

02: Differential Equations & Domains

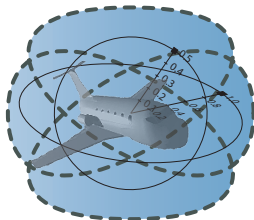
15-424: Foundations of Cyber-Physical Systems

André Platzer

aplatzer@cs.cmu.edu

Computer Science Department

Carnegie Mellon University, Pittsburgh, PA



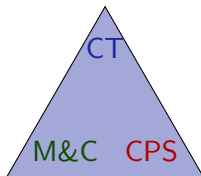
- 1 Learning Objectives
- 2 Introduction
- 3 Differential Equations
- 4 Examples of Differential Equations
- 5 Domains of Differential Equations

- 1 Learning Objectives
- 2 Introduction
- 3 Differential Equations
- 4 Examples of Differential Equations
- 5 Domains of Differential Equations

Learning Objectives

Differential Equations & Domains

semantics of differential equations
descriptive power of differential equations
syntax versus semantics



continuous dynamics
differential equations
evolution domains
first-order logic

continuous operational effects

- 1 Learning Objectives
- 2 Introduction**
- 3 Differential Equations
- 4 Examples of Differential Equations
- 5 Domains of Differential Equations

Example (Vector field and one solution of a differential equation)

$$\begin{pmatrix} y'(t) = f(t, y) \\ y(t_0) = y_0 \end{pmatrix}$$

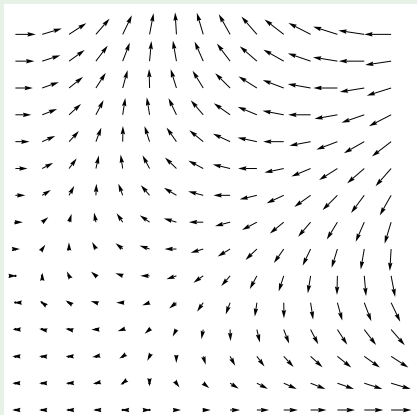
Intuition:

Example (Vector field and one solution of a differential equation)

$$\begin{cases} y'(t) = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

Intuition:

- 1 At each point in space, plot the value of $f(t, y)$ as a vector

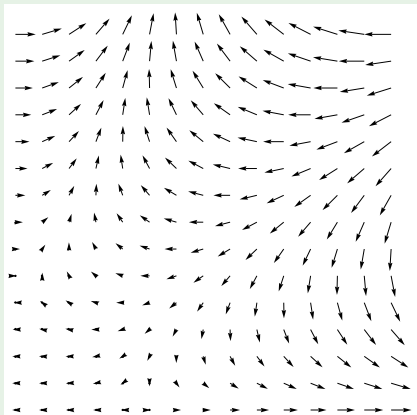


Example (Vector field and one solution of a differential equation)

$$\begin{cases} y'(t) = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

Intuition:

- 1 At each point in space, plot the value of $f(t, y)$ as a vector
- 2 Start at initial state y_0 at initial time t_0

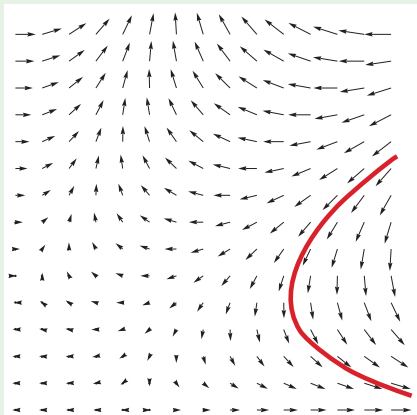


Example (Vector field and one solution of a differential equation)

$$\begin{cases} y'(t) = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

Intuition:

- 1 At each point in space, plot the value of $f(t, y)$ as a vector
- 2 Start at initial state y_0 at initial time t_0
- 3 Follow the direction of the vector

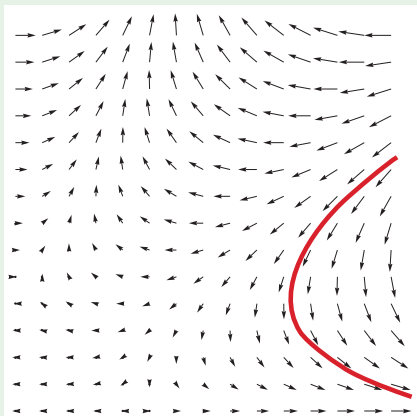


Example (Vector field and one solution of a differential equation)

$$\begin{pmatrix} y'(t) = f(t, y) \\ y(t_0) = y_0 \end{pmatrix}$$

Intuition:

- 1 At each point in space, plot the value of $f(t, y)$ as a vector
 - 2 Start at initial state y_0 at initial time t_0
 - 3 Follow the direction of the vector
- The diagram should show infinitely many vectors ...

