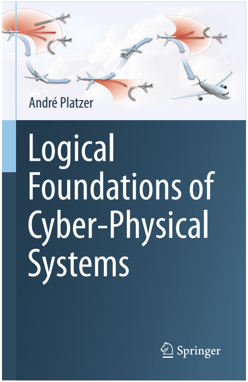
02: Differential Equations & Domains

Logical Foundations of Cyber-Physical Systems



André Platzer

Carnegie Mellon University
Computer Science Department

Outline

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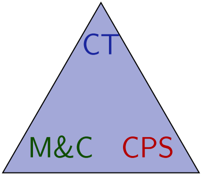
Learning Objectives

Differential Equations & Domains

semantics of differential equations
descriptive power of differential equations
syntax versus semantics



continuous dynamics
differential equations
evolution domains
first-order logic

continuous operational effects

Outline

Example (Vector field and one solution of a differential equation)

$$\begin{pmatrix} y'(t) = f(t,y) \\ y(t_0) = y_0 \end{pmatrix}$$

Intuition:

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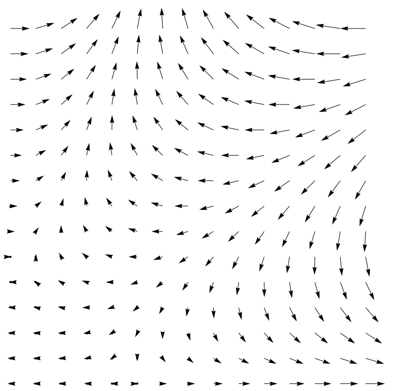

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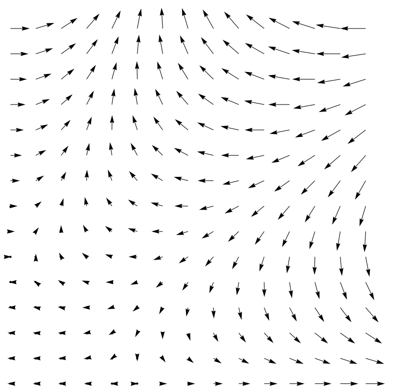

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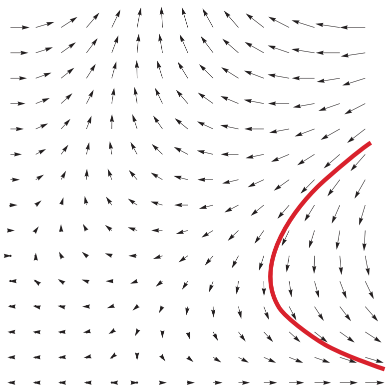
2. Start at initial state y_0 at initial time t_0
3. Follow the direction of the vector



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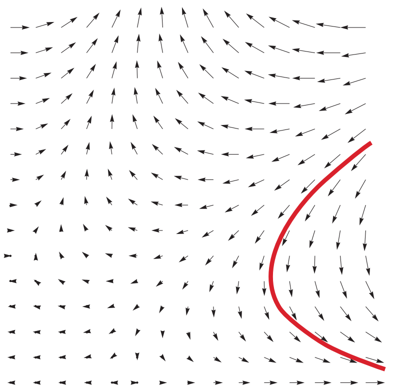
↯ The diagram should really show infinitely many vectors . . .



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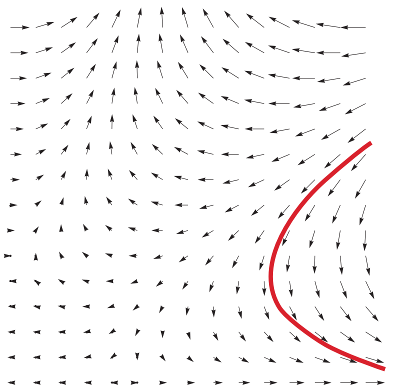

Your car's ODE: $x' = v, v' = a$

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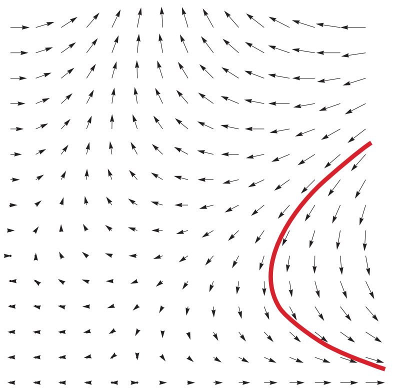

