# 09: Reactions & Delays

## 15-424: Foundations of Cyber-Physical Systems

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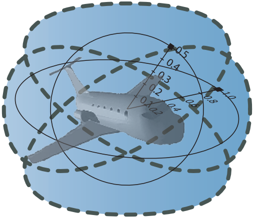

# Outline

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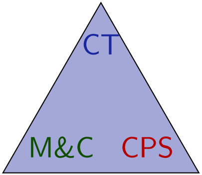
using loop invariants
design time-triggered control
design-by-invariant




modeling CPS
designing controls
time-triggered control
reaction delays
discrete sensing

semantics of time-triggered control
operational effect
finding control constraints
model-predictive control

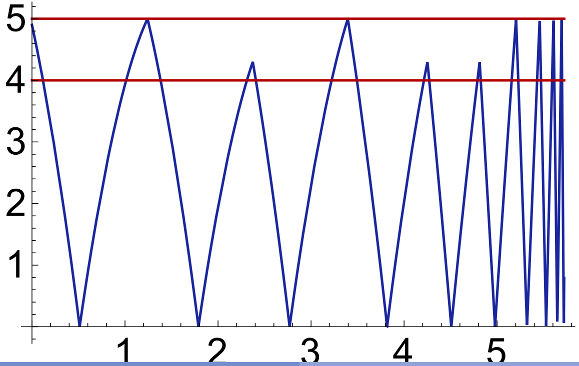
# Outline

# Quantum's Ping Pong Proof Invariants

**Proposition (Quantum can play ping pong safely)**

$$0 \le x \wedge x \le 5 \wedge v \le 0 \wedge g > 0 \wedge 1 \ge c \ge 0 \wedge f \ge 0 \rightarrow$$
$$\Big[\Big(\big(\{x' = v, v' = -g \,\&\, x \ge 0 \wedge x \le 5\} \cup \{x' = v, v' = -g \,\&\, x \ge 5\}\big);$$
$$\text{if}(x=0)\ v := -cv \text{ else if}(4 \le x \le 5 \wedge v \ge 0)\ v := -fv\Big)^*\Big](0 \le x \le 5)$$

**Proof** @invariant$(0 \le x \le 5 \wedge (x = 5 \rightarrow v \le 0))$



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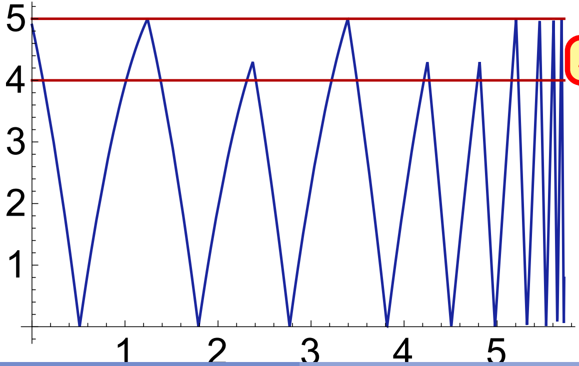


Just can't implement . . .

# Physical vs. Controller Events

## Physical vs. Controller Events

1. Justifiable: Physical events (on ground $x = 0$)
2. Justifiable: Physical evolution domains (above ground $x \geq 0$)
3. Questionable: Controller evolution domain ($x \leq 5$)
4. Unlike physics, controllers won't run *all* the time. Just often.
5. Controllers cannot sense and compute all the time.

# Back to the Drawing Desk: Quantum the Ping Pong Ball

**Conjecture (Quantum can play ping pong safely)**

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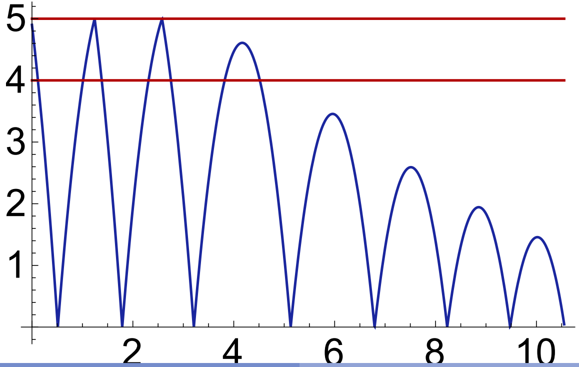
**Proof?** Ask René Descartes



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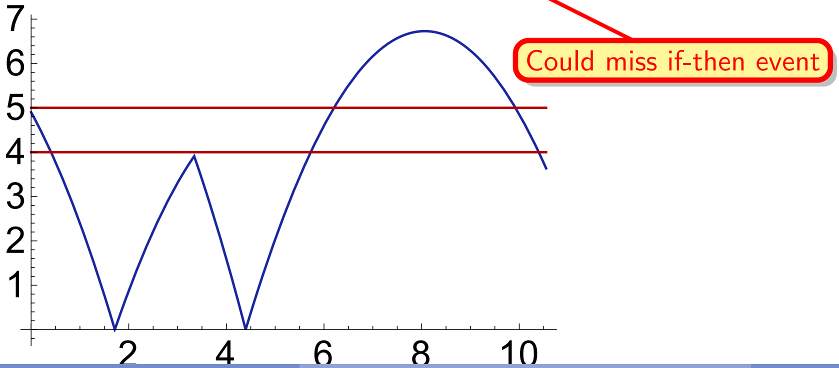

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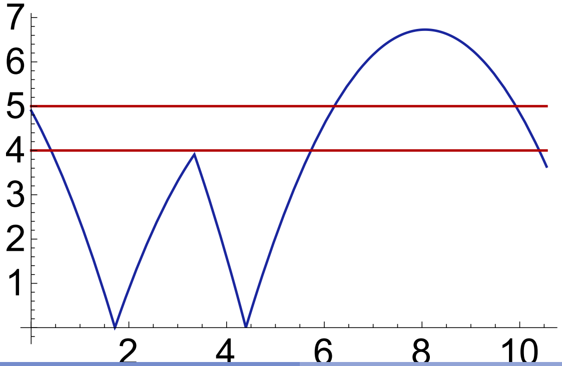

