20: Virtual Substitution & Real Equations

15-424: Foundations of Cyber-Physical Systems

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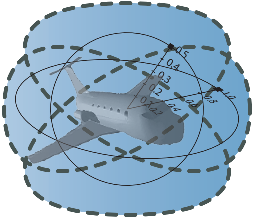


Outline

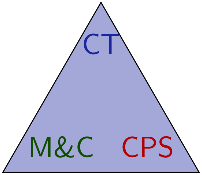
1. Learning Objectives
2. Real Arithmetic
   - Evaluating Real Arithmetic
   - Framing the Miracle
   - QE Example
   - Quantifier Elimination
   - QE Framework
   - Virtual Substitution by Example
   - Linear Virtual Substitution
   - Quadratic Virtual Substitution
3. Virtual Substitution
   - Square Root Expression Algebra
   - Virtual Square Root Comparisons
   - Example
4. Summary

Outline

Learning Objectives

Virtual Substitution & Real Equations

rigorous arithmetical reasoning
miracle of quantifier elimination
logical trinity for reals
switch between syntax & semantics at will
virtual substitution lemma
bridge gap between semantics and inexpressibles



analytic complexity
modeling tradeoffs

verifying CPS at scale

Outline

Evaluating Real Arithmetic Formulas

$x^2 > 2 \land 2x < 3 \lor x^3 < x^2$

Evaluating Real Arithmetic Formulas

When $\omega(x) = 2$

$\omega[\![x^2 > 2 \land 2x < 3 \lor x^3 < x^2]\!]$

Evaluating Real Arithmetic Formulas

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When $\nu(x) = -1$

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Evaluating Real Arithmetic Formulas

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Are the following formulas valid, i.e. true in all states?

$$x^2 > 2 \land 2x < 3 \lor x^3 < x^2$$
$$\forall x\,(x^2 > 2 \land 2x < 3 \lor x^3 < x^2)$$
$$\exists x\,(x^2 > 2 \land 2x < 3 \lor x^3 < x^2)$$

Evaluating Real Arithmetic Formulas

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Evaluating Real Arithmetic Formulas

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Framing the Miracle: Quiz

Is validity of formulas
decidable/semidecidable/undecidable/not semidecidable for:

1. Propositional logic
2. FOL uninterpreted
3. $\text{FOL}_{\mathbb{N}}[+, \cdot, =]$
4. $\text{FOL}_{\mathbb{R}}[+, \cdot, =, <]$
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- ✓ $\text{FOL}_{\mathbb{R}}[+,=,\wedge,\exists]$ — decidable Gaussian elim. [179 CE]
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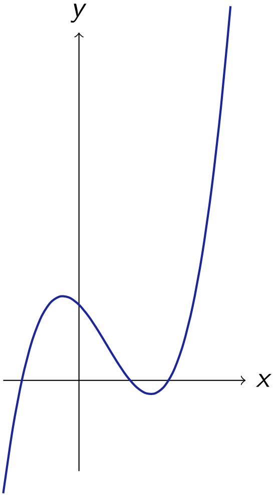
×	$FOL_{\mathbb{R}}[+,\cdot,sin,=,<]$ $\sin x=0$	not semidecidable

Quantifier Elimination $\leftrightsquigarrow$ Projection



$$F \equiv \exists y \, (y \geq 0 \land 1 - x - 1.83x^2 + 1.66x^3 > y)$$

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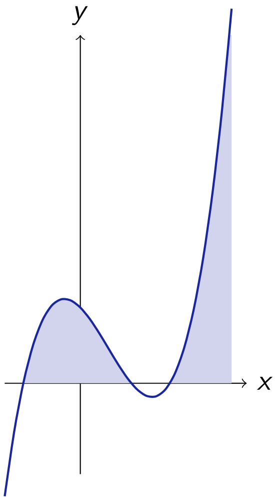

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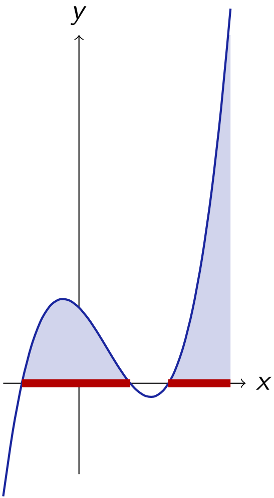

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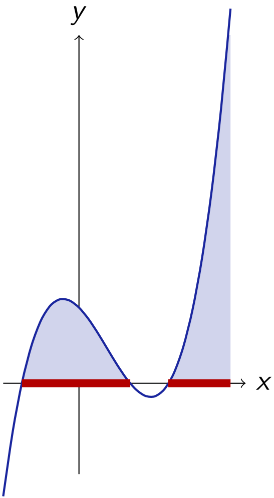

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QE

$$\text{QE}(F) \equiv 0.75 < x \land x < 0.68 \lor x > 1.17$$

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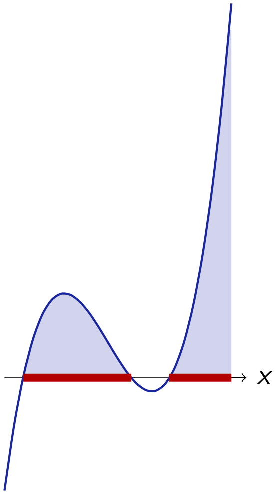


