Inductive Extensions to dL

Ananya Kumar

Carnegie Mellon University

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Motivation

- In dL, we often approximate functions
- Example: approximating arc length distance
- Use Taylor Series for approximation

Taylor Series

- Want to show:

$$e^x \geq 1 + x + \frac{x^2}{2} + ... + \frac{x^i}{i!}$$

Finite Case

- Want to show:

$$e^x \geq 1 + x + \frac{x^2}{2}$$

Model

- Capture $y = e^x$ using a differential equation

$$x = 0, y = 1 \vdash [(y' = y, x' = 1)]y \geq 1 + x + \frac{x^2}{2}$$

Proof

- Base case: use differential ghost

$$x = 0, y = 1 \vdash [(y' = y, x' = 1)]y \geq 1$$

Proof

- Cut base case into diff eq

$$x = 0, y = 1 \vdash [(y' = y, x' = 1 \ \& \ y \geq 1)]y \geq 1 + x$$

Proof

- Use a cut

$$y \geq 1 + x \vdash [(y' = y, x' = 1 \,\&\, y \geq 1)]y \geq 1 + x$$

Proof

- Use dI

$$\vdash [(y' = y, x' = 1 \ \& \ y \geq 1)] y' \geq 1$$

Proof

- Finish up using dE
- Repeat this same process to prove

$$e^x \geq 1 + x + \frac{x^2}{2}$$

Proof

- Repeat this same process to prove

$$e^x \geq 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

Proof

- Repeat this same process to prove

$$e^x \geq 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

Problems

- Very tedious
- Can repeatedly programmatically generate tactic
- Changing Taylor Series approximation requires reproving

Different Model

- Directly write a dL expression for:

$$e^x \geq 1 + x + \frac{x^2}{2} + ... + \frac{x^i}{i!}$$

Different Model

- Model:

$$
\begin{aligned}
x = 0, y = 1 \vdash [&(y' = y, x' = 1);\\
&s := 0; t := 1; i := 0;\\
&(s := s + t; i := i + 1; t := (t \cdot x)/i)^*]\\
&s \leq y
\end{aligned}
$$

Different Model

- Model:

$$x = 0, y = 1 \vdash [(y' = y, x' = 1);$$
$$s := 0; t := 1; i := 0;$$

Trace

- $s = 0,\ t = 1,\ i = 0$

Different Model

- Model:

$$x = 0, y = 1 \vdash [(y' = y, x' = 1);\\ s := 0; t := 1; i := 0;\\ (s := s + t; i := i + 1; t := (t \cdot x)/i)^0]$$

Trace

- $s = 0,\ t = 1,\ i = 0$

Different Model

- Model:

$$
\begin{aligned}
x = 0, y = 1 \vdash [&(y' = y, x' = 1);\\
&s := 0; t := 1; i := 0;\\
&(s := s + t; i := i + 1; t := (t \cdot x)/i)^1]
\end{aligned}
$$

Trace

- $s = 0 + 1,\ t = x,\ i = 1$

Different Model

- Model:

$$x = 0, y = 1 \vdash [(y' = y, x' = 1);$$
$$s := 0; t := 1; i := 0;$$
$$(s := s + t; i := i + 1; t := (t \cdot x)/i)^2]$$

Trace

- $s = 0 + 1 + x,\ t = x^2/2,\ i = 2$

Different Model

- Model:

$$x = 0, y = 1 \vdash [(y' = y, x' = 1);\\ s := 0; t := 1; i := 0;\\ (s := s + t; i := i + 1; t := (t \cdot x)/i)^3]$$

Trace

- $s = 0 + 1 + x + x^2/2,\ t = x^3/6,\ i = 3$

Proof

- Could try inducting on loop
- IH: $s \geq y$ is too weak
- Want to add connection between s' and s, and t' and t, x
- But $s' = t' = 0$

Problem

- s does not capture the expression $1 + x + \ldots$
- Only captures value
- So derivative of s is not connected to x

Alternative Model

- First loop, then apply differential equation

$$
\begin{aligned}
x = 0, y = 1 \vdash [&s := 0; t := 1; i := 0;\\
&(s := s + t; i := i + 1; t := (t \cdot x)/i)^*\\
&(y' = y, x' = 1);]\\
&s \leq y
\end{aligned}
$$

Alternative Model

- Cannot capture what we want
- s and t do not change with x and y

$$\begin{aligned} x = 0, y = 1 \vdash [&s := 0; t := 1; i := 0; \\ &(s := s + t; i := i + 1; t := (t \cdot x)/i)^* \\ &(y' = y, x' = 1);] \\ &s \leq y \end{aligned}$$

Symbol Trees

- Introduce symbol trees to dL
- Times(x, 3) represents polynomial $3x$
- Plus(Times(x, x), Plus(x, 4)) represents $x^2 + x + 4$

Symbol Trees

- Symbol trees are lazily evaluated
- Symbolically differentiated

Types: Encoded in Syntax

- s is string containing alphabets, underscores, and single quotations (')
- $R(s)$ is a real variable
- $S(s)$ is a symbol tree variable
- dL formula e is valid if $\omega \models e$ for all ω

Symbol Tree Term

- Symbol tree term inductively defined:

$$\text{SLeaf}(``R(s)") \mid \text{RLeaf}(r) \mid \text{Plus}(T_1, T_2) \mid \text{Times}(T_1, T_2) \mid \text{Div}(T_1, r) \mid \text{Neg}(T_1) \mid \text{Deriv}(T_1)$$