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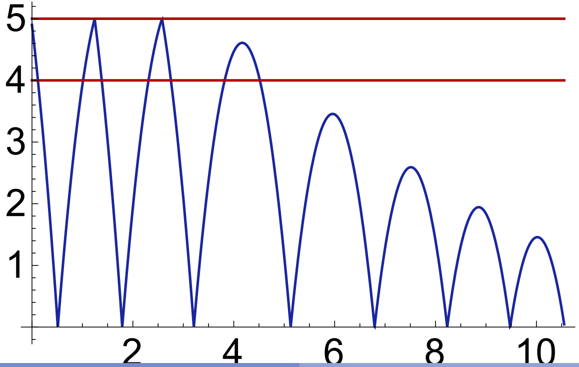

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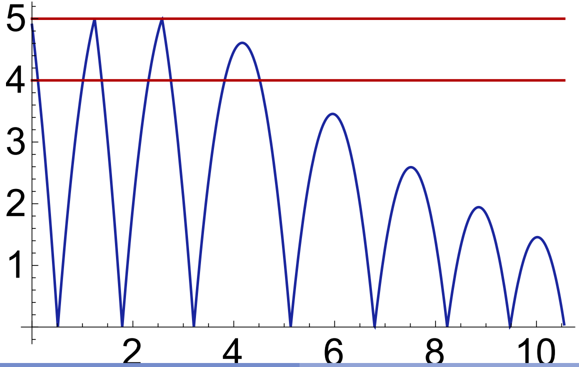

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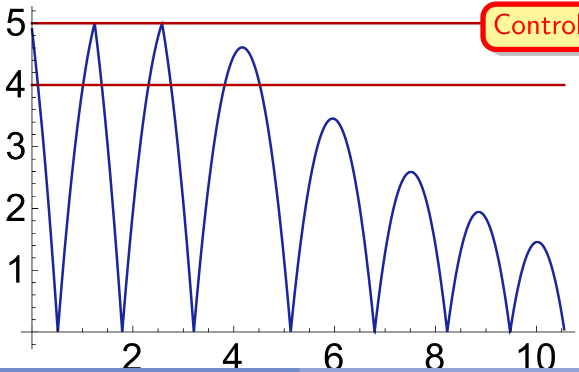
Control action before physics



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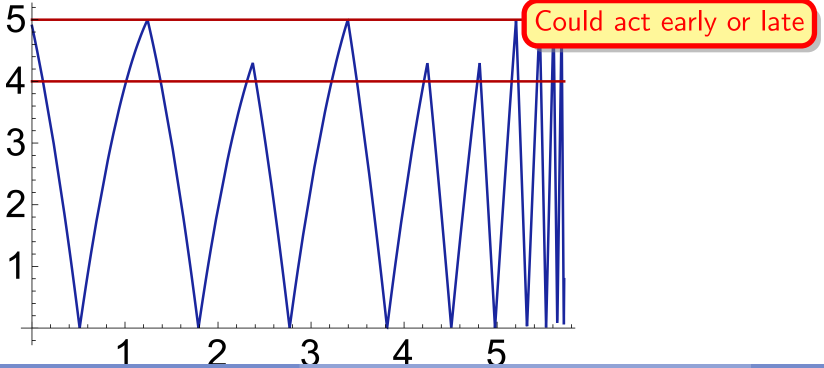

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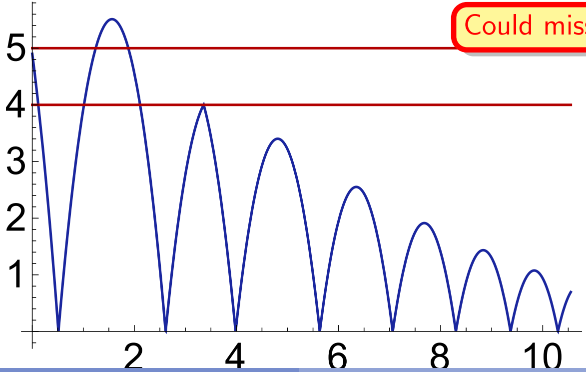
Could miss event off control cycle



# Delays May Miss Events

**Delays vs. Events**

1. Periodically/frequently monitoring for an event with a polling frequency / reaction time
2. Delays may make the controller miss events.
3. Discrepancy between event-triggered idea vs. real time-triggered implementation.
4. Issues indicate poor event abstraction
5. Slow controllers monitoring small regions of a fast moving system.
6. Controller needs to be aware of its own delay

# Cartesian Doubt: René Descartes's Cartesian Demon 1641

**Outwit the Cartesian Demon**

Skeptical about the truth of all beliefs until justification has been found.

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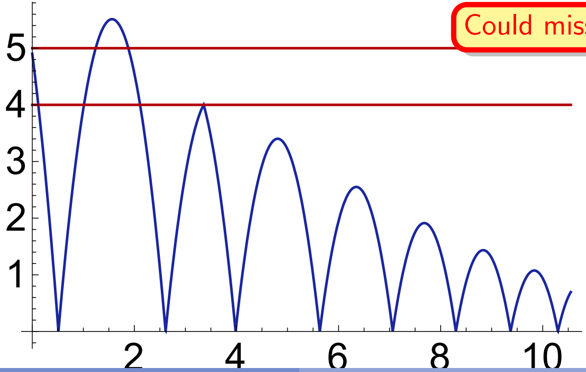