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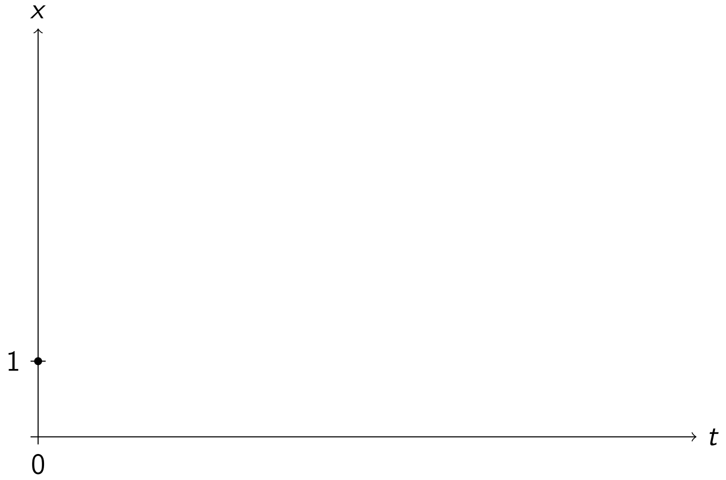
Well it's a wee bit more complicated

Intuition for Differential Equations



$$\begin{pmatrix} x'(t) = \frac{1}{4}x(t) \\ x(0) = 1 \end{pmatrix}$$

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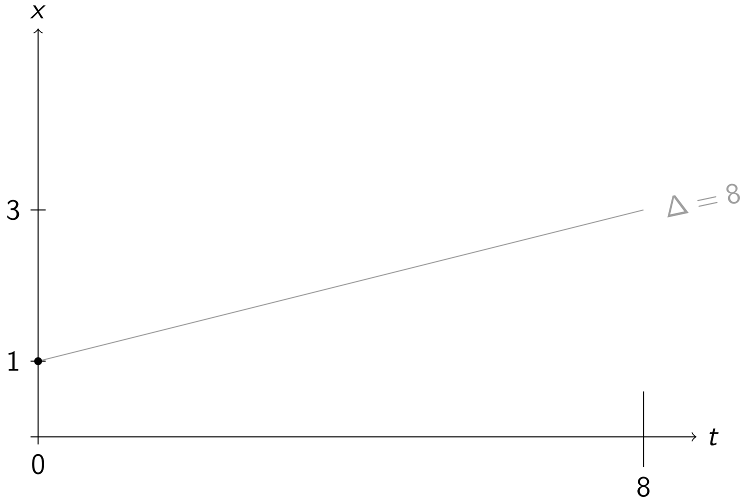

$$\begin{pmatrix} x'(t) = \frac{1}{4}x(t) \\ x(0) = 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} x(t+\Delta) := x(t) + \frac{1}{4}x(t)\Delta \\ x(0) := 1 \end{pmatrix}$$

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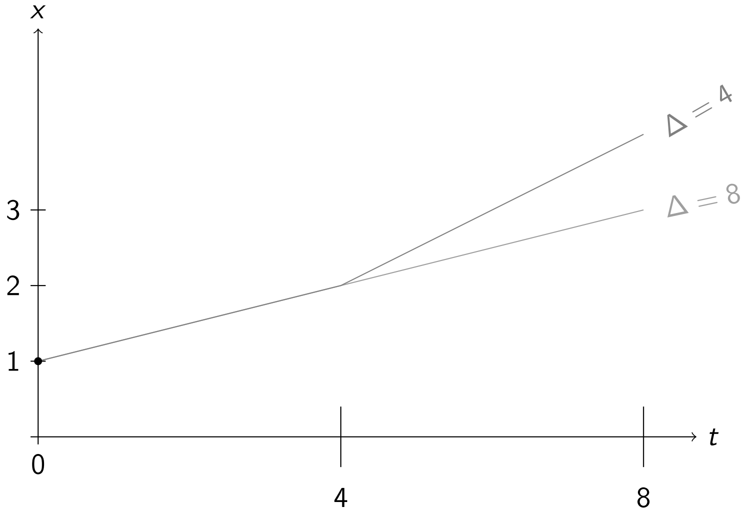

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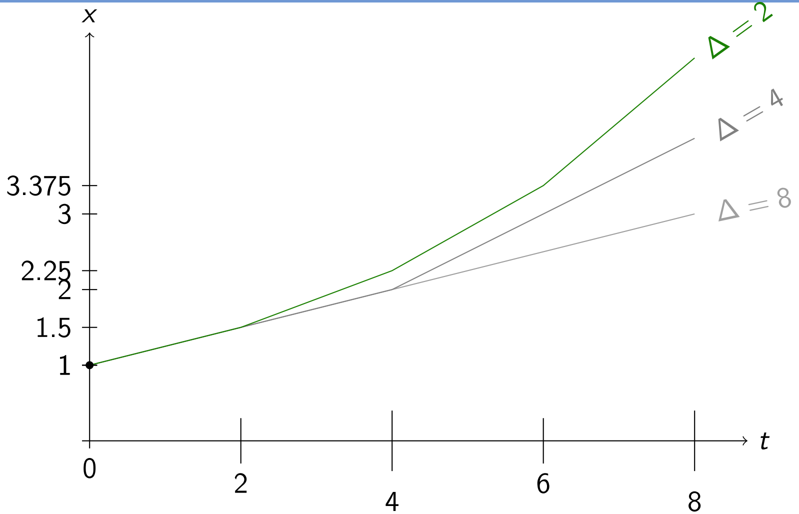