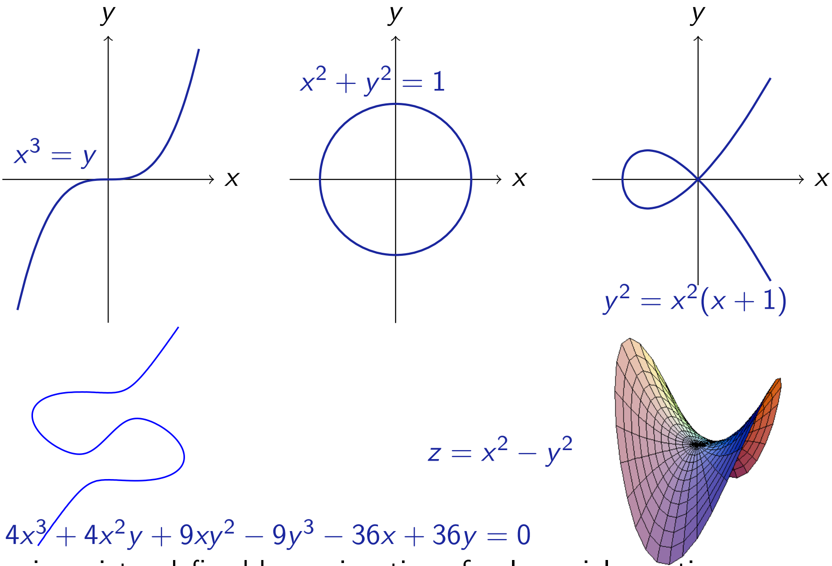
QE

$$\text{QE}(F) \equiv 0.75 < x \land x < 0.68 \lor x > 1.17$$

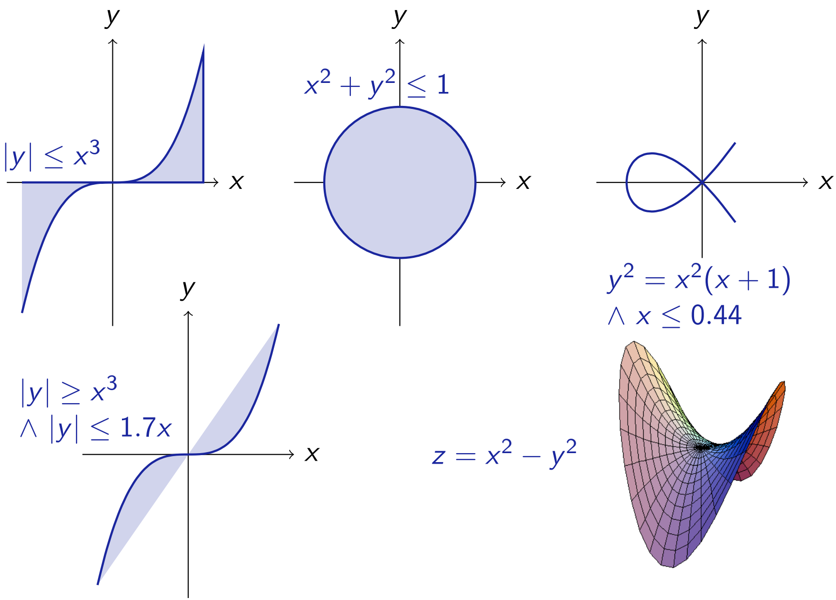
If all but one variable fixed: Finite union of intervals.

Univariate polynomials have finitely many roots. Sign changes finitely often.

Polynomial Equations $\rightsquigarrow$ Algebraic Varieties



Algebraic variety: defined by conjunction of polynomial equations



y
x
$|y| \leq x^3$
y
x
$x^2 + y^2 \leq 1$
y
x
$y^2 = x^2(x+1)$
$\land\ x \leq 0.44$
y
x
$|y| \geq x^3$
$\land\ |y| \leq 1.7x$
$z = x^2 - y^2$

Quantifier Elimination in Real Arithmetic

Theorem (Tarski'31)

First-order logic of real arithmetic is decidable since it admits quantifier elimination, i.e. with each formula ϕ*, a quantifier-free formula* $\mathrm{QE}(\phi)$ *can be associated effectively that is equivalent, i.e.* $\phi \leftrightarrow \mathrm{QE}(\phi)$ *is valid.*

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Theorem (Complexity, Davenport&Heintz'88,Weispfenning'88)

(Time and space) complexity of QE *for* $\mathbb{R}$ *is doubly exponential in the number of quantifier (alternations).*

$$2^{2^{O(n)}}$$

Quantifier Elimination Examples

- $\mathrm{QE}(\exists x\,(2x^2 + y \leq 5)) \equiv$

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  $\text{QE}(\forall y\,(y \leq 5)) \equiv -100 \leq 5 \land 5 \leq 5 \land 100 \leq 5 \equiv \textit{false}$
- $\text{QE}(\exists x\,(a = b + x^2)) \equiv$