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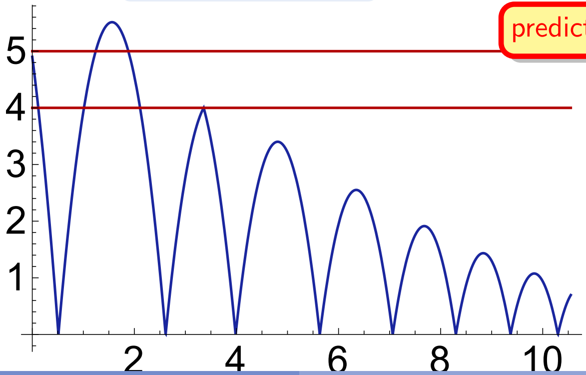
Proof? Ask René Descartes

predict 1s: $x + v - \frac{g}{2} > 5$



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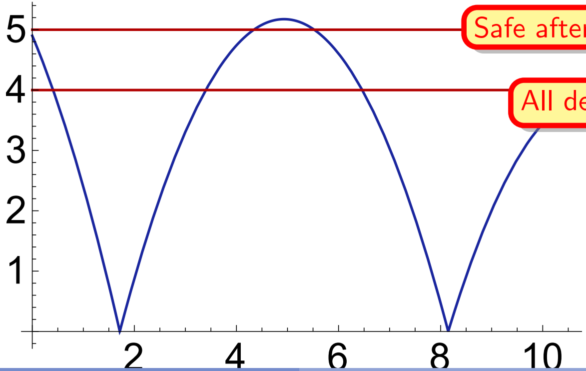


Safe after 1 s but not until then

All depends on sampling

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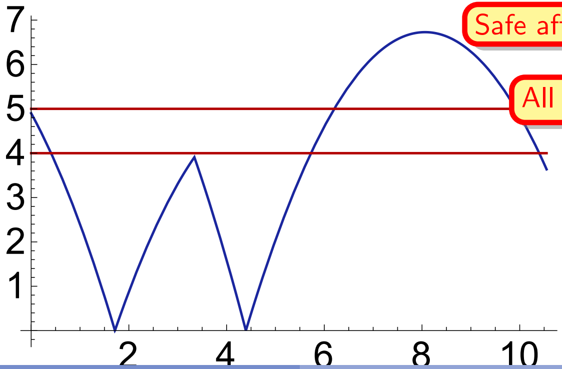
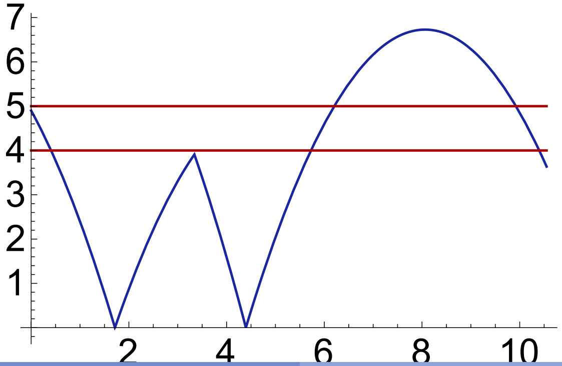


# Quantum Discovers Design-by-Invariant

## Design-by-Invariant

$2gx = 2gH - v^2 \land x \geq 0 \ \land c = 1 \land g > 0$

bouncing ball invariant



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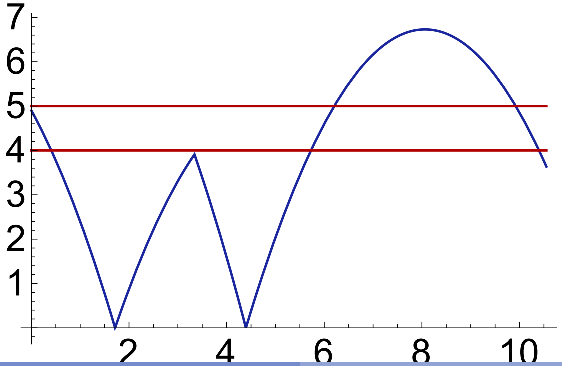
$2gx = 2gH - v^2 \land x \geq 0 \ \land c = 1 \land g = 1$

simplify arithmetic



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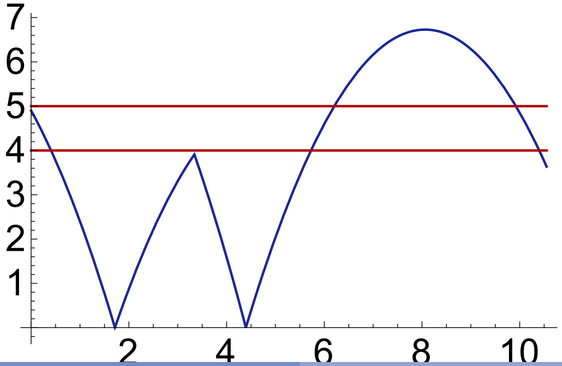

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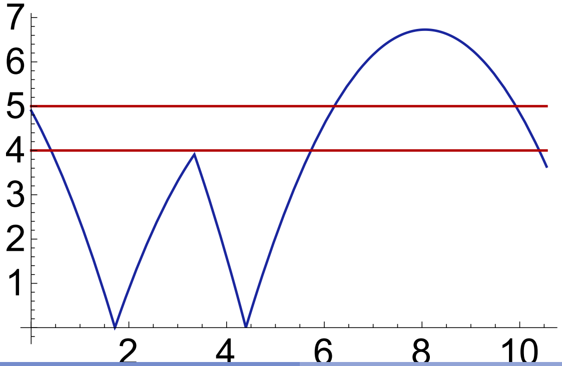
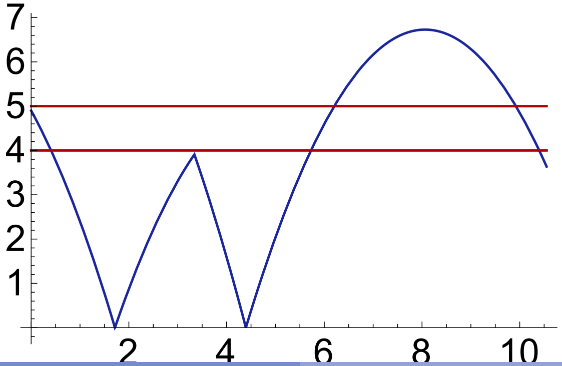
$2x = 2 \cdot {\color{red}H} - v^2 \land x \geq 0$