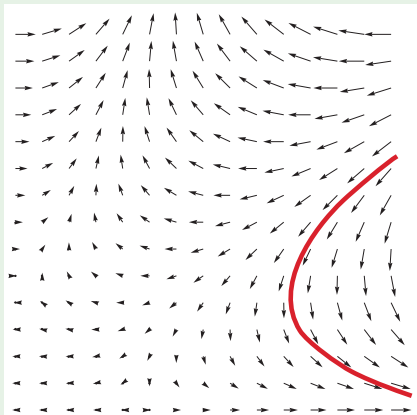


Example (Vector field and one solution of a differential equation)

$$\begin{cases} y'(t) = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

Intuition:

- 1 At each point in space, plot the value of $f(t, y)$ as a vector
 - 2 Start at initial state y_0 at initial time t_0
 - 3 Follow the direction of the vector
- The diagram should show infinitely many vectors ...



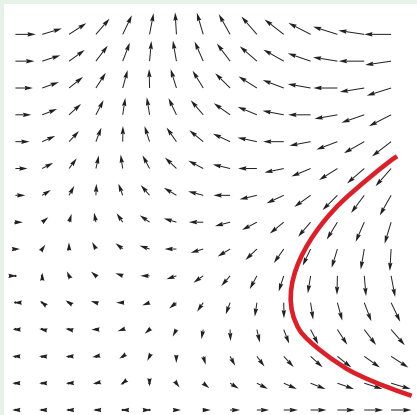
Your car's ODE $x' = v, v' = a$

Example (Vector field and one solution of a differential equation)

$$\begin{cases} y'(t) = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

Intuition:

- 1 At each point in space, plot the value of $f(t, y)$ as a vector
 - 2 Start at initial state y_0 at initial time t_0
 - 3 Follow the direction of the vector
- The diagram should show infinitely many vectors ...