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- $\mathrm{QE}(\exists x\,(a = b + x^2)) \equiv a \ge b$

Framework: Logical Normalization for QE

$$\begin{aligned}
\mathrm{QE}(A \land B) &\equiv \\
\mathrm{QE}(A \lor B) &\equiv \\
\mathrm{QE}(\neg A) &\equiv \\
\mathrm{QE}(\forall x\, A) &\equiv \\
\mathrm{QE}(\exists x\, A) &\equiv && A \text{ has quantifiers}
\end{aligned}$$

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\mathrm{QE}(A \land B) &\equiv \mathrm{QE}(A) \land \mathrm{QE}(B) \\
\mathrm{QE}(A \lor B) &\equiv \mathrm{QE}(A) \lor \mathrm{QE}(B) \\
\mathrm{QE}(\neg A) &\equiv \neg\, \mathrm{QE}(A) \\
\mathrm{QE}(\forall x\, A) &\equiv \mathrm{QE}(\neg \exists x\, \neg A) \\
\mathrm{QE}(\exists x\, A) &\equiv \mathrm{QE}(\exists x\ \mathrm{QE}(A)) && A \text{ has quantifiers}
\end{aligned}$$

Framework: Logical Normalization for QE

$$
\begin{aligned}
\mathrm{QE}(A \land B) &\equiv \mathrm{QE}(A) \land \mathrm{QE}(B) \\
\mathrm{QE}(A \lor B) &\equiv \mathrm{QE}(A) \lor \mathrm{QE}(B) \\
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\mathrm{QE}(\forall x\, A) &\equiv \mathrm{QE}(\neg \exists x\, \neg A) \\
\mathrm{QE}(\exists x\, A) &\equiv \mathrm{QE}(\exists x\, \mathrm{QE}(A)) && A \text{ has quantifiers} \\
\mathrm{QE}(\exists x\, (A \lor B)) &\equiv \\
\mathrm{QE}(\exists x\, \neg(A \land B)) &\equiv \\
\mathrm{QE}(\exists x\, \neg(A \lor B)) &\equiv \\
\mathrm{QE}(\exists x\, \neg\neg A) &\equiv
\end{aligned}
$$

Framework: Logical Normalization for QE

$$\begin{aligned}
\mathrm{QE}(A \land B) &\equiv \mathrm{QE}(A) \land \mathrm{QE}(B) \\
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\mathrm{QE}(\exists x\, (A \lor B)) &\equiv \mathrm{QE}(\exists x\, A) \lor \mathrm{QE}(\exists x\, B) \\
\mathrm{QE}(\exists x\, \neg(A \land B)) &\equiv \mathrm{QE}(\exists x\, (\neg A \lor \neg B)) && \text{with cost} \\
\mathrm{QE}(\exists x\, \neg(A \lor B)) &\equiv \mathrm{QE}(\exists x\, (\neg A \land \neg B)) && \text{with cost} \\
\mathrm{QE}(\exists x\, \neg\neg A) &\equiv \mathrm{QE}(\exists x\, A)
\end{aligned}$$

$$
\begin{aligned}
\mathrm{QE}(A \land B) &\equiv \mathrm{QE}(A) \land \mathrm{QE}(B) \\
\mathrm{QE}(A \lor B) &\equiv \mathrm{QE}(A) \lor \mathrm{QE}(B) \\
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\mathrm{QE}(\exists x\, A) &\equiv \mathrm{QE}(\exists x\ \mathrm{QE}(A)) && A \text{ has quantifiers} \\
\mathrm{QE}(\exists x\, (A \lor B)) &\equiv \mathrm{QE}(\exists x\, A) \lor \mathrm{QE}(\exists x\, B) \\
\mathrm{QE}(\exists x\, \neg(A \land B)) &\equiv \mathrm{QE}(\exists x\, (\neg A \lor \neg B)) && \text{with cost} \\
\mathrm{QE}(\exists x\, \neg(A \lor B)) &\equiv \mathrm{QE}(\exists x\, (\neg A \land \neg B)) && \text{with cost} \\
\mathrm{QE}(\exists x\, \neg\neg A) &\equiv \mathrm{QE}(\exists x\, A) \\
\mathrm{QE}(\exists x\, (A \land (B \lor C))) &\equiv \\
\mathrm{QE}(\exists x\, ((A \lor B) \land C)) &\equiv
\end{aligned}
$$