## Design-by-Invariant

$2x = 2 \cdot 5 - v^2 \land x \geq 0$

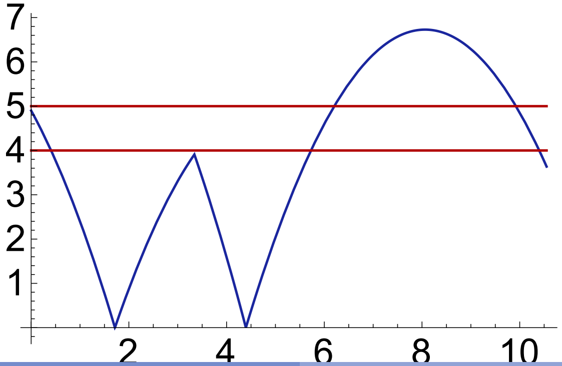
critical height



## Design-by-Invariant

$2x > 2 \cdot 5 - v^2 \land x \geq 0$

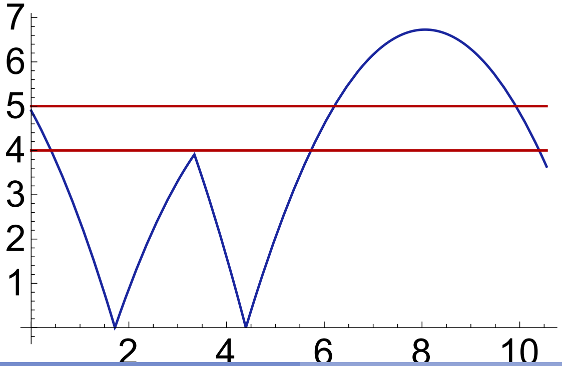
potential exceeds safe height



## Design-by-Invariant

$2x > 2 \cdot 5 - v^2 \wedge x \geq 0$

use invariant for control



# Quantum the Time-triggered Ping Pong Ball

**Conjecture (Quantum can play ping pong safely)**

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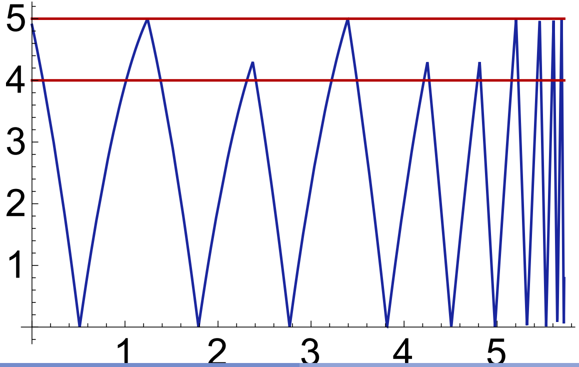

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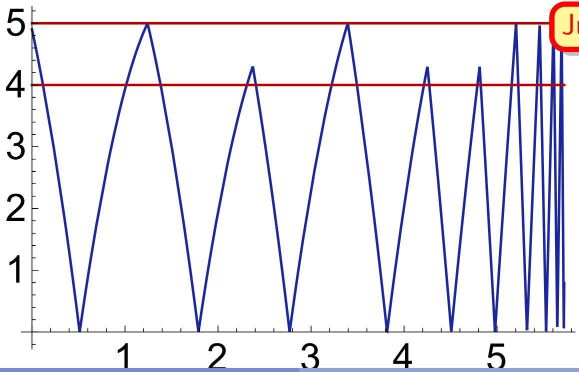
Proof? Ask René Descartes

Just for simplicity



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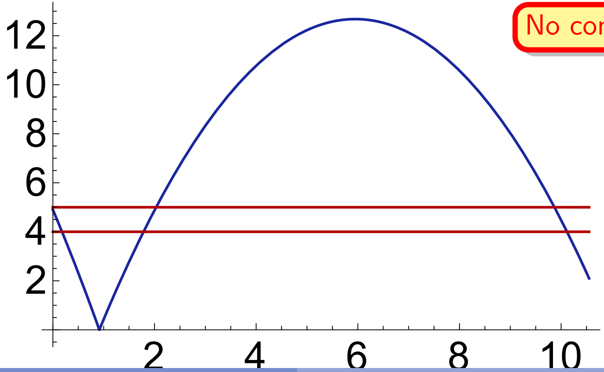
No control near ground



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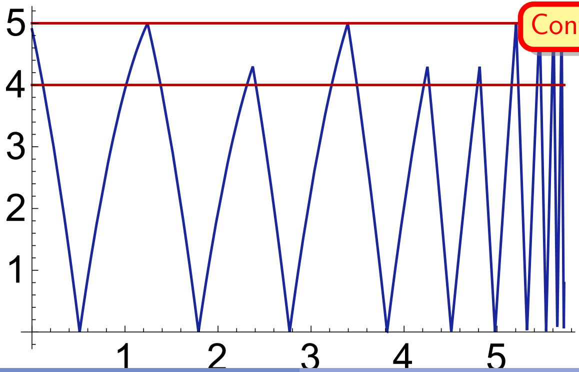


Control despite ground

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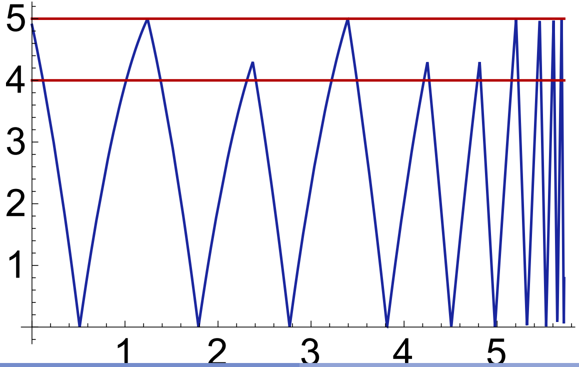

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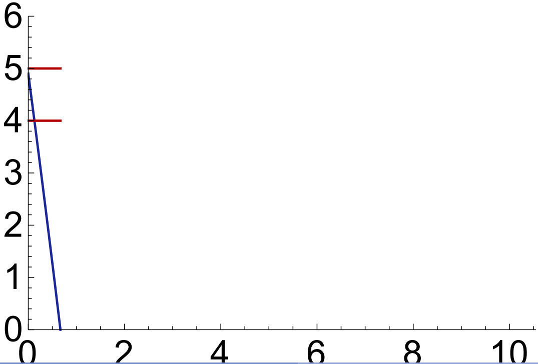