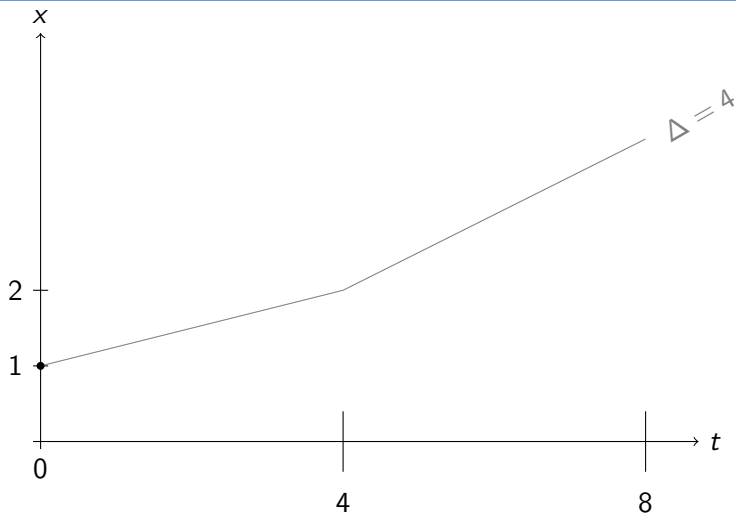


Your car's ODE

$$x' = v, v' = a$$

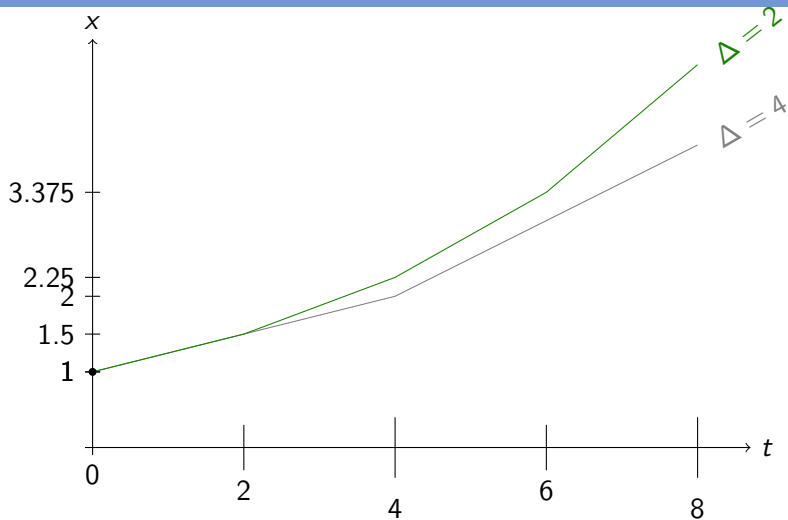
Well it's a wee bit more complicated

Intuition for Differential Equations



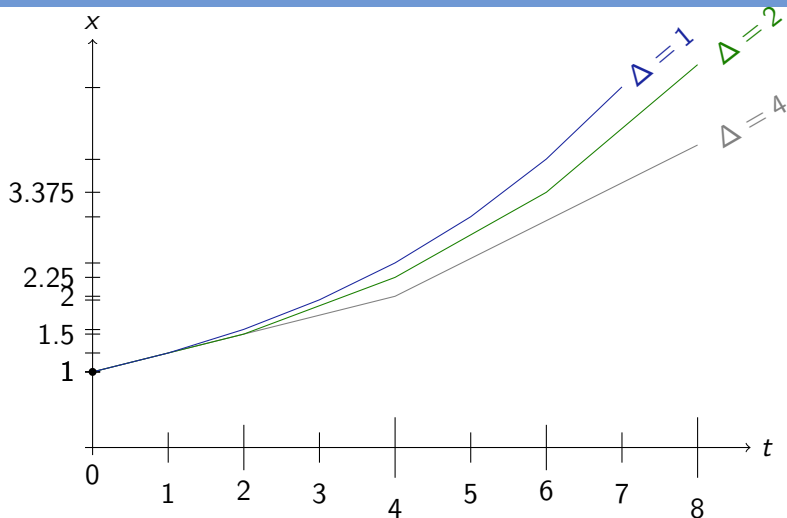
$$\left(\begin{array}{l} x'(t) = \frac{1}{4}x(t) \\ x(0) = 1 \end{array} \right)$$

Intuition for Differential Equations



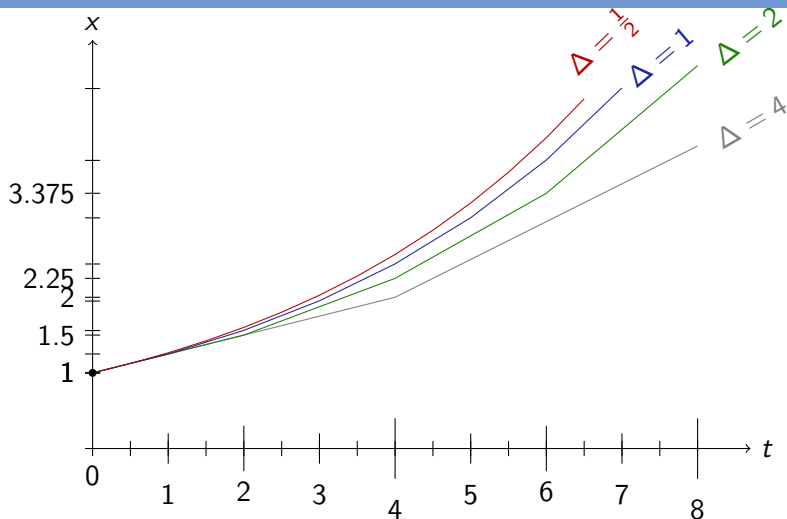
$$\left(\begin{array}{l} x'(t) = \frac{1}{4}x(t) \\ x(0) = 1 \end{array} \right) \rightsquigarrow \left(\begin{array}{l} x(t + \Delta) := x(t) + \frac{1}{4}x(t)\Delta \\ x(t_0) := 1 \end{array} \right)$$

Intuition for Differential Equations



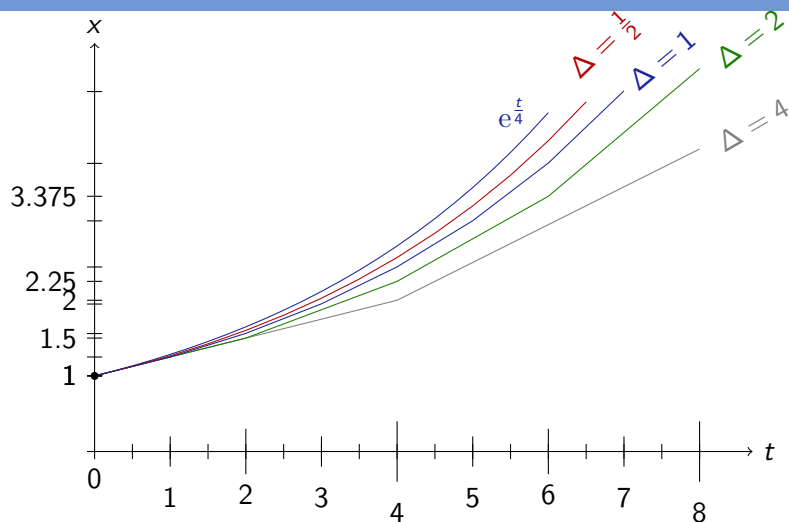
$$\left(\begin{array}{l} x'(t) = \frac{1}{4}x(t) \\ x(0) = 1 \end{array} \right) \rightsquigarrow \left(\begin{array}{l} x(t + \Delta) := x(t) + \frac{1}{4}x(t)\Delta \\ x(t_0) := 1 \end{array} \right)$$