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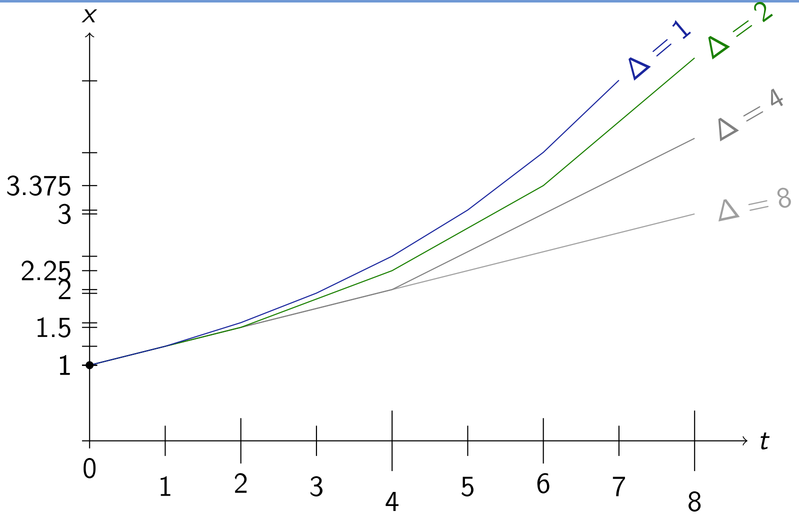

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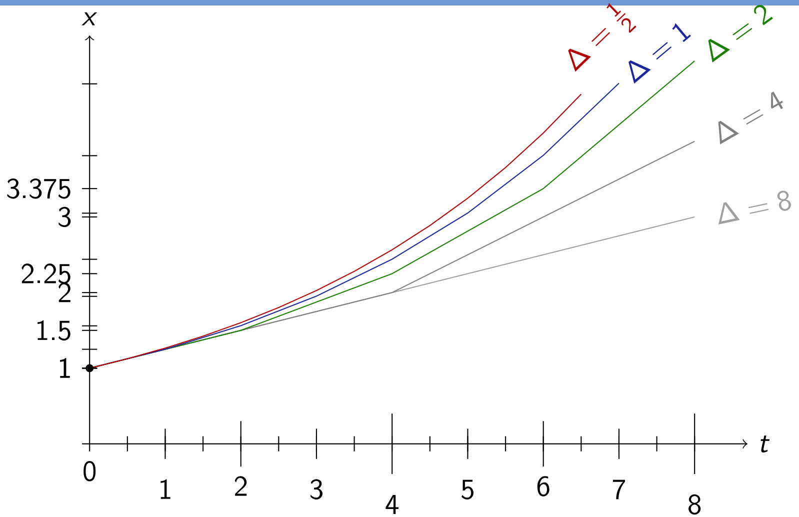

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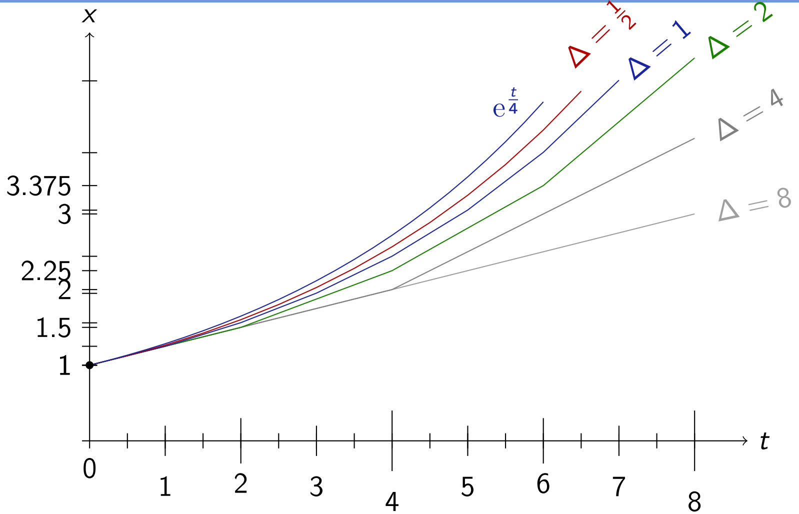

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Outline

1. What exactly is a vector field?
2. What does it mean to describe directions of evolution at *every* point in space?
3. Could these directions possibly contradict each other?