Framework: Logical Normalization for QE

$$\begin{aligned}
\mathrm{QE}(A \land B) &\equiv \mathrm{QE}(A) \land \mathrm{QE}(B) \\
\mathrm{QE}(A \lor B) &\equiv \mathrm{QE}(A) \lor \mathrm{QE}(B) \\
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\mathrm{QE}(\exists x\, A) &\equiv \mathrm{QE}(\exists x\ \mathrm{QE}(A)) && A \text{ has quantifiers} \\
\mathrm{QE}(\exists x\, (A \lor B)) &\equiv \mathrm{QE}(\exists x\, A) \lor \mathrm{QE}(\exists x\, B) \\
\mathrm{QE}(\exists x\, \neg(A \land B)) &\equiv \mathrm{QE}(\exists x\, (\neg A \lor \neg B)) && \text{with cost} \\
\mathrm{QE}(\exists x\, \neg(A \lor B)) &\equiv \mathrm{QE}(\exists x\, (\neg A \land \neg B)) && \text{with cost} \\
\mathrm{QE}(\exists x\, \neg\neg A) &\equiv \mathrm{QE}(\exists x\, A) \\
\mathrm{QE}(\exists x\, (A \land (B \lor C))) &\equiv \mathrm{QE}(\exists x\, ((A \land B) \lor (A \land C))) && \text{if need be} \\
\mathrm{QE}(\exists x\, ((A \lor B) \land C)) &\equiv \mathrm{QE}(\exists x\, ((A \land C) \lor (B \land C))) && \text{if need be}
\end{aligned}$$

Framework: Logical Normalization for QE

Normal Form $\text{QE}(\exists x\,(A_1 \land \ldots \land A_k))$ with atomic A_i

$$
\begin{aligned}
\text{QE}(A \land B) &\equiv \text{QE}(A) \land \text{QE}(B) \\
\text{QE}(A \lor B) &\equiv \text{QE}(A) \lor \text{QE}(B) \\
\text{QE}(\neg A) &\equiv \neg\, \text{QE}(A) \\
\text{QE}(\forall x\, A) &\equiv \text{QE}(\neg \exists x\, \neg A) \\
\text{QE}(\exists x\, A) &\equiv \text{QE}(\exists x\ \text{QE}(A)) && A \text{ has quantifiers} \\
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\text{QE}(\exists x\, \neg(A \land B)) &\equiv \text{QE}(\exists x\,(\neg A \lor \neg B)) && \text{with cost} \\
\text{QE}(\exists x\, \neg(A \lor B)) &\equiv \text{QE}(\exists x\,(\neg A \land \neg B)) && \text{with cost} \\
\text{QE}(\exists x\, \neg\neg A) &\equiv \text{QE}(\exists x\, A) \\
\text{QE}(\exists x\,(A \land (B \lor C))) &\equiv \text{QE}(\exists x\,((A \land B) \lor (A \land C))) && \text{if need be} \\
\text{QE}(\exists x\,((A \lor B) \land C)) &\equiv \text{QE}(\exists x\,((A \land C) \lor (B \land C))) && \text{if need be}
\end{aligned}
$$

Framework: Arithmetical Normalization for QE

Normal Form $\text{QE}(\exists x\,(p_1 \sim_i 0 \land \ldots \land p_k \sim_k 0))$ and $\sim_i \in \{>, =, \geq, \neq\}$

$$
\begin{aligned}
p = q &\equiv p - q = 0\\
p \geq q &\equiv p - q \geq 0\\
p > q &\equiv p - q > 0\\
p \neq q &\equiv p - q \neq 0\\
p \leq q &\equiv q - p \geq 0\\
p < q &\equiv q - p > 0\\
\neg(p \geq q) &\equiv p < q\\
\neg(p > q) &\equiv p \leq q\\
\neg(p = q) &\equiv p \neq q\\
\neg(p \neq q) &\equiv p = q
\end{aligned}
$$

Quantifier Elimination by Virtual Substitution

Virtual Substitution

$$\exists x\, F \leftrightarrow \bigvee_{t \in T} A_t \land F_x^t$$

where terms T substituted (virtually) into F depend on F
where A_t are quantifier-free additional compatibility conditions

Needs simplifier for intermediate results

Quantifier Elimination by Virtual Substitution

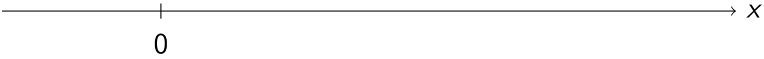
Virtual Substitution

Quantifier $\exists x\, F \leftrightarrow \bigvee_{t \in T} A_t \wedge F_x^t$ Quantifier-free

where terms T substituted (virtually) into F depend on F
where A_t are quantifier-free additional compatibility conditions

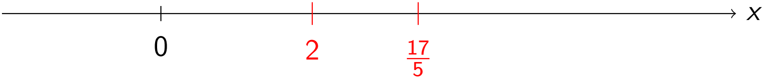
Needs simplifier for intermediate results

Virtual Substitution by Example



Can we get rid of the quantifier without changing the semantics?

$$\exists x (x > 2 \land x < \frac{17}{5})$$



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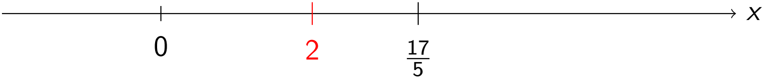
Virtual Substitution by Example

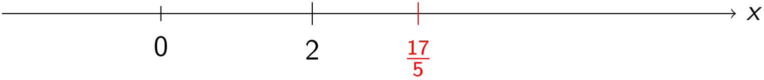


Can we get rid of the quantifier without changing the semantics?

$$\begin{aligned} & \exists x (x > 2 \land x < \tfrac{17}{5}) \\ \equiv \quad & (2 > 2 \land 2 < \tfrac{17}{5}) \end{aligned}$$

boundary case "$x = 2$"