Intuition for Differential Equations



$$\left(\begin{array}{l} x'(t) = \frac{1}{4}x(t) \\ x(0) = 1 \end{array} \right) \rightsquigarrow \left(\begin{array}{l} x(t + \Delta) := x(t) + \frac{1}{4}x(t)\Delta \\ x(t_0) := 1 \end{array} \right)$$

Intuition for Differential Equations



$$\left(\begin{array}{l} x'(t) = \frac{1}{4}x(t) \\ x(0) = 1 \end{array} \right) \rightsquigarrow \left(\begin{array}{l} x(t + \Delta) := x(t) + \frac{1}{4}x(t)\Delta \\ x(t_0) := 1 \end{array} \right)$$

- 1 Learning Objectives
- 2 Introduction
- 3 Differential Equations**
- 4 Examples of Differential Equations
- 5 Domains of Differential Equations

The Meaning of Differential Equations

- 1 What exactly is a vector field?
- 2 What does it mean to describe directions of evolution at every point in space?
- 3 Could directions possibly contradict each other?

Importance of meaning

The physical impacts of CPSs do not leave much room for failure, so we immediately want to get into the mood of consistently studying the behavior and exact meaning of all relevant aspects of CPS.

Definition (Ordinary Differential Equation, ODE)

$f : D \rightarrow \mathbb{R}^n$ on domain $D \subseteq \mathbb{R} \times \mathbb{R}^n$ (i.e., open connected). Then $Y : I \rightarrow \mathbb{R}^n$ is *solution* of initial value problem (IVP)

$$\begin{pmatrix} y'(t) = f(t, y) \\ y(t_0) = y_0 \end{pmatrix}$$

on interval $I \subseteq \mathbb{R}$, iff, for all times $t \in I$,

Definition (Ordinary Differential Equation, ODE)

$f : D \rightarrow \mathbb{R}^n$ on domain $D \subseteq \mathbb{R} \times \mathbb{R}^n$ (i.e., open connected). Then $Y : I \rightarrow \mathbb{R}^n$ is *solution* of initial value problem (IVP)

$$\begin{pmatrix} y'(t) = f(t, y) \\ y(t_0) = y_0 \end{pmatrix}$$

on interval $I \subseteq \mathbb{R}$, iff, for all times $t \in I$,

- 1 defined $(t, Y(t)) \in D$

Definition (Ordinary Differential Equation, ODE)

$f : D \rightarrow \mathbb{R}^n$ on domain $D \subseteq \mathbb{R} \times \mathbb{R}^n$ (i.e., open connected). Then $Y : I \rightarrow \mathbb{R}^n$ is *solution* of initial value problem (IVP)

$$\begin{pmatrix} y'(t) = f(t, y) \\ y(t_0) = y_0 \end{pmatrix}$$

on interval $I \subseteq \mathbb{R}$, iff, for all times $t \in I$,

- 1 defined $(t, Y(t)) \in D$
- 2 time-derivative $Y'(t)$ exists and $Y'(t) = f(t, Y(t))$.

Definition (Ordinary Differential Equation, ODE)

$f : D \rightarrow \mathbb{R}^n$ on domain $D \subseteq \mathbb{R} \times \mathbb{R}^n$ (i.e., open connected). Then $Y : I \rightarrow \mathbb{R}^n$ is *solution* of initial value problem (IVP)

$$\begin{pmatrix} y'(t) = f(t, y) \\ y(t_0) = y_0 \end{pmatrix}$$

on interval $I \subseteq \mathbb{R}$, iff, for all times $t \in I$,

- 1 defined $(t, Y(t)) \in D$
- 2 time-derivative $Y'(t)$ exists and $Y'(t) = f(t, Y(t))$.
- 3 initial value $Y(t_0) = y_0$

Definition (Ordinary Differential Equation, ODE)

$f : D \rightarrow \mathbb{R}^n$ on domain $D \subseteq \mathbb{R} \times \mathbb{R}^n$ (i.e., open connected). Then $Y : I \rightarrow \mathbb{R}^n$ is *solution* of initial value problem (IVP)

$$\begin{pmatrix} y'(t) = f(t, y) \\ y(t_0) = y_0 \end{pmatrix}$$

on interval $I \subseteq \mathbb{R}$, iff, for all times $t \in I$,

- 1 defined $(t, Y(t)) \in D$
- 2 time-derivative $Y'(t)$ exists and $Y'(t) = f(t, Y(t))$.
- 3 initial value $Y(t_0) = y_0$

If $f \in C(D, \mathbb{R}^n)$, then $Y \in C^1(I, \mathbb{R}^n)$.