Importance of meaning

The physical impacts of CPSs do not leave much room for failure. We immediately want to get into the habit of studying the behavior and exact meaning of all relevant aspects of CPS.

Definition (Ordinary Differential Equation, ODE)

$f : D \to \mathbb{R}^n$ on domain $D \subseteq \mathbb{R} \times \mathbb{R}^n$ (i.e., open connected set). Then $Y : I \to \mathbb{R}^n$ is *solution* of initial value problem (IVP)

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Differential Equations & Initial Value Problems

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If $f \in C(D, \mathbb{R}^n)$, then $Y \in C^1(I, \mathbb{R}^n)$.
If f continuous, then Y continuously differentiable.

Outline

Example: A Constant Differential Equation

Example (Initial value problem)

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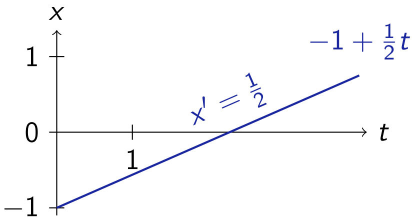

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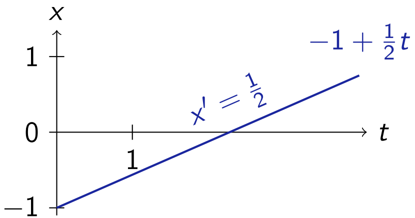

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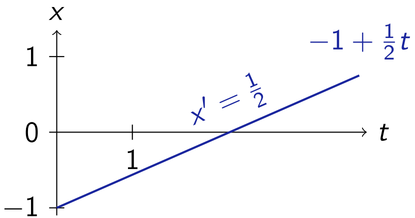

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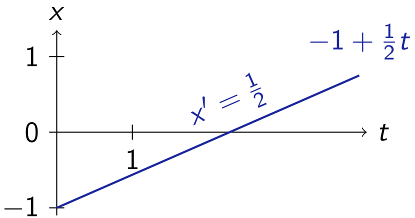

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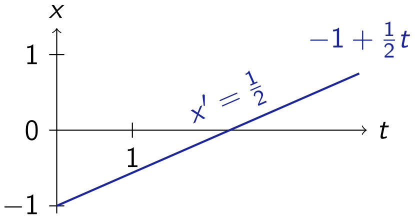
Check by inserting solution into ODE+IVP.

$$\begin{pmatrix} (x(t))' = (\frac{1}{2}t - 1)' = \frac{1}{2} \\ x(0) = \frac{1}{2} \cdot 0 - 1 = -1 \end{pmatrix}$$

□



Example: A Linear Differential Equation from before

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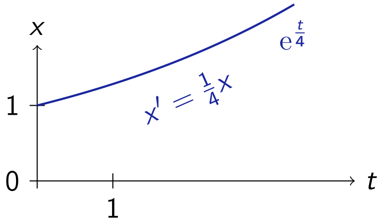

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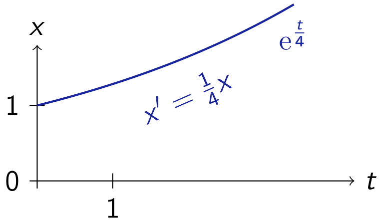

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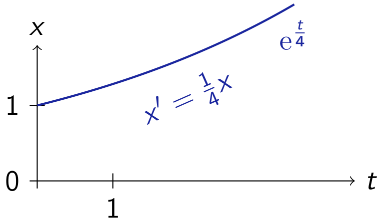

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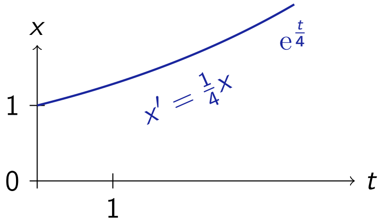

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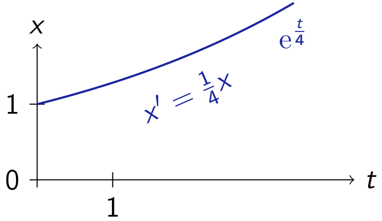
$$\begin{pmatrix} (x(t))' = (e^{\frac{t}{4}})' = e^{\frac{t}{4}}(\frac{t}{4})' = e^{\frac{t}{4}}\frac{1}{4} = \frac{1}{4}x(t) \\ x(0) = e^{\frac{0}{4}} = 1 \end{pmatrix}$$

□



Example: Linear Dynamics

Example (Initial value problem)

$$\begin{pmatrix} v'(t) = w(t) \\ w'(t) = -v(t) \\ v(0) = 0 \\ w(0) = 1 \end{pmatrix} \quad \text{has solution}$$

