

Assignment 3: Proofs and Differential Invariants (Part 1)
15-424/15-624/15-824 Foundations of Cyber-Physical Systems
Course TA: Brandon Bohrer (bbohrer@cs)

Due: 11:59pm, Tuesday 2/28 **TUESDAY BEFORE EXAM**
 Total Points: 60

1. **A Differential Proof Rule.** Determine if the following proof rule is sound or unsound, then prove your conjecture using a semantic argument.

$$\text{(PART A)} \frac{\Gamma \vdash J, \Delta \quad \Gamma, H \vdash [x' := f(x)](J)', \Delta \quad J \vdash F}{\Gamma \vdash [x' = f(x) \& H]F, \Delta}$$

Hint: Proof rules prove a little differently than axioms. For an axiom $\phi \rightarrow \psi$, we had to show that for any state ν , $\nu \models \phi \rightarrow \psi$. This means that when we show $\nu \models \psi$, all we get to assume is $\nu \models \phi$ for the same state ν .

The soundness of a proof rule $\frac{\vdash \phi}{\vdash \psi}$ says that if the top is valid, the bottom is valid. To prove soundness of $\frac{\vdash \phi}{\vdash \psi}$, you assume that for all ν , $\nu \models \phi$. Then you try to prove that for all ν , $\nu \models \psi$. Because your assumption contains a forall, you can assume, for example, $\omega \models \phi$ when showing $\nu \models \psi$. This gives you more flexibility in the proof than when proving an axiom.

2. **Differential Axioms.** Determine if the following axioms are sound or unsound, then give a semantics proof or disproof for each axiom. Your proofs can and should take advantage of semantic definitions and lemmas from lecture, but do not have to be incredibly long.

- (a) $(x^2)' = 2xx'$
- (b) $(y^5)' = 5y^4$
- (c) $([x' = f(x) \& H]P) \rightarrow (H \rightarrow P)$
- (d) $(x)' = x'$
- (e) $(\theta_1 \cdot \theta_2)' = (\theta_1)' \cdot \theta_2 + \theta_1 \cdot (\theta_2)'$

Note the notation in Question (d): $(x)'$ is the differential of the term x , while x' is a differential symbol.

3. **Practice Using Differential Invariants.** Prove each of the following statements using a differential invariant and any other proof rules presented in class that are needed to prove the property. Do not use differential solution proof rules/axioms.

(a) $xy = 0 \rightarrow [\{x' = -10xy, y' = 10y^2\}]xy = 0$

(b) $y \neq 0 \wedge \frac{x}{y^2} = 1 \rightarrow [\{x' = 2x, y' = y \ \& \ y \neq 0\}] \frac{x}{y^2} = 1$

(c) $x^4 + y^5 = 10 \rightarrow [\{x' = 10y^4, y' = -8x^3\}] x^4 + y^5 = 10$

(d) $x + y \neq z \rightarrow [\{x' = 2x, y' = 4x, z' = 6x\}] x + y \neq z$

(e) $x \geq 0 \wedge y \geq 0 \rightarrow [x' = y, y' = 1](x \geq 0 \wedge y \geq 0)$