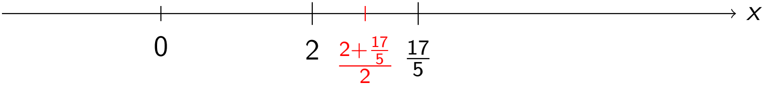
Virtual Substitution by Example



Can we get rid of the quantifier without changing the semantics?

$$
\begin{array}{lll}
 & \exists x (x > 2 \land x < \frac{17}{5}) & \\
\equiv & (2 > 2 \land 2 < \frac{17}{5}) & \text{boundary case "}x = 2\text{"} \\
\lor & (\frac{17}{5} > 2 \land \frac{17}{5} < \frac{17}{5}) & \text{boundary case "}x = \frac{17}{5}\text{"}
\end{array}
$$

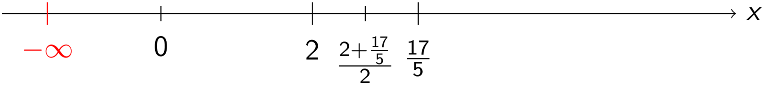
Virtual Substitution by Example



Can we get rid of the quantifier without changing the semantics?

$$
\begin{array}{lll}
 & \exists x (x > 2 \land x < \frac{17}{5}) & \\
\equiv & (2 > 2 \land 2 < \frac{17}{5}) & \text{boundary case "}x = 2\text{"} \\
\lor & (\frac{17}{5} > 2 \land \frac{17}{5} < \frac{17}{5}) & \text{boundary case "}x = \frac{17}{5}\text{"} \\
\lor & (\frac{2+\frac{17}{5}}{2} > 2 \land \frac{2+\frac{17}{5}}{2} < \frac{17}{5}) & \text{intermediate case "}x = \frac{2+\frac{17}{5}}{2}\text{"}
\end{array}
$$

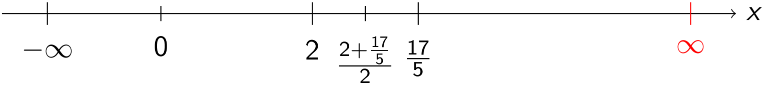
Virtual Substitution by Example



Can we get rid of the quantifier without changing the semantics?

$$
\begin{array}{lll}
 & \exists x (x > 2 \land x < \frac{17}{5}) & \\
\equiv & (2 > 2 \land 2 < \frac{17}{5}) & \text{boundary case "} x = 2 \text{"} \\
\lor & (\frac{17}{5} > 2 \land \frac{17}{5} < \frac{17}{5}) & \text{boundary case "} x = \frac{17}{5} \text{"} \\
\lor & (\frac{2+\frac{17}{5}}{2} > 2 \land \frac{2+\frac{17}{5}}{2} < \frac{17}{5}) & \text{intermediate case "} x = \frac{2+\frac{17}{5}}{2} \text{"} \\
\lor & (-\infty > 2 \land -\infty < \frac{17}{5}) & \text{extremal case "} x = -\infty \text{"}
\end{array}
$$

Virtual Substitution by Example



Can we get rid of the quantifier without changing the semantics?

$$
\begin{array}{lll}
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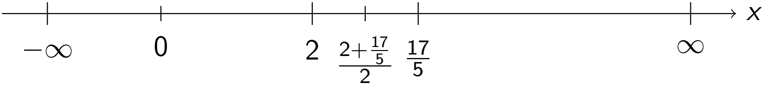
\equiv & (2 > 2 \land 2 < \frac{17}{5}) & \text{boundary case "}x = 2\text{"} \\
\lor & (\frac{17}{5} > 2 \land \frac{17}{5} < \frac{17}{5}) & \text{boundary case "}x = \frac{17}{5}\text{"} \\
\lor & (\frac{2+\frac{17}{5}}{2} > 2 \land \frac{2+\frac{17}{5}}{2} < \frac{17}{5}) & \text{intermediate case "}x = \frac{2+\frac{17}{5}}{2}\text{"} \\
\lor & (-\infty > 2 \land -\infty < \frac{17}{5}) & \text{extremal case "}x = -\infty\text{"} \\
\lor & (\infty > 2 \land \infty < \frac{17}{5}) & \text{extremal case "}x = \infty\text{"}
\end{array}
$$

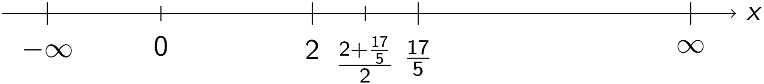
Virtual Substitution by Example



Can we get rid of the quantifier without changing the semantics?

$$
\begin{array}{cll}
 & \exists x (x > 2 \land x < \frac{17}{5}) & \\
\equiv & (2 > 2 \land 2 < \frac{17}{5}) & \text{boundary case "}x = 2\text{"} \\
\lor & (\frac{17}{5} > 2 \land \frac{17}{5} < \frac{17}{5}) & \text{boundary case "}x = \frac{17}{5}\text{"} \\
\lor & (\frac{2+\frac{17}{5}}{2} > 2 \land \frac{2+\frac{17}{5}}{2} < \frac{17}{5}) & \text{intermediate case "}x = \frac{2+\frac{17}{5}}{2}\text{"} \\
\lor & (-\infty > 2 \land -\infty < \frac{17}{5}) & \text{extremal case "}x = -\infty\text{"} \\
\lor & (\infty > 2 \land \infty < \frac{17}{5}) & \text{extremal case "}x = \infty\text{"} \\
\equiv & \mathit{true} & \text{evaluate}
\end{array}
$$