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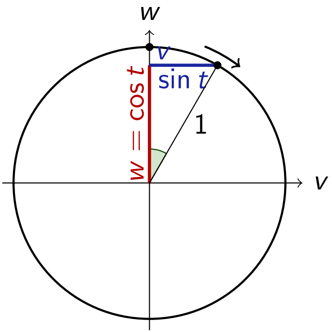
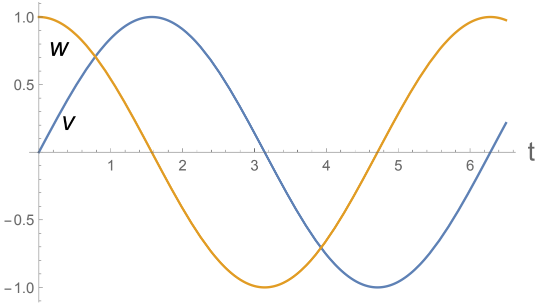
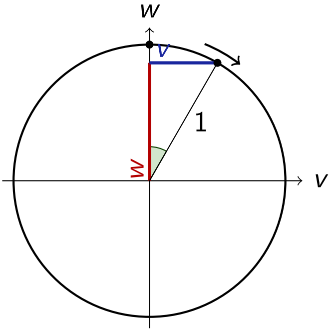
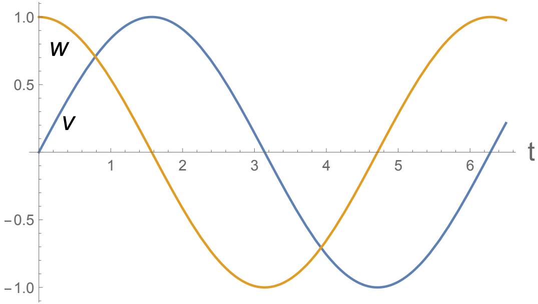




Example (Initial value problem)

$$\begin{pmatrix} v'(t) = \omega w(t) \\ w'(t) = -\omega v(t) \\ v(0) = 0 \\ w(0) = 1 \end{pmatrix}$$ has solution





Example: Rotational Dynamics

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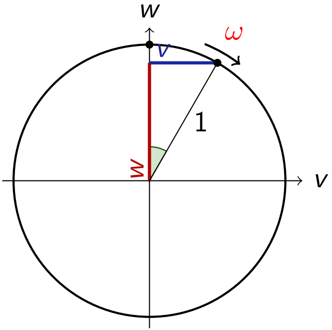
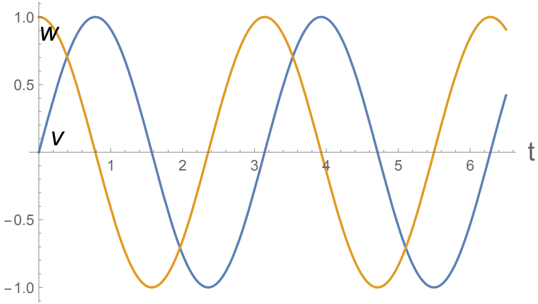
$$\begin{pmatrix} v'(t) = \omega w(t) \\ w'(t) = -\omega v(t) \\ v(0) = 0 \\ w(0) = 1 \end{pmatrix} \quad \text{has solution} \quad \begin{pmatrix} v(t) = \sin(\omega t) \\ w(t) = \cos(\omega t) \end{pmatrix}$$



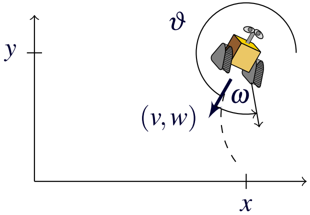


Example (Initial value problem)

$$\begin{pmatrix} x'(t) = v(t) \\ y'(t) = w(t) \\ v'(t) = \omega w(t) \\ w'(t) = -\omega v(t) \\ x(0) = x_0 \\ y(0) = y_0 \\ v(0) = v_0 \\ w(0) = w_0 \end{pmatrix}$$

Example (Initial value problem)

$$\left(\begin{array}{rl} x'(t) &= v(t)\\ y'(t) &= w(t)\\ v'(t) &= \omega w(t)\\ w'(t) &= -\omega v(t)\\ x(0) &= x_0\\ y(0) &= y_0\\ v(0) &= v_0\\ w(0) &= w_0 \end{array}\right)$$