Definition (Ordinary Differential Equation, ODE)

$f : D \rightarrow \mathbb{R}^n$ on domain $D \subseteq \mathbb{R} \times \mathbb{R}^n$ (i.e., open connected). Then $Y : I \rightarrow \mathbb{R}^n$ is *solution* of initial value problem (IVP)

$$\begin{pmatrix} y'(t) = f(t, y) \\ y(t_0) = y_0 \end{pmatrix}$$

on interval $I \subseteq \mathbb{R}$, iff, for all times $t \in I$,

- 1 defined $(t, Y(t)) \in D$
- 2 time-derivative $Y'(t)$ exists and $Y'(t) = f(t, Y(t))$.
- 3 initial value $Y(t_0) = y_0$

If $f \in C(D, \mathbb{R}^n)$, then $Y \in C^1(I, \mathbb{R}^n)$.

If f continuous, then Y continuously differentiable.

- 1 Learning Objectives
- 2 Introduction
- 3 Differential Equations
- 4 Examples of Differential Equations**
- 5 Domains of Differential Equations

Example: A Constant Differential Equation

Example (Initial value problem)

$$\begin{cases} x'(t) = 5 \\ x(0) = 2 \end{cases}$$

has a solution

Example: A Constant Differential Equation

Example (Initial value problem)

$$\begin{cases} x'(t) = 5 \\ x(0) = 2 \end{cases}$$

has a solution $x(t) = 5t + 2$

Example: A Constant Differential Equation

Example (Initial value problem)

$$\begin{cases} x'(t) = 5 \\ x(0) = 2 \end{cases}$$

has a solution $x(t) = 5t + 2$

Check by inserting solution into ODE+IVP.

$$\begin{cases} (x(t))' = (5t + 2)' = 5 \\ x(0) = 5 \cdot 0 + 2 = 2 \end{cases}$$



Example: A Linear Differential Equation from before

Example (Initial value problem)

$$\begin{pmatrix} x'(t) = \frac{1}{4}x(t) \\ x(0) = 1 \end{pmatrix}$$

has a solution

Example: A Linear Differential Equation from before

Example (Initial value problem)

$$\begin{pmatrix} x'(t) = \frac{1}{4}x(t) \\ x(0) = 1 \end{pmatrix}$$

has a solution $x(t) = e^{\frac{t}{4}}$

Example: A Linear Differential Equation from before

Example (Initial value problem)

$$\begin{pmatrix} x'(t) = \frac{1}{4}x(t) \\ x(0) = 1 \end{pmatrix}$$

has a solution $x(t) = e^{\frac{t}{4}}$

Check by inserting solution into ODE+IVP.

$$\begin{pmatrix} (x(t))' = (e^{\frac{t}{4}})' = e^{\frac{t}{4}}(\frac{t}{4})' = e^{\frac{t}{4}}\frac{1}{4} = \frac{1}{4}x(t) \\ x(0) = e^{\frac{0}{4}} = 1 \end{pmatrix}$$



ODE Examples

ODE	Solution
$x' = 1, x(0) = x_0$	$x(t) = x_0 + t$

ODE Examples

ODE	Solution
$x' = 1, x(0) = x_0$	$x(t) = x_0 + t$
$x' = 5, x(0) = x_0$	$x(t) = x_0 + 5t$

ODE Examples

ODE	Solution
$x' = 1, x(0) = x_0$	$x(t) = x_0 + t$
$x' = 5, x(0) = x_0$	$x(t) = x_0 + 5t$
$x' = x, x(0) = x_0$	$x(t) = x_0 e^t$

ODE Examples

ODE	Solution
$x' = 1, x(0) = x_0$	$x(t) = x_0 + t$
$x' = 5, x(0) = x_0$	$x(t) = x_0 + 5t$
$x' = x, x(0) = x_0$	$x(t) = x_0 e^t$
$x' = x^2, x(0) = x_0$	$x(t) = \frac{x_0}{1 - tx_0}$