Virtual Substitution by Example



Can we get rid of the quantifier without changing the semantics?

$$
\begin{array}{lll}
 & \exists x\,(x > 2 \land x < \frac{17}{5}) & \\
\equiv & (2 > 2 \land 2 < \frac{17}{5}) & \text{boundary case ``} x = 2\text{''} \\
\lor & (\frac{17}{5} > 2 \land \frac{17}{5} < \frac{17}{5}) & \text{boundary case ``} x = \frac{17}{5}\text{''} \\
\lor & (\frac{2+\frac{17}{5}}{2} > 2 \land \frac{2+\frac{17}{5}}{2} < \frac{17}{5}) & \text{intermediate case ``} x = \frac{2+\frac{17}{5}}{2}\text{''} \\
\lor & (-\infty > 2 \land -\infty < \frac{17}{5}) & \text{extremal case ``} x = -\infty\text{''} \\
\lor & (\infty > 2 \land \infty < \frac{17}{5}) & \text{extremal case ``} x = \infty\text{''} \\
\equiv & \mathit{true} & \text{evaluate}
\end{array}
$$

- ∞ is not in $\mathrm{FOL}_{\mathbb{R}}$
- Interior points aren't always terms in $\mathrm{FOL}_{\mathbb{R}}$ if nonlinear
- Substituting them into formulas requires attention

Linear Virtual Substitution

Theorem (Virtual Substitution: Linear Equation)

$$\exists x \, (bx + c = 0 \land F) \leftrightarrow$$

Linear Virtual Substitution

Theorem (Virtual Substitution: Linear Equation)

$$\exists x\, (bx + c = 0 \land F) \;\leftrightarrow\; F_x^{-c/b}$$

Linear solution

Linear Virtual Substitution

Theorem (Virtual Substitution: Linear Equation)

$$\exists x\,(bx + c = 0 \land F) \;\leftrightarrow\; b \neq 0 \land F_x^{-c/b}$$

Don't divide by 0

Linear Virtual Substitution

Theorem (Virtual Substitution: Linear Equation)

$$b \neq 0 \rightarrow \left(\exists x \left(bx + c = 0 \land F\right) \leftrightarrow b \neq 0 \land F_x^{-c/b}\right)$$

Only actually linear solution if $b \neq 0$

Linear Virtual Substitution

Theorem (Virtual Substitution: Linear Equation $x \notin b, c$)

$$b \neq 0 \rightarrow \left(\exists x \left(bx + c = 0 \wedge F\right) \leftrightarrow b \neq 0 \wedge F_x^{-c/b}\right) \quad \text{if } x \notin b, c$$

Only linear if no x in b, c

Linear Virtual Substitution

Theorem (Virtual Substitution: Linear Equation $x \notin b, c$)

$$b \neq 0 \rightarrow \left(\exists x \left(bx + c = 0 \land F\right) \leftrightarrow b \neq 0 \land F_x^{-c/b}\right) \quad \textit{if } x \notin b, c$$

Conditional equivalence, so conditions may need to be checked or case-split

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation)

$$\exists x\,(ax^2 + bx + c = 0 \land F) \leftrightarrow$$

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation)

$$\exists x\,(ax^2 + bx + c = 0 \land F) \leftrightarrow$$

$$F_x^{(-b+\sqrt{b^2-4ac})/(2a)}$$

Quadratic solution

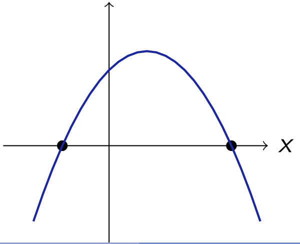
Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation)

$$\exists x\,(ax^2 + bx + c = 0 \land F) \leftrightarrow$$

$$\left(F_x^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_x^{(-b-\sqrt{b^2-4ac})/(2a)}\right)$$

Or negative square root solution



Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation)

$$\exists x\,(ax^2 + bx + c = 0 \land F) \leftrightarrow$$

$$a \neq 0 \land \quad \big(F_x^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_x^{(-b-\sqrt{b^2-4ac})/(2a)}\big)$$

Don't divide by 0

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation)

$$\exists x\,(ax^2 + bx + c = 0 \land F) \leftrightarrow$$

$$a \neq 0 \land b^2 - 4ac \geq 0 \land \left(F_x^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_x^{(-b-\sqrt{b^2-4ac})/(2a)}\right)$$