ODE Examples

ODE	Solution
$x' = 1, x(0) = x_0$	$x(t) = x_0 + t$
$x' = 5, x(0) = x_0$	$x(t) = x_0 + 5t$
$x' = x, x(0) = x_0$	$x(t) = x_0 e^t$
$x' = x^2, x(0) = x_0$	$x(t) = \frac{x_0}{1-tx_0}$
$x' = \frac{1}{x}, x(0) = 1$	$x(t) = \sqrt{1+2t} \dots$
$y'(x) = -2xy, y(0) = 1$	$y(x) = e^{-x^2}$
$x'(t) = tx, x(0) = x_0$	$x(t) = x_0 e^{\frac{t^2}{2}}$
$x' = \sqrt{x}, x(0) = x_0$	$x(t) = \frac{t^2}{4} \pm t\sqrt{x_0} + x_0$
$x' = y, y' = -x, x(0) = 0, y(0) = 1$	$x(t) = \sin t, y(t) = \cos t$
$x' = 1 + x^2, x(0) = 0$	$x(t) = \tan t$
$x'(t) = \frac{2}{t^3} x(t)$	$x(t) = e^{-\frac{1}{t^2}}$ non-analytic
$x' = x^2 + x^4$	???
$x'(t) = e^{t^2}$	non-elementary

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$x' = 1, x(0) = x_0$	$x(t) = x_0 + t$
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$x' = y, y' = -x, x(0) = 0, y(0) = 1$	$x(t) = \sin t, y(t) = \cos t$
$x' = 1 + x^2, x(0) = 0$	$x(t) = \tan t$
$x'(t) = \frac{2}{t^3}x(t)$	$x(t) = e^{-\frac{1}{t^2}}$ non-analytic
$x' = x^2 + x^4$	???
$x'(t) = e^{t^2}$	non-elementary

Descriptive power of differential equations

1. Solutions of differential equations can be much more involved than the differential equations themselves.
2. Representational and descriptive power of differential equations!
3. Simple differential equations can describe quite complicated physical processes.
4. Local description as the direction into which the system evolves.

Outline

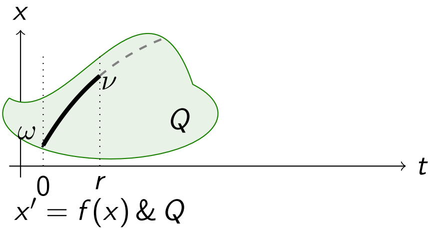
Enable Cyber to interact with Physics

Definition (Evolution domain constraints)

A *differential equation* $x' = f(x)$ *with evolution domain* Q is denoted by

$$x' = f(x) \,\&\, Q$$

conjunctive notation (&) signifies that the system obeys the differential equation $x' = f(x)$ **and** the evolution domain Q.



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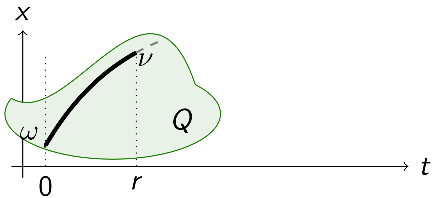

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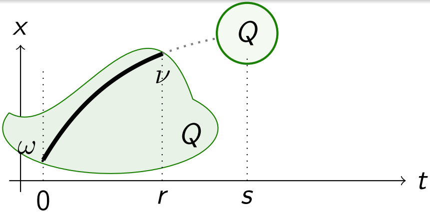

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conjunctive notation (&) signifies that the system obeys the differential equation $x' = f(x)$ **and** the evolution domain Q.

$x' = v, v' = a, t' = 1 \,\&\, t \leq \varepsilon$	stops at clock ε at the latest
$x' = v, v' = a, t' = 1 \,\&\, v \geq 0$	stops before velocity negative
$x' = y, y' = x + y^2 \,\&\, \mathit{true}$	no constraint

Evolution Domain Constraints

Define: Terms   Cyber to interact with Physics   Define: Formulas

Definition (Evolution domain constraints)

A *differential equation* $x' = f(x)$ *with evolution domain* Q is denoted by

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$x' = y, y' = x + y^2 \,\&\, \mathit{true}$	no constraint

Terms: Syntax

Definition (Syntax of terms)

A *term* e is a polynomial term defined by the grammar:

$$e, \tilde{e} ::= x \mid c \mid e + \tilde{e} \mid e \cdot \tilde{e}$$

where $e, \tilde{e}$ are terms, $x \in \mathcal{V}$ is a variable, $c \in \mathbb{Q}$ a rational number constant

Definition (Syntax of terms)

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Variable Constant Add Multiply

Terms: Syntax & Semantics

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Definition (Semantics of terms) $(\llbracket\cdot\rrbracket : \mathrm{Trm} \to (\mathcal{S} \to \mathbb{R}))$

