Dependency Analysis for Hybrid Programs

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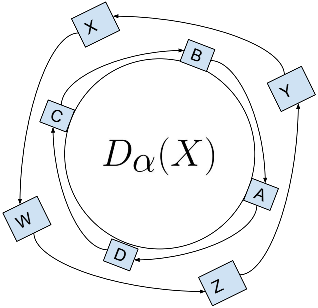

Motivation

- Semantics of hybrid programs α: $(u, w) \in [\![\alpha]\!]$, e.g.:

$$(u, w) \in [\![x := e]\!] \iff w = u_x^{u[\![e]\!]}$$

- What relationships can we find between states u and w?

Dependency Analysis

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- Ideally, we want to minimize Y, but consider:

$$?P; x := y \cup x := z$$

- Aim: syntactic, compositional analysis of α for a "small" Y

Definition

For each set X, $D_\alpha(X)$ is a set of variables that X might depend on:

$$D_{y:=e}(X) = \bigcup_{x \in X} \begin{cases} FV(e) & \text{if } x = y \\ \{x\} & \text{otherwise} \end{cases}$$

$$D_{\alpha;\beta}(X) = D_\alpha(D_\beta(X))$$

$$D_{\alpha \cup \beta}(X) = D_\alpha(X) \cup D_\beta(X)$$

Definition (tests)

Tests are slightly weird:

$$D_{?Q}(X) = X \cup FV(Q)$$

Compare to its semantics:

$$(u, u) \in [\![?Q]\!] \iff u \in [\![Q]\!]$$

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What does x depend on?

$$?k = 1; x := y \cup x := z$$

Definition (loops)

Semantics of loops (α^*):

$$\alpha^0 \cup \alpha^1 \cup \alpha^2 \cup \cdots$$

$$\alpha^n \equiv \underbrace{\alpha; \ldots; \alpha}_{n\ times}$$

Loops can be unfolded:

$$D_{\alpha^*}(X) = \bigcup_{i \in \mathbb{N}} D^i_\alpha(X)$$

$$D^0_\alpha(S) = S$$

$$D^{i+1}_\alpha(S) = D_\alpha(D^i_\alpha(S))$$

Definition (ODEs)

ODEs in vector form ($\vec{x}' = \vec{e}$), first attempt:

$$D_{\vec{x}'=\vec{e}}(X) = \begin{cases} X & \text{if } X \cap \vec{x} = \emptyset \\ X \cup \vec{x} \cup \bigcup_{e \in \vec{e}} FV(e) & \text{otherwise} \end{cases}$$

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Extension to handle evolution domain constraints:

$$D_{\vec{x}'=\vec{e}\&Q}(X) = D_{\vec{x}'=\vec{e}}(X) \cup FV(Q)$$

Definition (ODEs)

This badly over-approximates the dependencies, e.g.:

$$x' = 1, y' = 1$$

Has solution:

$$x = x_0 + t, y = y_0 + t$$

But analysis says x, y are interdependent

Definition (ODEs)

Not easy to get right, consider:

$$x' = 1, y' = y^2$$

If $y_0 = 0$, $x = x_0 + t, y = 0$ is the solution
If $y_0 > 0$, $t < \frac{1}{y_0}$, so $x \in [x_0, x_0 + \frac{1}{y_0})$

Global solutions

Assume $\vec{x}' = \vec{e}$ has a global solution for all time, then we can instead use:

$$D_{\vec{x}'=\vec{e}}(X) = T$$

where T is the smallest set satisfying $X \subseteq T$ and $\vec{x}_i \in T \rightarrow FV(\vec{e}_i) \in T$, i.e. the transitive closure of all variables mentioned in X across the system.

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When do global solutions exist?

Linearity

If ODE is linear, i.e. $\vec{x}' = A\vec{x}$, then $e^{tA}\vec{x_0}$ is a global solution.

Simple to check if ODE is linear, e.g.:

$$x' = v, v' = a \& v \geq 0$$

x depends on $\{x, v, a\}$ and v depends on $\{v, a\}$

Linearity, revisited

Consider the non-linear system:

$$x' = 1, y' = xy$$

Solution: $x = x_0 + t, y = y_0 e^{x_0 t + \frac{t^2}{2}}$, but analysis says x depends on $\{x, y\}$

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This works even if there is no solution for all time:

$$x' = x^2, y' = xy$$

Correctness

Proposition (Coincidence for terms and formulas)

If u, v agree on $FV(e)$, then $u[\![e]\!] = v[\![e]\!]$, and if u, v agree on $FV(Q)$, then $u \in [\![Q]\!] \iff v \in [\![Q]\!]$.

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Theorem (Restricted coincidence)

If u, v agree on $D_\alpha(X)$ and $(u, u') \in [\![\alpha]\!]$, there exists $(v, v') \in [\![\alpha]\!]$ such that u', v' agree on X.

Proof.