Real solution if square root exists by discriminant

Quadratic Virtual Substitution

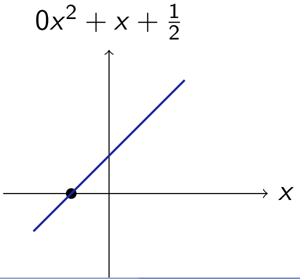
Theorem (Virtual Substitution: Quadratic Equation)

$$\exists x\,(ax^2 + bx + c = 0 \land F) \leftrightarrow$$

$$a = 0 \land b \neq 0 \land F_x^{-c/b}$$

$$\lor\ a \neq 0 \land b^2 - 4ac \geq 0 \land \left(F_x^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_x^{(-b-\sqrt{b^2-4ac})/(2a)}\right)$$

Instead linear solution if $a = 0$ (may case-split)



Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation)

$$a \neq 0 \lor b \neq 0 \lor c \neq 0 \rightarrow$$
$$\Big(\exists x\,(ax^2 + bx + c = 0 \land F) \leftrightarrow$$
$$a = 0 \land b \neq 0 \land F_x^{-c/b}$$
$$\lor\ a \neq 0 \land b^2 - 4ac \geq 0 \land \big(F_x^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_x^{(-b-\sqrt{b^2-4ac})/(2a)}\big)\Big)$$

Only equivalent solution if not all 0 which gives trivial equation

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$a \neq 0 \lor b \neq 0 \lor c \neq 0 \rightarrow$$
$$\Big(\exists x\,(ax^2 + bx + c = 0 \land F) \leftrightarrow$$
$$a = 0 \land b \neq 0 \land F_x^{-c/b}$$
$$\lor\ a \neq 0 \land b^2 - 4ac \geq 0 \land \big(F_x^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_x^{(-b-\sqrt{b^2-4ac})/(2a)}\big)\Big)$$

Only linear or quadratic if no x in a, b, c

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$a \neq 0 \lor b \neq 0 \lor c \neq 0 \rightarrow$$
$$\Big(\exists x \, (ax^2 + bx + c = 0 \land F) \leftrightarrow$$
$$a = 0 \land b \neq 0 \land F_x^{-c/b}$$
$$\lor \; a \neq 0 \land b^2 - 4ac \geq 0 \land \left(F_x^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_x^{(-b-\sqrt{b^2-4ac})/(2a)}\right)\Big)$$

1. Quantifier-free equivalent

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$a \neq 0 \lor b \neq 0 \lor c \neq 0 \rightarrow$$
$$\Big(\exists x\,(ax^2 + bx + c = 0 \land F) \leftrightarrow$$
$$a = 0 \land b \neq 0 \land F_x^{-c/b}$$
$$\lor\ a \neq 0 \land b^2 - 4ac \geq 0 \land \big(F_x^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_x^{(-b-\sqrt{b^2-4ac})/(2a)}\big)\Big)$$

1. Quantifier-free equivalent
2. Just not a formula . . .

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$a \neq 0 \lor b \neq 0 \lor c \neq 0 \rightarrow$$
$$\Big(\exists x\, (ax^2 + bx + c = 0 \land F) \leftrightarrow$$
$$a = 0 \land b \neq 0 \land F_x^{-c/b}$$
$$\lor\ a \neq 0 \land b^2 - 4ac \geq 0 \land \big(F_x^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_x^{(-b-\sqrt{b^2-4ac})/(2a)}\big)\Big)$$

1. Quantifier-free equivalent
2. Just not a formula . . .
3. $(-b + \sqrt{b^2 - 4ac})/(2a)$ is not in $\text{FOL}_{\mathbb{R}}$ and neither is $-c/b$

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$a \neq 0 \lor b \neq 0 \lor c \neq 0 \rightarrow$$
$$\Big(\exists x\,(ax^2 + bx + c = 0 \land F) \leftrightarrow$$
$$a = 0 \land b \neq 0 \land F_x^{-c/b}$$
$$\lor\, a \neq 0 \land b^2 - 4ac \geq 0 \land \left(F_x^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_x^{(-b-\sqrt{b^2-4ac})/(2a)}\right)\Big)$$

1. Quantifier-free equivalent
2. Just not a formula ...
3. $(-b + \sqrt{b^2 - 4ac})/(2a)$ is not in $\mathrm{FOL}_{\mathbb{R}}$ and neither is $-c/b$
4. Virtual substitution $F_{\bar{x}}^{(a+b\sqrt{c})/d}$ acts as if it were to substitute $(a + b\sqrt{c})/d$ for x in F

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$a \neq 0 \lor b \neq 0 \lor c \neq 0 \rightarrow$$
$$\Big(\exists x\,(ax^2 + bx + c = 0 \land F) \leftrightarrow$$
$$a = 0 \land b \neq 0 \land F_{\bar{x}}^{-c/b}$$
$$\lor\, a \neq 0 \land b^2 - 4ac \geq 0 \land \big(F_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \lor F_{\bar{x}}^{(-b-\sqrt{b^2-4ac})/(2a)}\big)\Big)$$

1. Quantifier-free equivalent
2. Just not a formula ...
3. $(-b + \sqrt{b^2 - 4ac})/(2a)$ is not in $\mathrm{FOL}_{\mathbb{R}}$ and neither is $-c/b$
4. Virtual substitution $F_{\bar{x}}^{(a+b\sqrt{c})/d}$ acts as if it were to substitute $(a + b\sqrt{c})/d$ for x in F ... but it's merely equivalent