The *value of term* e in state $\omega : \mathcal{V} \to \mathbb{R}$ is a real number denoted $\omega\llbracket e\rrbracket$ and is defined by induction on the structure of e:

$$\begin{aligned}
\omega\llbracket x\rrbracket &= \omega(x) && \text{if } x \in \mathcal{V} \text{ is a variable}\\
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$$\omega[\![e + \tilde{e}]\!] = \omega[\![e]\!] + \omega[\![\tilde{e}]\!] \quad \text{addition of reals}$$

$$\omega[\![e \cdot \tilde{e}]\!] = \omega[\![e]\!] \cdot \omega[\![\tilde{e}]\!] \quad \text{multiplication of reals}$$

$\omega[\![4 + x \cdot 2]\!] =$ if $\omega(x) = 5$

Terms: Syntax & Semantics

Definition (Syntax of terms)

A *term* e is a polynomial term defined by the grammar:

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\end{aligned}$$

$$\omega[\![4 + x \cdot 2]\!] = \omega[\![4]\!] + \omega[\![x]\!] \cdot \omega[\![2]\!] = 4 + \omega(x) \cdot 2 = 14 \qquad \text{if } \omega(x) = 5$$

Terms: Syntax & Semantics

Definition (Syntax of terms)

A *term* e is a polynomial term defined by the grammar:

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$\omega\llbracket e \cdot \tilde{e} \rrbracket = \omega\llbracket e \rrbracket \cdot \omega\llbracket \tilde{e} \rrbracket$	multiplication of reals

What about $x - y$?

Terms: Syntax & Semantics

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\end{aligned}$$

What about $x - y$? Defined as $x + (-1) \cdot y$

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What about x^4?

Terms: Syntax & Semantics

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What about x^4? Defined as $x \cdot x \cdot x \cdot x$

Terms: Syntax & Semantics

Definition (Syntax of terms)

A *term* e is a polynomial term defined by the grammar:

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What about x^n?

Terms: Syntax & Semantics

Definition (Syntax of terms)

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\end{aligned}$$

What about x^n? Defined as $x \cdot x \cdot x \cdot x \cdot x \cdot \ldots$, wait when do we stop???

First-Order Logic Formulas: Syntax

Definition (Syntax of first-order logic formulas)

The *formulas* of *FOL of real arithmetic* are defined by the grammar:

$$P, Q ::= e \geq \tilde{e} \mid e = \tilde{e} \mid \neg P \mid P \wedge Q \mid P \vee Q \mid P \rightarrow Q \mid P \leftrightarrow Q \mid \forall x\, P \mid \exists x\, P$$

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Greater-or-equal · Not · And · Or · Imply · Equiv · All · Exists

First-Order Logic Formulas: Syntax & Semantics

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Definition (Semantics of first-order logic formulas)

First-order formula P is true in state ω, written $\omega \models P$, defined inductively:

$\omega \models e = \tilde{e}$	iff $\omega[\![e]\!] = \omega[\![\tilde{e}]\!]$
$\omega \models e \geq \tilde{e}$	iff $\omega[\![e]\!] \geq \omega[\![\tilde{e}]\!]$
$\omega \models \neg P$	iff $\omega \not\models P$, i.e., if it is not the case that $\omega \models P$
$\omega \models P \wedge Q$	iff $\omega \models P$ and $\omega \models Q$
$\omega \models P \vee Q$	iff $\omega \models P$ or $\omega \models Q$
$\omega \models P \rightarrow Q$	iff $\omega \not\models P$ or $\omega \models Q$
$\omega \models \forall x\, P$	iff $\omega_x^d \models P$ for all $d \in \mathbb{R}$
$\omega \models \exists x\, P$	iff $\omega_x^d \models P$ for some $d \in \mathbb{R}$

$$\omega_x^d(y) = \begin{cases} d & \text{if } y{=}x \\ \omega(y) & \text{if } y{\neq}x \end{cases}$$

First-Order Logic Formulas: Syntax & Semantics

$\omega \models P$ formula P is true in state ω

$\models P$ formula P is *valid*, i.e., true in all states ω, i.e., $\omega \models P$ for all ω

$[\![P]\!] = \{\omega : \omega \models P\}$ set of all states in which P is true

Definition (Semantics of first-order logic formulas)

First-order formula P is true in state ω, written $\omega \models P$, defined inductively:

$\omega \models e = \tilde{e}$ iff $\omega[\![e]\!] = \omega[\![\tilde{e}]\!]$

$\omega \models e \geq \tilde{e}$ iff $\omega[\![e]\!] \geq \omega[\![\tilde{e}]\!]$

$\omega \models \neg P$ iff $\omega \not\models P$, i.e., if it is not the case that $\omega \models P$