Real Tree Term

- Real term inductively defined

$$R(s) \mid c \mid e_1 + e_2 \mid e_1 \cdot e_2 \mid e_1/e_2 \mid E[T]$$

Assignments

- 2 types of assignments
- $R(x) := r$
- $S(x) := s$

Derivative Semantics

- If $\omega[[T]] = \text{RLeaf}(r)$, then $\omega[[\text{Deriv}(T)]] = \text{RLeaf}(0)$
- If $\omega[[T]] = \text{SLeaf}(\text{“}R(s)\text{”})$, then $\omega[[\text{Deriv}(T)]] = \text{SLeaf}(\text{“}R(s)'\text{”})$
- If $\omega[[T]] = \text{Plus}(T_1, T_2)$, then $\omega[[\text{Deriv}(T)]] = \text{Plus}(\omega[[\text{Deriv}(T_1)]], \omega[[\text{Deriv}(T_2)]])$

Equality

- $=$ means symbol trees are syntactically identical

$$\frac{T_1 = T_1' \quad T_2 = T_2'}{\text{Times}(T_1, T_2) = \text{Times}(T_1', T_2')} \qquad \frac{T_1 = T_1' \quad T_2 = T_2'}{\text{Plus}(T_1, T_2) = \text{Plus}(T_1', T_2')}$$

$$\frac{T_1 = T_1' \quad r = r' \quad r, r' \neq 0 \in \mathbb{R}}{\text{Divide}(T_1, r) = \text{Divide}(T_1', r')}$$

Equivalence

- $\equiv$ is weaker than $=$

$$\frac{}{\mathsf{Times}(T_1, T_2) \equiv \mathsf{Times}(T_2, T_1)} \qquad \frac{}{\mathsf{Plus}(T_1, T_2) \equiv \mathsf{Plus}(T_2, T_1)}$$

$$\frac{}{\mathsf{Times}(T_1, \mathsf{RLeaf}(0)) \equiv \mathsf{RLeaf}(0)} \qquad \frac{}{\mathsf{Plus}(T_1, \mathsf{RLeaf}(0)) = T_1}$$

$$\frac{}{\mathsf{Deriv}(\mathsf{Plus}(T_1, T_2)) \equiv \mathsf{Plus}(\mathsf{Deriv}(T_1), \mathsf{Deriv}(T_2))}$$

Equivalence

- Equivalence in symbol trees translates to equivalence in evaluation

$$\frac{\gamma \vdash T_1 \equiv T_2}{\gamma \vdash E[T_1] = E[T_2]}$$

Evaluation Rules

- Additional rules to compute $E[T]$
- Translate composition in symbol trees to reals

$$\frac{\gamma \vdash E[T_1] = r_1 \quad \gamma \vdash E[T_2] = r_2}{\gamma \vdash E[\mathrm{Plus}(T_1, T_2)] = r_1 + r_2}$$

$$\frac{\gamma \vdash E[T_1] = r_1 \quad \gamma \vdash E[T_2] = r_2}{\gamma \vdash E[\mathrm{Times}(T_1, T_2)] = r_1 \cdot r_2}$$

dI

- dI rule applies as usual, with additional definition

$$(E[T])' = E[\text{Deriv}(T)]$$

New Model

$$R(x) = 0, R(y) = 1$$

New Model

$$R(x) = 0, R(y) = 1, R(i) = 0,$$
$$S(s) = \mathsf{RLeaf}(0), S(t) = \mathsf{RLeaf}(1)$$
$$\vdash$$

New Model

$$R(x) = 0, R(y) = 1, R(i) = 0,$$
$$S(s) = \text{RLeaf}(0), S(t) = \text{RLeaf}(1)$$
$$\vdash$$
$$[(S(s) := \text{Plus}(S(s), S(t));$$
$$R(i) := R(i) + 1;$$
$$S(t) := \text{Divide}(\text{Times}(S(t), \text{SLeaf}(\text{``}R(x)\text{''})), \text{RLeaf}(R(i))))^*]$$

New Model

$$
\begin{aligned}
&R(x) = 0, R(y) = 1, R(i) = 0,\\
&S(s) = \mathrm{RLeaf}(0), S(t) = \mathrm{RLeaf}(1)\\
&\vdash\\
&[(S(s) := \mathrm{Plus}(S(s), S(t));\\
&\;\; R(i) := R(i) + 1;\\
&\;\; S(t) := \mathrm{Divide}(\mathrm{Times}(S(t), \mathrm{SLeaf}(\text{“}R(x)\text{”})), \mathrm{RLeaf}(R(i))))^*]\\
&[(R(y)' = R(y), R(x)' = 1)]
\end{aligned}
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New Model

$$
\begin{aligned}
& R(x) = 0, R(y) = 1, R(i) = 0, \\
& S(s) = \mathrm{RLeaf}(0), S(t) = \mathrm{RLeaf}(1) \\
& \vdash \\
& [(S(s) := \mathrm{Plus}(S(s), S(t)); \\
& \quad R(i) := R(i) + 1; \\
& \quad S(t) := \mathrm{Divide}(\mathrm{Times}(S(t), \mathrm{SLeaf}(\text{``}R(x)\text{''})), \mathrm{RLeaf}(R(i))))^{*}] \\
& [(R(y)' = R(y), R(x)' = 1)] \\
& E[S(s)] \leq R(y)
\end{aligned}
$$

Enables Proof

- We use a loop invariant
- Uses symbolic derivative
- Use differential cut and dI like in original proof

Big Caveats

- We have not proved soundness for symbol trees
- Taylor Series: 1.5 page proof sketch, including invariant
- Not formally verified

Conclusion

- Aim: Explore a cool idea and showed its applications
- Future work: Prove soundness, find other applications
- Thanks to Andre and Brandon for advice