ODE	Solution
$x' = 1, x(0) = x_0$	$x(t) = x_0 + t$
$x' = 5, x(0) = x_0$	$x(t) = x_0 + 5t$
$x' = x, x(0) = x_0$	$x(t) = x_0 e^t$
$x' = x^2, x(0) = x_0$	$x(t) = \frac{x_0}{1 - tx_0}$
$x' = \frac{1}{x}, x(0) = 1$	$x(t) = \sqrt{1 + 2t} \dots$

ODE	Solution
$x' = 1, x(0) = x_0$	$x(t) = x_0 + t$
$x' = 5, x(0) = x_0$	$x(t) = x_0 + 5t$
$x' = x, x(0) = x_0$	$x(t) = x_0 e^t$
$x' = x^2, x(0) = x_0$	$x(t) = \frac{x_0}{1 - tx_0}$
$x' = \frac{1}{x}, x(0) = 1$	$x(t) = \sqrt{1 + 2t} \dots$
$y'(x) = -2xy, y(0) = 1$	$y(x) = e^{-x^2}$

ODE	Solution
$x' = 1, x(0) = x_0$	$x(t) = x_0 + t$
$x' = 5, x(0) = x_0$	$x(t) = x_0 + 5t$
$x' = x, x(0) = x_0$	$x(t) = x_0 e^t$
$x' = x^2, x(0) = x_0$	$x(t) = \frac{x_0}{1 - tx_0}$
$x' = \frac{1}{x}, x(0) = 1$	$x(t) = \sqrt{1 + 2t} \dots$
$y'(x) = -2xy, y(0) = 1$	$y(x) = e^{-x^2}$
$x'(t) = tx, x(0) = x_0$	$x(t) = x_0 e^{\frac{t^2}{2}}$

ODE Examples

ODE	Solution
$x' = 1, x(0) = x_0$	$x(t) = x_0 + t$
$x' = 5, x(0) = x_0$	$x(t) = x_0 + 5t$
$x' = x, x(0) = x_0$	$x(t) = x_0 e^t$
$x' = x^2, x(0) = x_0$	$x(t) = \frac{x_0}{1 - tx_0}$
$x' = \frac{1}{x}, x(0) = 1$	$x(t) = \sqrt{1 + 2t} \dots$
$y'(x) = -2xy, y(0) = 1$	$y(x) = e^{-x^2}$
$x'(t) = tx, x(0) = x_0$	$x(t) = x_0 e^{\frac{t^2}{2}}$
$x' = \sqrt{x}, x(0) = x_0$	$x(t) = \frac{t^2}{4} \pm t\sqrt{x_0} + x_0$

ODE Examples

ODE	Solution
$x' = 1, x(0) = x_0$	$x(t) = x_0 + t$
$x' = 5, x(0) = x_0$	$x(t) = x_0 + 5t$
$x' = x, x(0) = x_0$	$x(t) = x_0 e^t$
$x' = x^2, x(0) = x_0$	$x(t) = \frac{x_0}{1 - tx_0}$
$x' = \frac{1}{x}, x(0) = 1$	$x(t) = \sqrt{1 + 2t} \dots$
$y'(x) = -2xy, y(0) = 1$	$y(x) = e^{-x^2}$
$x'(t) = tx, x(0) = x_0$	$x(t) = x_0 e^{\frac{t^2}{2}}$
$x' = \sqrt{x}, x(0) = x_0$	$x(t) = \frac{t^2}{4} \pm t\sqrt{x_0} + x_0$
$x' = y, y' = -x, x(0) = 0, y(0) = 1$	$x(t) = \sin t, y(t) = \cos t$

ODE Examples

ODE	Solution
$x' = 1, x(0) = x_0$	$x(t) = x_0 + t$
$x' = 5, x(0) = x_0$	$x(t) = x_0 + 5t$
$x' = x, x(0) = x_0$	$x(t) = x_0 e^t$
$x' = x^2, x(0) = x_0$	$x(t) = \frac{x_0}{1 - tx_0}$
$x' = \frac{1}{x}, x(0) = 1$	$x(t) = \sqrt{1 + 2t} \dots$
$y'(x) = -2xy, y(0) = 1$	$y(x) = e^{-x^2}$
$x'(t) = tx, x(0) = x_0$	$x(t) = x_0 e^{\frac{t^2}{2}}$
$x' = \sqrt{x}, x(0) = x_0$	$x(t) = \frac{t^2}{4} \pm t\sqrt{x_0} + x_0$
$x' = y, y' = -x, x(0) = 0, y(0) = 1$	$x(t) = \sin t, y(t) = \cos t$
$x' = 1 + x^2, x(0) = 0$	$x(t) = \tan t$

