftrightarrow p(e)$$

$$\begin{array}{l}
\frac{[?]}{A \rightarrow [x'' = -g]((x = 0 \rightarrow B(x, -cv)) \wedge (x \geq 0 \rightarrow B(x, v)))} \\
\frac{[:=]}{A \rightarrow [x'' = -g]((x = 0 \rightarrow [v := -cv]B(x, v)) \wedge (x \geq 0 \rightarrow B(x, v)))} \\
\frac{[?], [?]}{A \rightarrow [x'' = -g]([?x = 0][v := -cv]B(x, v) \wedge [?x \geq 0]B(x, v))} \\
\frac{[:]}{A \rightarrow [x'' = -g]([?x = 0; v := -cv]B(x, v) \wedge [?x \geq 0]B(x, v))} \\
\frac{[\cup]}{A \rightarrow [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x, v)} \\
\frac{[:]}{A \rightarrow [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x, v)}
\end{array}$$

$$A \stackrel{\text{def}}{\equiv} 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x, v) \stackrel{\text{def}}{\equiv} 0 \leq x \wedge x \leq H$$

$$\{x'' = -g\} \stackrel{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

$$['] \quad [x' = f(x)]p(x) \leftrightarrow \forall t \geq 0 [x := y(t)]p(x)$$

$$\begin{array}{l}
 [i] \quad \frac{}{A \rightarrow \forall t \geq 0 [x := H - \frac{g}{2}t^2; v := -gt]((x=0 \rightarrow B(x, -cv)) \wedge (x \geq 0 \rightarrow B(x, v)))} \\
 ['] \quad \frac{}{A \rightarrow [x'' = -g]((x=0 \rightarrow B(x, -cv)) \wedge (x \geq 0 \rightarrow B(x, v)))} \\
 [:=] \quad \frac{}{A \rightarrow [x'' = -g]((x=0 \rightarrow [v := -cv]B(x, v)) \wedge (x \geq 0 \rightarrow B(x, v)))} \\
 [?], ['] \quad \frac{}{A \rightarrow [x'' = -g]([?x = 0][v := -cv]B(x, v) \wedge [?x \geq 0]B(x, v))} \\
 [i] \quad \frac{}{A \rightarrow [x'' = -g]([?x = 0; v := -cv]B(x, v) \wedge [?x \geq 0]B(x, v))} \\
 [U] \quad \frac{}{A \rightarrow [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x, v)} \\
 [i] \quad \frac{}{A \rightarrow [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x, v)}
 \end{array}$$

$$A \stackrel{\text{def}}{=} 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x, v) \stackrel{\text{def}}{=} 0 \leq x \wedge x \leq H$$

$$\{x'' = -g\} \stackrel{\text{def}}{=} \{x' = v, v' = -g\}$$

$$[;] [\alpha; \beta]P \leftrightarrow [\alpha][\beta]P$$

$$\begin{array}{l}
 \text{[:=]} \frac{A \rightarrow \forall t \geq 0 [x := H - \frac{g}{2}t^2][v := -gt]((x=0 \rightarrow B(x, -cv)) \wedge (x \geq 0 \rightarrow B(x, v)))}{[;]} \\
 \text{[;]} \frac{A \rightarrow \forall t \geq 0 [x := H - \frac{g}{2}t^2; v := -gt]((x=0 \rightarrow B(x, -cv)) \wedge (x \geq 0 \rightarrow B(x, v)))}{[;]} \\
 \text{[']} \frac{A \rightarrow [x'' = -g]((x=0 \rightarrow B(x, -cv)) \wedge (x \geq 0 \rightarrow B(x, v)))}{[;]} \\
 \text{[:=]} \frac{A \rightarrow [x'' = -g]((x=0 \rightarrow [v := -cv]B(x, v)) \wedge (x \geq 0 \rightarrow B(x, v)))}{[;]} \\
 \text{[?], [?]} \frac{A \rightarrow [x'' = -g]([?x = 0][v := -cv]B(x, v) \wedge [?x \geq 0]B(x, v))}{[;]} \\
 \text{[;]} \frac{A \rightarrow [x'' = -g]([?x = 0; v := -cv]B(x, v) \wedge [?x \geq 0]B(x, v))}{[;]} \\
 \text{[U]} \frac{A \rightarrow [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x, v)}{[;]} \\
 \text{[;]} \frac{A \rightarrow [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x, v)}{[;]}
 \end{array}$$

$$A \stackrel{\text{def}}{=} 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x, v) \stackrel{\text{def}}{=} 0 \leq x \wedge x \leq H$$

$$\{x'' = -g\} \stackrel{\text{def}}{=} \{x' = v, v' = -g\}$$

$$\begin{array}{l}
 \text{[:=]} \frac{}{A \rightarrow \forall t \geq 0 [x := H - \frac{g}{2}t^2] ((x=0 \rightarrow B(x, -c(-gt))) \wedge (x \geq 0 \rightarrow B(x, -gt)))} \\
 \text{[:=]} \frac{}{A \rightarrow \forall t \geq 0 [x := H - \frac{g}{2}t^2] [v := -gt] ((x=0 \rightarrow B(x, -cv)) \wedge (x \geq 0 \rightarrow B(x, v)))} \\
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 \text{[?], [?]} \frac{}{A \rightarrow [x'' = -g] ([?x = 0] [v := -cv] B(x, v) \wedge [?x \geq 0] B(x, v))} \\
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$$A \stackrel{\text{def}}{\equiv} 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