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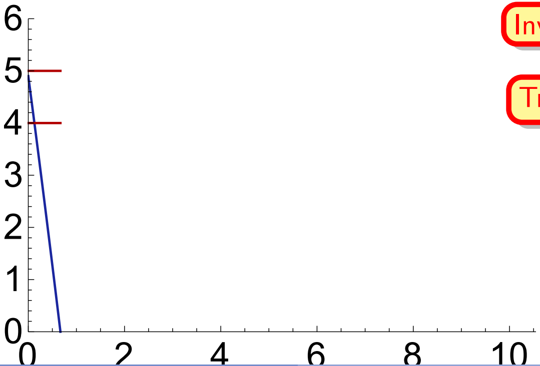
Invariants are **invariants**!

True ever $\rightsquigarrow$ true always



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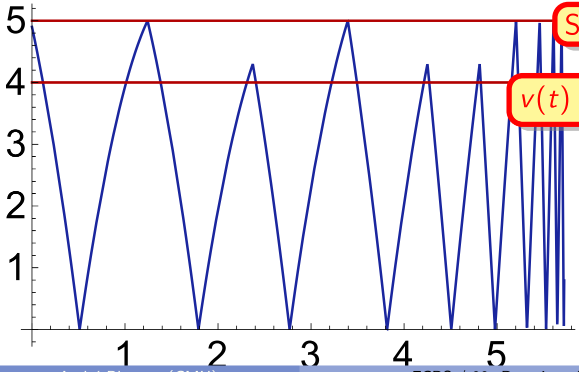
**Proof?** Ask René Descartes




Slow turn around

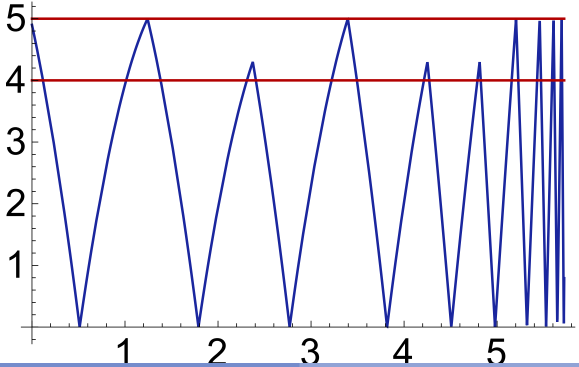
$v(t) = v - gt = v - t < 0$

# Quantum the Time-triggered Ping Pong Ball

**Conjecture (Quantum can play ping pong safely)**

$0 \le x \land x \le 5 \land v \le 0 \land g = 1 \land 1 = c \land f = 1 \rightarrow$
$\big[\big(\text{if}(x{=}0)\ v := -cv; \text{if}((x{>}5\frac{1}{2}{-}v \lor 2x{>}2{\cdot}5{-}v^2{\land}v{<}1){\land}v{\ge}0)\ v := -fv;$
$t := 0; \{x' = v, v' = -g, t' = 1\,\&\, x \ge 0 \land t{\le}1\}\big)^*\big](0{\le}x{\le}5)$

**Proof?** Ask René Descartes who says yes



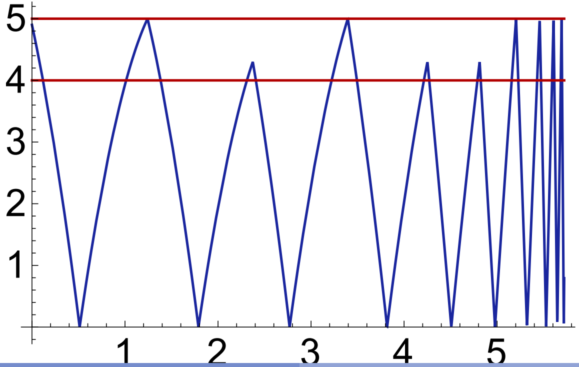
# Outline

# Quantum's Time-triggered Ping Pong Proof Invariants

**Proposition (Quantum can play ping pong safely in real-time)**

$$0 \le x \land x \le 5 \land v \le 0 \land g{=}1{>}0 \land 1{=}c{\ge}0 \land 1{=}f{\ge}0 \rightarrow \\ \Big[\big(\text{if}(x{=}0)\ v := -cv; \text{if}((x{>}5\tfrac{1}{2}{-}v \lor 2x{>}2{\cdot}5{-}v^2{\land}v{<}1){\land}v{\ge}0)\ v := -fv; \\ t := 0; \{x' = v, v' = -g, t' = 1\ \&\ x \ge 0 \land t \le 1\}\big)^*\Big](0{\le}x{\le}5)$$

**Proof**



# Quantum's Time-triggered Ping Pong Proof Invariants

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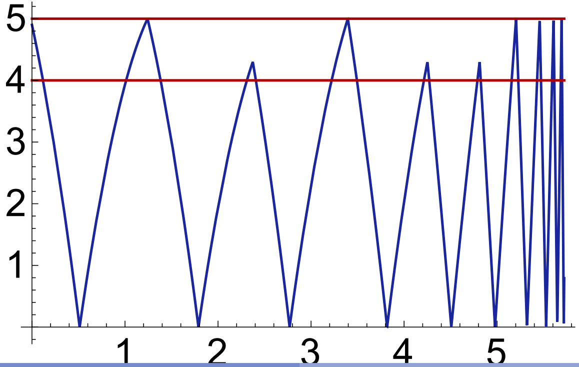
$$t := 0; \{x' = v, v' = -g, t' = 1 \,\&\, x \ge 0 \land t \le 1\}\big)^*\big](0{\le}x{\le}5)$$