ODE Examples

ODE	Solution
$x' = 1, x(0) = x_0$	$x(t) = x_0 + t$
$x' = 5, x(0) = x_0$	$x(t) = x_0 + 5t$
$x' = x, x(0) = x_0$	$x(t) = x_0 e^t$
$x' = x^2, x(0) = x_0$	$x(t) = \frac{x_0}{1 - tx_0}$
$x' = \frac{1}{x}, x(0) = 1$	$x(t) = \sqrt{1 + 2t} \dots$
$y'(x) = -2xy, y(0) = 1$	$y(x) = e^{-x^2}$
$x'(t) = tx, x(0) = x_0$	$x(t) = x_0 e^{\frac{t^2}{2}}$
$x' = \sqrt{x}, x(0) = x_0$	$x(t) = \frac{t^2}{4} \pm t\sqrt{x_0} + x_0$
$x' = y, y' = -x, x(0) = 0, y(0) = 1$	$x(t) = \sin t, y(t) = \cos t$
$x' = 1 + x^2, x(0) = 0$	$x(t) = \tan t$
$x'(t) = \frac{2}{t^3} x(t)$	$x(t) = e^{-\frac{1}{t^2}}$ non-analytic

ODE	Solution
$x' = 1, x(0) = x_0$	$x(t) = x_0 + t$
$x' = 5, x(0) = x_0$	$x(t) = x_0 + 5t$
$x' = x, x(0) = x_0$	$x(t) = x_0 e^t$
$x' = x^2, x(0) = x_0$	$x(t) = \frac{x_0}{1 - tx_0}$
$x' = \frac{1}{x}, x(0) = 1$	$x(t) = \sqrt{1 + 2t} \dots$
$y'(x) = -2xy, y(0) = 1$	$y(x) = e^{-x^2}$
$x'(t) = tx, x(0) = x_0$	$x(t) = x_0 e^{\frac{t^2}{2}}$
$x' = \sqrt{x}, x(0) = x_0$	$x(t) = \frac{t^2}{4} \pm t\sqrt{x_0} + x_0$
$x' = y, y' = -x, x(0) = 0, y(0) = 1$	$x(t) = \sin t, y(t) = \cos t$
$x' = 1 + x^2, x(0) = 0$	$x(t) = \tan t$
$x'(t) = \frac{2}{t^3} x(t)$	$x(t) = e^{-\frac{1}{t^2}}$ non-analytic
$x' = x^2 + x^4$???

ODE	Solution
$x' = 1, x(0) = x_0$	$x(t) = x_0 + t$
$x' = 5, x(0) = x_0$	$x(t) = x_0 + 5t$
$x' = x, x(0) = x_0$	$x(t) = x_0 e^t$
$x' = x^2, x(0) = x_0$	$x(t) = \frac{x_0}{1 - tx_0}$
$x' = \frac{1}{x}, x(0) = 1$	$x(t) = \sqrt{1 + 2t} \dots$
$y'(x) = -2xy, y(0) = 1$	$y(x) = e^{-x^2}$
$x'(t) = tx, x(0) = x_0$	$x(t) = x_0 e^{\frac{t^2}{2}}$
$x' = \sqrt{x}, x(0) = x_0$	$x(t) = \frac{t^2}{4} \pm t\sqrt{x_0} + x_0$
$x' = y, y' = -x, x(0) = 0, y(0) = 1$	$x(t) = \sin t, y(t) = \cos t$
$x' = 1 + x^2, x(0) = 0$	$x(t) = \tan t$
$x'(t) = \frac{2}{t^3} x(t)$	$x(t) = e^{-\frac{1}{t^2}}$ non-analytic
$x' = x^2 + x^4$???
$x'(t) = e^{t^2}$	non-elementary

ODE	Solution
$x' = 1, x(0) = x_0$	$x(t) = x_0 + t$
$x' = 5, x(0) = x_0$	$x(t) = x_0 + 5t$
$x' = x, x(0) = x_0$	$x(t) = x_0 e^t$
$x' = x^2, x(0) = x_0$	$x(t) = \frac{x_0}{1 - tx_0}$
$x' = \frac{1}{x}, x(0) = 1$	$x(t) = \sqrt{1 + 2t} \dots$
$y'(x) = -2xy, y(0) = 1$	$y(x) = e^{-x^2}$
$x'(t) = tx, x(0) = x_0$	$x(t) = x_0 e^{\frac{t^2}{2}}$
$x' = \sqrt{x}, x(0) = x_0$	$x(t) = \frac{t^2}{4} \pm t\sqrt{x_0} + x_0$
$x' = y, y' = -x, x(0) = 0, y(0) = 1$	$x(t) = \sin t, y(t) = \cos t$
$x' = 1 + x^2, x(0) = 0$	$x(t) = \tan t$
$x'(t) = \frac{2}{t^3} x(t)$	$x(t) = e^{-\frac{1}{t^2}}$ non-analytic
$x' = x^2 + x^4$???
$x'(t) = e^{t^2}$	non-elementary