$\omega \models P \land Q$ iff $\omega \models P$ and $\omega \models Q$

$\omega \models P \lor Q$ iff $\omega \models P$ or $\omega \models Q$

$\omega \models P \rightarrow Q$ iff $\omega \not\models P$ or $\omega \models Q$

$\omega \models \forall x\, P$ iff $\omega_x^d \models P$ for all $d \in \mathbb{R}$

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First-Order Logic Formulas: Syntax & Semantics

$\omega \models P$ formula P is true in state ω

$\models P$ formula P is *valid*, i.e., true in all states ω, i.e., $\omega \models P$ for all ω

$[\![P]\!] = \{\omega : \omega \models P\}$ set of all states in which P is true

$\exists y\,(y^2 \leq x)$ for $\omega(x) = 5$ and $\nu(x) = -5$

Definition (Semantics of first-order logic formulas)

First-order formula P is true in state ω, written $\omega \models P$, defined inductively:

$\omega \models e = \tilde{e}$ iff $\omega[\![e]\!] = \omega[\![\tilde{e}]\!]$

$\omega \models e \geq \tilde{e}$ iff $\omega[\![e]\!] \geq \omega[\![\tilde{e}]\!]$

$\omega \models \neg P$ iff $\omega \not\models P$, i.e., if it is not the case that $\omega \models P$

$\omega \models P \land Q$ iff $\omega \models P$ and $\omega \models Q$

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First-Order Logic Formulas: Syntax & Semantics

$\omega \models P$ formula P is true in state ω

$\models P$ formula P is *valid*, i.e., true in all states ω, i.e., $\omega \models P$ for all ω

$[\![P]\!] = \{\omega : \omega \models P\}$ set of all states in which P is true

$\omega \models \exists y\,(y^2 \leq x)$ but $\nu \not\models \exists y\,(y^2 \leq x)$ for $\omega(x) = 5$ and $\nu(x) = -5$

Definition (Semantics of first-order logic formulas)

First-order formula P is true in state ω, written $\omega \models P$, defined inductively:

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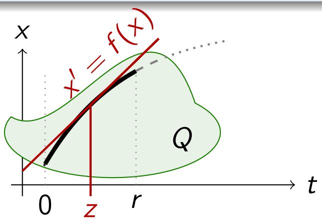
$$\omega_x^d(y) = \begin{cases} d & \text{if } y = x \\ \omega(y) & \text{if } y \neq x \end{cases}$$

Definition (Semantics of differential equations)

A function $\varphi : [0, r] \to \mathcal{S}$ of some duration $r \geq 0$ satisfies the differential equation $x' = f(x) \,\&\, Q$, written $\varphi \models x' = f(x) \land Q$, iff:

1. $\varphi(z)(x') = \frac{\mathrm{d}\varphi(t)(x)}{\mathrm{d}t}(z)$ exists at all times $0 \leq z \leq r$
2. $\varphi(z) \models x' = f(x) \land Q$ for all times $0 \leq z \leq r$
3. $\varphi(z) = \varphi(0)$ except at x, x'



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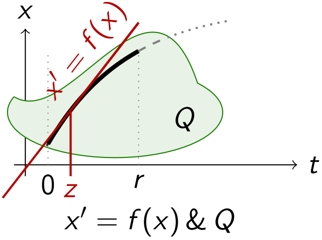

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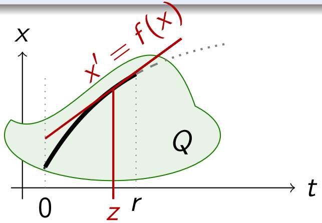

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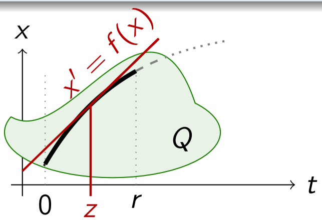


$x' = f(x) \,\&\, Q$

Outline

Summary: Differential Equations & Domains

Definition (Syntax of terms)

$$e, \tilde{e} \ ::= \ x \mid c \mid e + \tilde{e} \mid e \cdot \tilde{e}$$

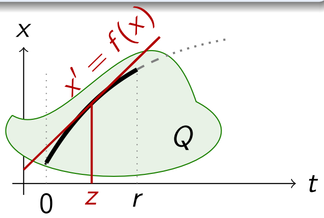
Definition (Syntax of first-order logic formulas)

$$P, Q ::= e \geq \tilde{e} \mid e = \tilde{e} \mid \neg P \mid P \wedge Q \mid P \vee Q \mid P \rightarrow Q \mid P \leftrightarrow Q \mid \forall x\, P \mid \exists x\, P$$

Definition (Syntax of continuous programs)

A *differential equation* $x' = f(x)$ *with evolution domain* Q is denoted by

$$x' = f(x) \,\&\, Q$$



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