**Proof** @invariant($2x = 2H - v^2 \land x \ge 0 \land x{\le}5$)



# Quantum's Time-triggered Ping Pong Proof Invariants

**Proposition (Quantum can play ping pong safely in real-time)**

$$2x = 2H - v^2 \land 0 \leq x \land x \leq 5 \land v \leq 0 \land g{=}1{>}0 \land 1{=}c{\geq}0 \land 1{=}f{\geq}0 \rightarrow$$
$$\Big[\Big(\text{if}(x{=}0)\ v := -cv; \text{if}((x{>}5\tfrac{1}{2}{-}v \lor 2x{>}2{\cdot}5{-}v^2{\land}v{<}1){\land}v{\geq}0)\ v := -fv;$$
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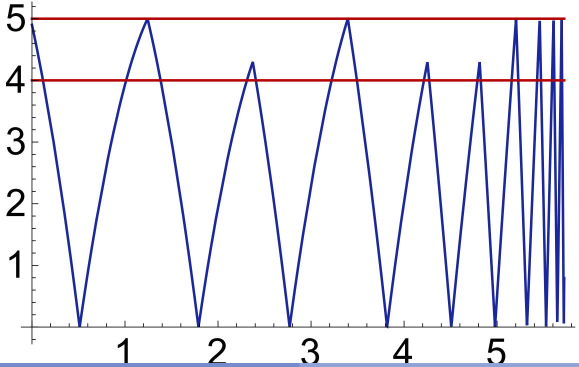
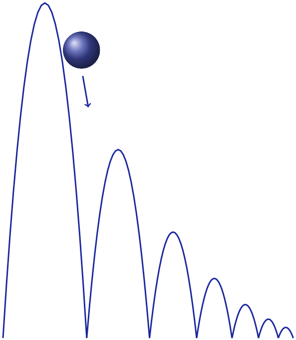


# Outline

# Summary: Time-triggered Control

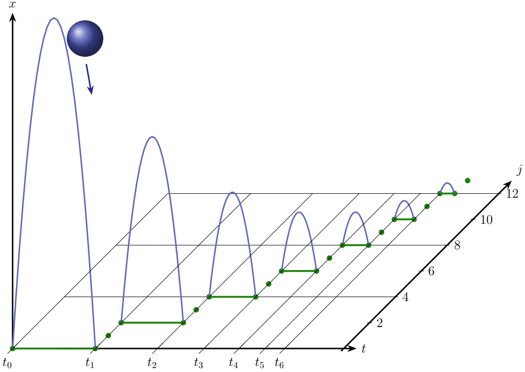
1. Common paradigm for designing real controllers
2. Periodical or pseudo-periodical control (jitter)
3. Expects delays, expects inertia
4. Implementation: discrete-time sensing
5. Predict events, not just if($eventnow(x)$) ...
6. Safe controllers know their own reaction delays
7. Burden of event detection brought to attention of CPS programmer
8. Time-triggered controls are implementable and more robust, but make design and verification more challenging!
9. Use knowledge gained from verified event-triggered model as a basis for designing a time-triggered controller



**Example (Quantum the Bouncing Ball)**

$$\big(\{x' = v, v' = -g \,\&\, x \geq 0\};$$
$$\text{if}(x = 0)\; v := -cv\big)^*$$

# How Quantum Met Achilles and His Tortoise

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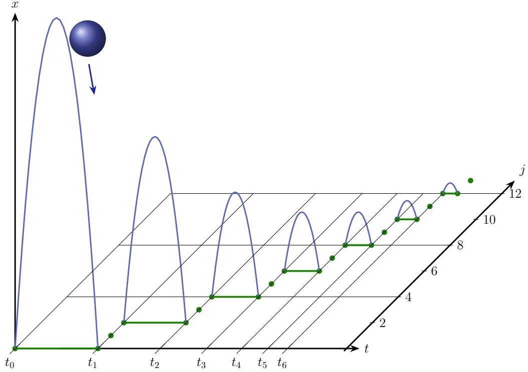


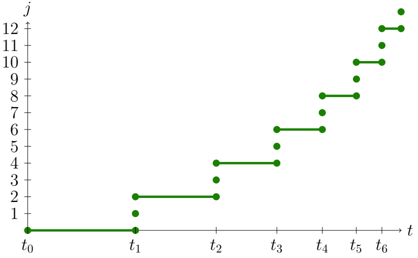


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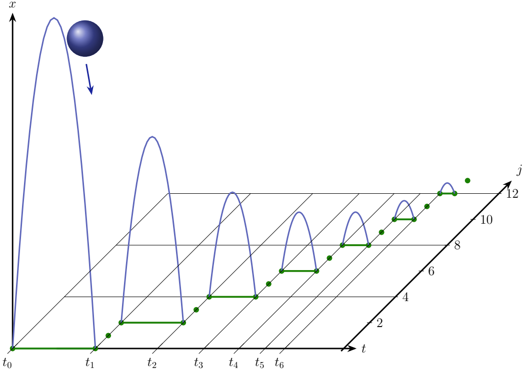
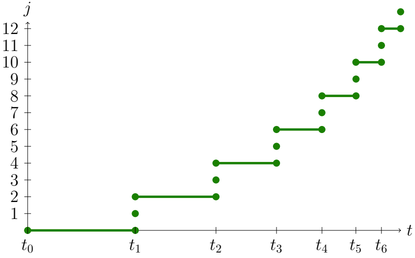
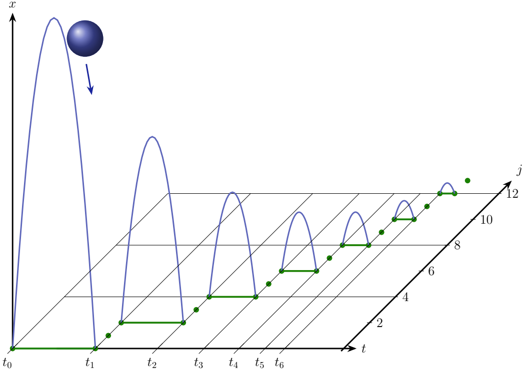