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$a \neq 0 \lor b \neq 0 \lor c \neq 0 \rightarrow$$
$$\Big(\exists x\, (ax^2 + bx + c = 0 \land F) \leftrightarrow$$
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1. Quantifier-free equivalent
2. Just not a formula ...
3. $(-b + \sqrt{b^2 - 4ac})/(2a)$ is not in $\mathrm{FOL}_{\mathbb{R}}$ and neither is $-c/b$
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Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

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Square Root Expression Algebra

Virtual Substitution into Polynomial

Virtually substitute $(a + b\sqrt{c})/d$ into a polynomial p:

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Algebra of terms $(a+b\sqrt{c})/d$ with polynomials $a, b, c, d \in \mathbb{Q}[x_1, .., x_n]$:

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Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$a \neq 0 \lor b \neq 0 \lor c \neq 0 \rightarrow$$
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Lemma (Virtual Substitution Lemma for $\sqrt{\cdot}$)

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Extended logic $F_x^{(a+b\sqrt{c})/d} \equiv F_{\bar{x}}^{(a+b\sqrt{c})/d}$ $\mathrm{FOL}_{\mathbb{R}}$

$$\omega_x^r \in [\![F]\!] \text{ iff } \omega \in [\![F_{\bar{x}}^{(a+b\sqrt{c})/d}]\!] \text{ where } r = (\omega[\![a]\!] + \omega[\![b]\!]\sqrt{\omega[\![c]\!]})/\omega[\![d]\!] \in \mathbb{R}$$

Example: Curiosity

$$a \neq 0 \rightarrow (\exists x\,(ax^2 + bx + c = 0 \wedge ax^2 + bx + c \leq 0) \leftrightarrow b^2 - 4ac \geq 0 \wedge true)$$

$$\begin{aligned}
&(ax^2 + bx + c)_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)}\\
=&a((-b + \sqrt{b^2 - 4ac})/(2a))^2 + b((-b + \sqrt{b^2 - 4ac})/(2a)) + c\\
=&a((b^2+b^2-4ac + (-b-b)\sqrt{b^2-4ac})/(4a^2)) + (-b^2 + b\sqrt{b^2-4ac})/(2a) + c\\
=&(ab^2+ab^2-4a^2c + (-ab-ab)\sqrt{b^2-4ac})/(4a^2) + (-b^2+2ac + b\sqrt{b^2-4ac})/(2a)\\
=&((ab^2+ab^2-4a^2c)2a + (-b^2+2ac)4a^2 + ((-ab-ab)2a+b4a^2)\sqrt{b^2-4ac})/(4a^2)\\
=&(\cancel{2a^2b^2} + \cancel{2a^2b^2} - \cancel{8a^3c} - \cancel{4a^2b^2} + \cancel{8a^3c} + (-\cancel{2a^2b}-\cancel{2a^2b}+\cancel{4a^2b})\sqrt{b^2-4ac})/(4a^2\\
=&(0 + 0\sqrt{b^2 - 4ac})/1 = 0
\end{aligned}$$

$$(ax^2+bx+c = 0)_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \equiv ((0 + 0\sqrt{..})/1 = 0) \equiv (0\cdot 1 = 0) \equiv true$$

$$(ax^2+bx+c \leq 0)_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \equiv (\underbrace{(0 + 0\sqrt{..})/1}_{0} \leq 0) \equiv (0\cdot 1 \leq 0) \equiv true$$

Example: Nonnegative Roots

$$a \neq 0 \rightarrow \big(\exists x\,(ax^2 + bx + c = 0 \land x \geq 0)$$
$$\leftrightarrow b^2 - 4ac \geq 0 \land (2ba \leq 0 \land 4ac \geq 0 \lor -2a \leq 0 \land 4ac \leq 0$$
$$\lor 2ba \leq 0 \land 4ac \geq 0 \lor 2a \leq 0 \land 4ac \leq 0)\big)$$

$$-(-b+\sqrt{b^2-4ac})/(2a) = ((-1+0\sqrt{b^2-4ac})/1)\cdot((-b+\sqrt{b^2-4ac})/(2a$$
$$= (b - \sqrt{b^2-4ac})/(2a)$$

$$(-x \leq 0)_{\bar{x}}^{(b-\sqrt{b^2-4ac})/(2a)}$$
$$\equiv \; b2a{\leq}0 \land \cancel{b^2}-(-1)^2(\cancel{b^2}-4ac){\geq}0 \lor -1\cdot 2a{\leq}0 \land \cancel{b^2}-(-1)^2(\cancel{b^2}-4ac){\leq}0$$
$$\equiv \; 2ba \leq 0 \land 4ac \geq 0 \lor -2a \leq 0 \land 4ac \leq 0$$

$$(-x \leq 0)_{\bar{x}}^{(b+\sqrt{b^2-4ac})/(2a)}$$
$$\equiv \; b2a \leq 0 \land \cancel{b^2} - 1^2(\cancel{b^2} - 4ac) \geq 0 \lor 1\cdot 2a \leq 0 \land \cancel{b^2} - 1^2(\cancel{b^2} - 4ac) \leq 0$$
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