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A Proof of a Single-hop Bouncing Ball

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Resolving abbreviations at the top premise yields provable arithmetic:

$$0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \rightarrow$$

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$$\left. \wedge \left(H - \frac{g}{2}t^2 \geq 0 \rightarrow 0 \leq H - \frac{g}{2}t^2 \wedge H - \frac{g}{2}t^2 \leq H \right) \right)$$

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Exciting!

We have just formally verified our very first CPS!

A Proof of a Single-hop Bouncing Ball

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Exciting!

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Okay, it was a grotesquely simplified single-hop bouncing ball.

But the axioms of our proof technique were completely general, so they carry us forward to true CPSs.

- 1 Learning Objectives
- 2 Approach & Reminder
- 3 Intermediate Conditions for CPS
- 4 Dynamic Axioms for Dynamical Systems
 - Nondeterministic Choices
 - Assignments
 - Differential Equations
 - Tests
 - Sequential Compositions
 - Loops
 - Soundness
 - Diamonds
- 5 First Bouncing Ball Proof
- 6 Summary

$$[:=] [x := e]p(x) \leftrightarrow p(e)$$

equations of truth

$$[?] [?Q]P \leftrightarrow (Q \rightarrow P)$$

$$['] [x' = f(x)]p(x) \leftrightarrow \forall t \geq 0 [x := y(t)]p(x) \quad (y'(t) = f(y))$$

$$[\cup] [\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$$

$$[;] [\alpha; \beta]P \leftrightarrow [\alpha][\beta]P$$

$$[*] [\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$$

$$\langle \cdot \rangle \langle \alpha \rangle P \leftrightarrow \neg[\alpha]\neg P$$

One axiom for each HP operator

Using an axiom from left to right simplifies the HP structure

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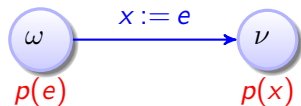
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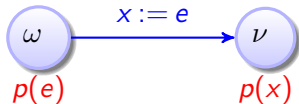
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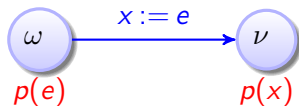
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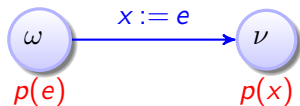


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- 1 $p(e)$ stands for the same formula as $p(x)$
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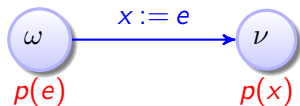
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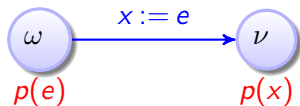
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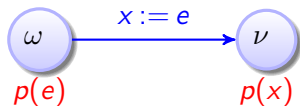
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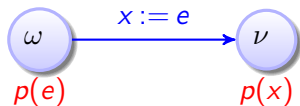
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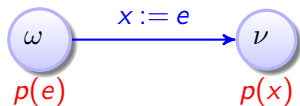
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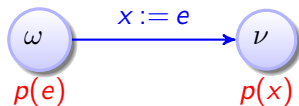
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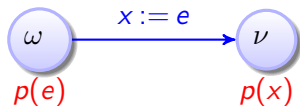
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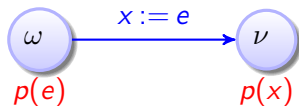
$$[:=] \quad [x := e]p(x) \leftrightarrow p(e)$$



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- ✓ $[x := x + y]x \leq y^2 \leftrightarrow x + y \leq y^2$ is instance of $[:=]$
- × $[x := x^2]\forall x (x \geq 0) \leftrightarrow \forall x (x^2 \geq 0)$ x bound by $\forall x$
- × $[x := x + y]\forall y (x \leq y^2) \leftrightarrow \forall y (x + y \leq y^2)$ var of $x+y$ bound by $\forall y$
- $[x := x + y][y := 5]x \geq 0 \leftrightarrow [y := 5]x + y \geq 0$
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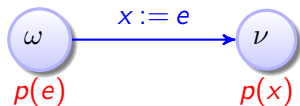
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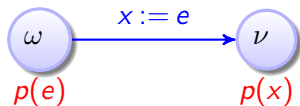
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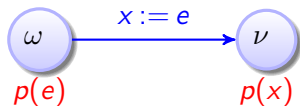
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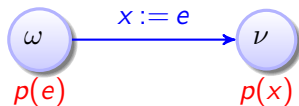
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If you bind a free variable, you go to logic jail!



André Platzer.

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J. Autom. Reas., 59(2):219–265, 2017.

[doi:10.1007/s10817-016-9385-1](https://doi.org/10.1007/s10817-016-9385-1).



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[doi:10.1109/LICS.2012.64](https://doi.org/10.1109/LICS.2012.64).



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