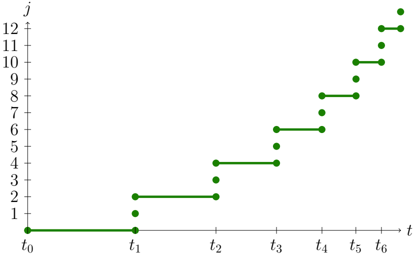




## Example (Quantum the Bouncing Ball experiences time)

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

# How Quantum Met Achilles and His Tortoise

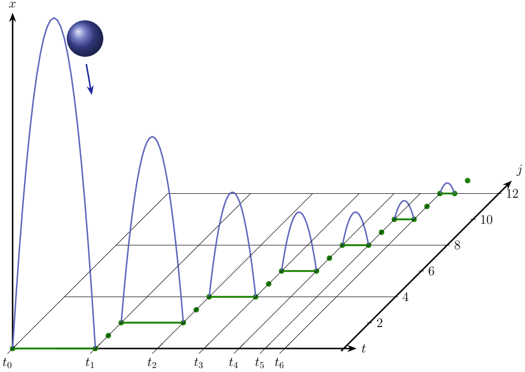


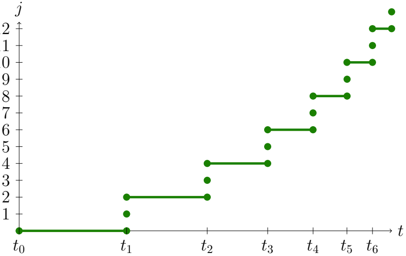




**Example (Quantum the Bouncing Ball experiences time)**

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i}$$

# How Quantum Met Achilles and His Tortoise

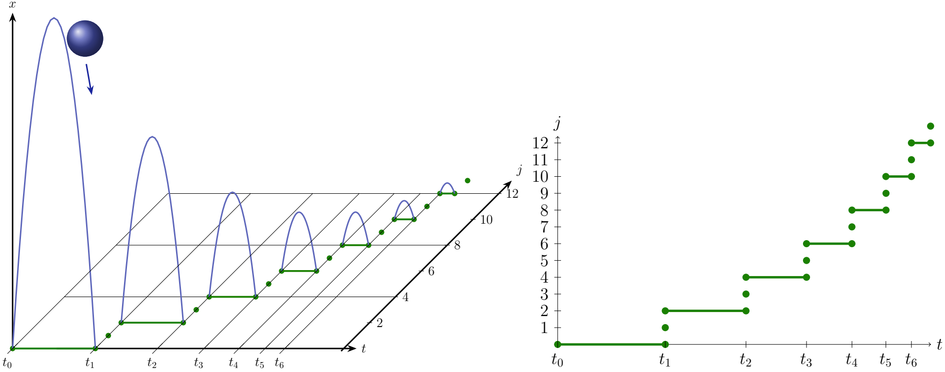



## Example (Quantum the Bouncing Ball experiences time)

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1 - \frac{1}{2}}$$

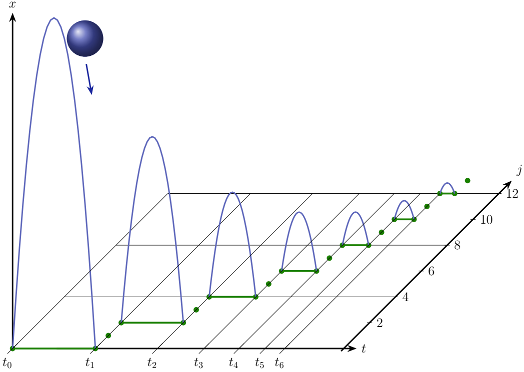
# How Quantum Met Achilles and His Tortoise

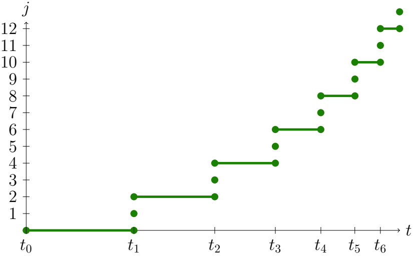



## Example (Quantum the Bouncing Ball experiences time)

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1-\frac{1}{2}} = 2$$

# How Quantum Met Achilles and His Tortoise




## Example (Quantum the Bouncing Ball experiences time)

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1 - \frac{1}{2}} = 2 < \infty$$

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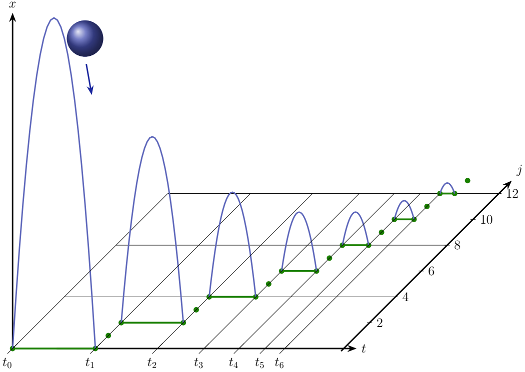


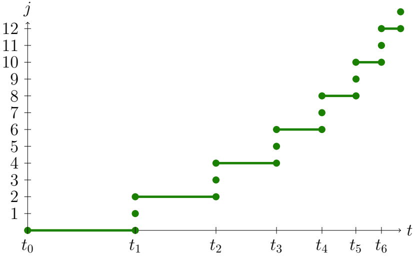


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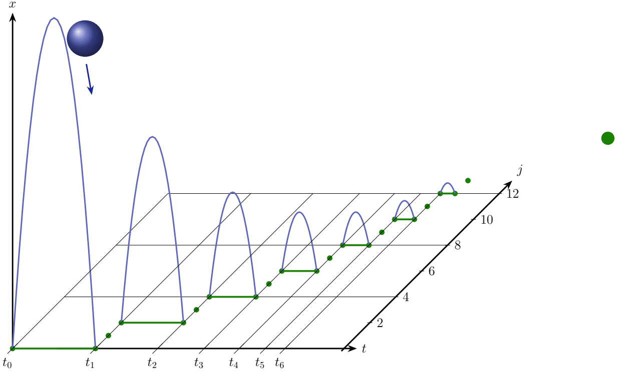