Descriptive power of differential equations

- 1 Solutions of differential equations can be much more involved than the differential equations themselves.
- 2 Representational and descriptive power of differential equations!
- 3 Simple differential equations can describe quite complicated physical processes.
- 4 Local description as the direction into which the system evolves.

- 1 Learning Objectives
- 2 Introduction
- 3 Differential Equations
- 4 Examples of Differential Equations
- 5 Domains of Differential Equations**

Evolution Domain Constraints

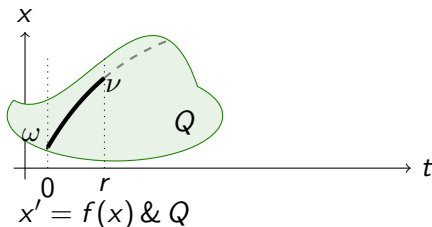
Enable Cyber to interact with Physics

Definition (Evolution domain constraints)

A differential equation $x' = f(x)$ with evolution domain Q is denoted by

$$x' = f(x) \& Q$$

conjunctive notation ($\&$) signifies that the system obeys the differential equation $x' = f(x)$ and the evolution domain Q .



Evolution Domain Constraints

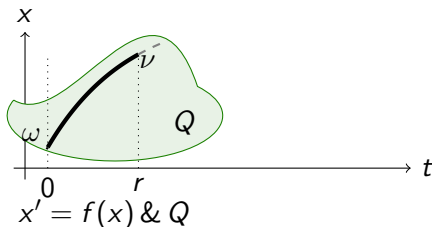
Enable Cyber to interact with Physics

Definition (Evolution domain constraints)

A differential equation $x' = f(x)$ with evolution domain Q is denoted by

$$x' = f(x) \& Q$$

conjunctive notation ($\&$) signifies that the system obeys the differential equation $x' = f(x)$ and the evolution domain Q .



Evolution Domain Constraints

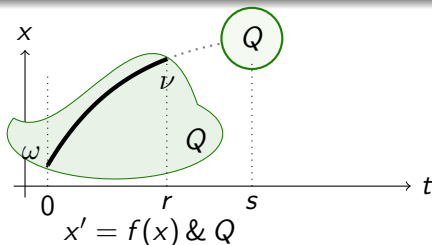
Enable Cyber to interact with Physics

Definition (Evolution domain constraints)

A differential equation $x' = f(x)$ with evolution domain Q is denoted by

$$x' = f(x) \& Q$$

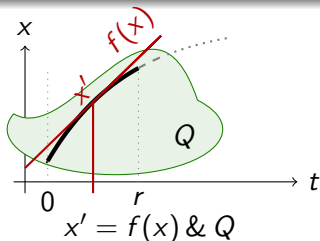
conjunctive notation ($\&$) signifies that the system obeys the differential equation $x' = f(x)$ and the evolution domain Q .



Definition (Semantics of differential equations)

A function $\varphi : [0, r] \rightarrow \mathcal{S}$ of some duration $r \geq 0$ satisfies the differential equation $x' = f(x) \ \& \ Q$, written $\varphi \models x' = f(x) \ \& \ Q$, iff:

- 1 $\varphi(\zeta)(x') = \frac{d\varphi(t)(x)}{dt}(\zeta)$ exists at for all times $0 \leq \zeta \leq r$
- 2 $\varphi(\zeta) \in \llbracket x' = f(x) \ \& \ Q \rrbracket$ for all times $0 \leq \zeta \leq r$
- 3 $\varphi(\zeta)(z) = \varphi(0)(z)$ for all variables $z \neq x$



Developed on the board:

- 1 First-order logic of real arithmetic
- 2 **The meaning of terms**
- 3 The meaning of formulas

See lecture notes for details [1].



André Platzer.

Foundations of cyber-physical systems.

Lecture Notes 15-424/624, Carnegie Mellon University, 2017.

URL: <http://www.cs.cmu.edu/~aplatzer/course/fcps17.html>.



André Platzer.

Logical Analysis of Hybrid Systems: Proving Theorems for Complex Dynamics.

Springer, Heidelberg, 2010.

doi:10.1007/978-3-642-14509-4.