By induction on α. □

Correctness

Corollary

If $P \to [\alpha]Q$, then there is a precondition R such that $P \to R$, $R \to [\alpha]Q$, where $FV(R) \subseteq D_\alpha(FV(Q))$.

Proof.

Let $R \equiv \exists \vec{x} P$, then for $u \in [\![R]\!]$ there exists $v \in [\![P]\!]$.
For any $(u, u') \in [\![\alpha]\!]$, previous theorem gives $(v, v') \in [\![\alpha]\!]$, and u', v' agree on $FV(Q)$.
By assumption, $v' \in [\![Q]\!]$, so $u' \in [\![Q]\!]$ by the coincidence lemma. □

Application: Invariants for free!

Task: find possible invariants for: $x' = -y, y' = x$

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Parametric invariant candidate p over $V = \{x, y\}$:

$$p = ax^2 + bxy + cy^2 + dx + ey + f$$

Set Lie derivative $p' = 0$:

$$\begin{aligned} p' &= 2axx' + b(x'y + y'x) + 2cyy' + dx' + ey' \\ &= -2axy + b(x^2 - y^2) + 2cxy - dy + ex \\ &= 0 \end{aligned}$$

Application: Differential Invariant Generation

Comparing coefficients: $a = c, b = 0, d = 0, e = 0$

Therefore, $ax^2 + ay^2 + f = 0$ is an invariant, for any constant a, f

Application: Differential Invariant Generation

How did we choose p (or equivalently V)?

Recall corollary: can always find $P \to [\alpha]Q$, $FV(P) \subseteq D_\alpha(FV(Q))$

Only search for invariants $\phi \to [\alpha]\phi$, where ϕ is closed under D_α.

Application: Differential Invariant Generation

Dependency also allows us to search in order, e.g.:

$$\alpha \equiv x_1' = d_1, x_2' = d_2, d_1' = -\omega d_2, d_2' = \omega d_1, t' = 1$$

$$D_\alpha(d_1) = \{\omega, d_1, d_2\}$$

$$D_\alpha(d_2) = \{\omega, d_1, d_2\}$$

$$D_\alpha(x_1) = \{\omega, d_2, d_1, x_1\}$$

$$D_\alpha(x_2) = \{\omega, d_2, d_1, x_2\}$$

$$D_\alpha(t) = \{t\}, D_\alpha(\omega) = \{\omega\}$$

Search for invariants in sets: $\{t\}, \{d_1, d_2\}, \{d_1, d_2, x_1, x_2\}$.

Results

ODE System	**Invariants Generated**
$v' = aw, w' = -v$	$f_0 + f_1(v^2 + aw^2) = 0$
$x' = v, v' = a, t' = 1$	$f_0 + f_1(-at + v) = 0$
$x' = x, t' = 1$	$f_0 + t \geq 0, f_1 + x^2 \geq 0$
α (prev. slide)	$f_0 + f_1(d_1^2 + d_2^2) = 0,$ $f_2 + f_3(d_2 - \omega x_1) + f_4(d_1 + \omega x_2) = 0,$ $f_5 + t \geq 0$

Results

Parametric invariant for motion equation: $f_0 + f_1(-at + v) = 0$

Set $f_1 = 1, f_0 = -v_0$: $v = v_0 + a * t$

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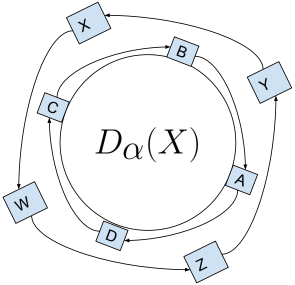
Another invariant:

$$f_0 + a^2 f_1 t^2 + \frac{1}{2} at(-2f_2 + f_3 t - 4f_1 v) + v(f_2 - f_3 t + f_1 v) + f_3 x = 0$$

Set $f_2 = f_1 = 0, f_3 = 1, f_0 = -x_0$: $x = x_0 + vt - \frac{1}{2}at^2$

Substituting previous equation: $x = x_0 + v_0 t + \frac{1}{2}at^2$

Conclusion



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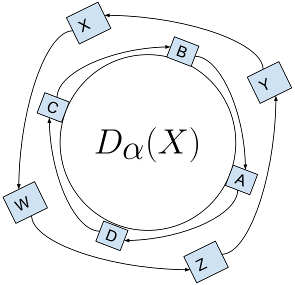

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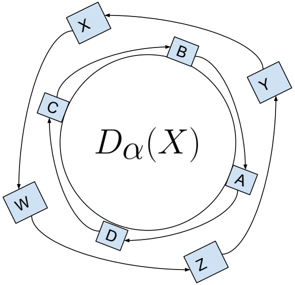

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$$f_0 + t \geq 0, f_1 + x^2 \geq 0$$
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$$f_2 + f_3(d_2 - \omega x_1) + f_4(d_1 + \omega x_2) = 0,$$
$$f_5 + t \geq 0$$

Questions?