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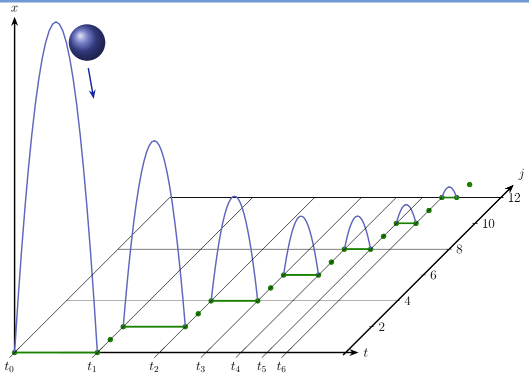



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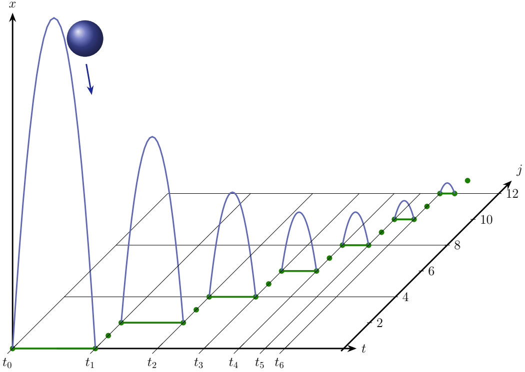



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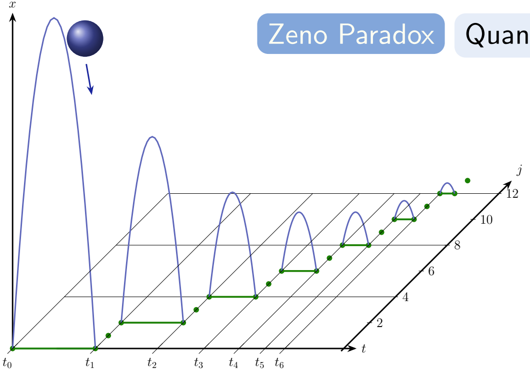
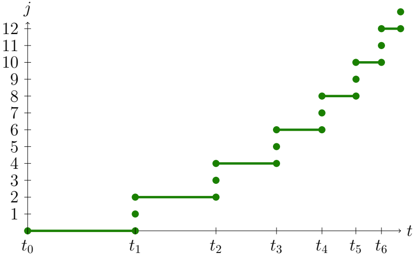
$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1 - \frac{1}{2}} = 2 < \infty$$

# How Quantum Met Achilles and His Tortoise

**Zeno Paradox** Quantum's model causes a time freeze




## Example (Quantum the Bouncing Ball experiences time)

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1-\frac{1}{2}} = 2 < \infty$$

# What to do with assignments

$$\frac{[x:=e]p(x) \leftrightarrow p(e)}{[x:=x^2]x \neq 0 \leftrightarrow x^2 \neq 0} \qquad e \rightsquigarrow x^2,\ p(\cdot) \rightsquigarrow \cdot \neq 0$$

$$\frac{[x:=e]p(x) \leftrightarrow p(e)}{[x:=x^2][y:=2x]x>0 \leftrightarrow [y:=2x^2]x^2>0} \qquad e \rightsquigarrow x^2,\ p(\cdot) \rightsquigarrow [y:=2\cdot](\cdot>0)$$

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$$\frac{[x:=e]p(x) \leftrightarrow p(e)}{[x:=5y][y:=2x](x>0) \leftrightarrow [y:=2(5y)](5y>0)} \qquad e \rightsquigarrow 5y,\ p(\cdot) \rightsquigarrow [y:=2\cdot](\cdot>0)$$

$$\frac{[x:=e]p(x) \leftrightarrow p(e)}{[x:=x^2][x' = 2x]x>0 \leftrightarrow [x' = 2x^2]x^2>0} \qquad e \rightsquigarrow x^2,\ p(\cdot) \rightsquigarrow [\cdot' = 2\cdot] \cdot >0$$

$$\frac{[x:=e]p(x) \leftrightarrow p(e)}{[x:=x^2][(x:=x+1)^*]x\geq 0 \leftrightarrow [(x:=x^2+1)^*]x^2\geq 0} \qquad e \rightsquigarrow x^2,\ p(\cdot) \rightsquigarrow [(x:= \cdot +1)^*] \cdot$$

# What to do with assignments and what not to do!

$$\frac{[x:=e]p(x) \leftrightarrow p(e)}{[x:=x^2]x \neq 0 \leftrightarrow x^2 \neq 0} \qquad e \rightsquigarrow x^2, p(\cdot) \rightsquigarrow \cdot \neq 0$$

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$$\frac{[x:=e]p(x) \leftrightarrow p(e)}{[x:=x^2][x'=2x]x>0 \leftrightarrow [x'=2x^2]x^2>0} \qquad e \rightsquigarrow x^2, p(\cdot) \rightsquigarrow [\cdot' = 2\cdot]\cdot>0$$

$$\frac{[x:=e]p(x) \leftrightarrow p(e)}{[x:=x^2][(x:=x+1)^*]x\geq 0 \leftrightarrow [(x:=x^2+1)^*]x^2\geq 0} \qquad e \rightsquigarrow x^2, p(\cdot) \rightsquigarrow [(x:=\cdot+1)^*]\cdot$$

# What else to do with assignments

$$[:=] \quad [x := e]p(x) \leftrightarrow p(e)$$

$$\frac{\Gamma, x = e \vdash P, \Delta}{\Gamma \vdash [x{:=}e]P, \Delta}$$

# What else to do with assignments and what not to do!

$$[:=] \quad [x := e]p(x) \leftrightarrow p(e)$$

$$\frac{\Gamma, x = e \vdash P, \Delta}{\Gamma \vdash [x{:=}e]P, \Delta} \quad \text{if } x \notin \Gamma, \Delta, e$$

André Platzer.
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André Platzer.
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