**Assignment 5: Play Around with Hybrid Games**
**15-424/15-624 Foundations of Cyber-Physical Systems**
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Due: **Beginning of class**, Thursday, 3/31/2016
Total Points: 60

1. **Games, games, games** Consider the following formulas: a) Under which circumstances can Angel win? b) Briefly describe her winning strategy. c) Can Demon change one symbol to make it so that Angel can never win?

   **A warm-up:** $\langle (x := 0 \cap x := 1)^{\times} \rangle x \geq 0$

   **Ups and Downs:** $\langle (((x := x + 1 \cup (x' = v)^d); (y := y - 1 \cup (y' = w)^d))^* \rangle |x - y| \leq 1$

   **Dangerous Choices:** $\langle (((n' = -1 \cup x := x - 1); (?n > 0)^d)^* \rangle (x < 0)$

   **A chase:** $\langle (w := w \cap w := -w); (v := v \cup v := -v); (x' = v)^d; y' = w \rangle x < y$

   **Guess my sign** $\langle (((w := w \cap w := -w); (v := v \cup v := -v))^{\times} \rangle v == w$

   **A (real) game of wit:** Let $\alpha \equiv (x_1 := x_1 - 1 \cup x_1 := x_1 - 2 \cup x_1 := x_1 - 3 \cup x_2 := x_2 - 1 \cup x_2 := x_2 - 2 \cup x_2 := x_2 - 3; (?x_1 \geq 0 \land x_2 \geq 0))$.

   The actual game:
   $x_1 = n, x_2 = n, t = 0 \vdash \langle (\alpha; \alpha^d)^* \rangle (x_1 = 0 \land x_2 = 0)$

2. **Taylorism.** Prove (using the axioms and proof rules of d$\mathcal{L}$) that

$$x = 1 \land t = 0 \rightarrow [\{x' = x, t' = 1 \& x \geq 1\}] x \geq 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24}$$

   Submit *either* a hand-written proof or a Bellerophon tactic that proves this.

3. **Convergence and Divergence.** Consider the infinite summation over function $f(n)$, with $f(n) \geq 0$ for all $n \in \mathbb{N}$:

$$\sum_{n=0}^{\infty} f(n)$$

   (a) Write a d$\mathcal{L}$ formula, containing one [] or $\langle\rangle$ modality, in d$\mathcal{L}$, which, if proved valid, would guarantee the sum converges.

   (b) Write a d$\mathcal{L}$ formula, containing one [] or $\langle\rangle$ modality, in d$\mathcal{L}$, which, if proved valid, would guarantee the sum diverges.

   *Hint:* for a refresher, `http://en.wikipedia.org/wiki/Convergent_series`

4. **Chord length vs. arc length.** Because you've already done lab 3, we know you know how fun circular dynamics are! So in this exercise and the following, we are going to prove yet *another* interesting property of circular dynamics! Let's dive right in!

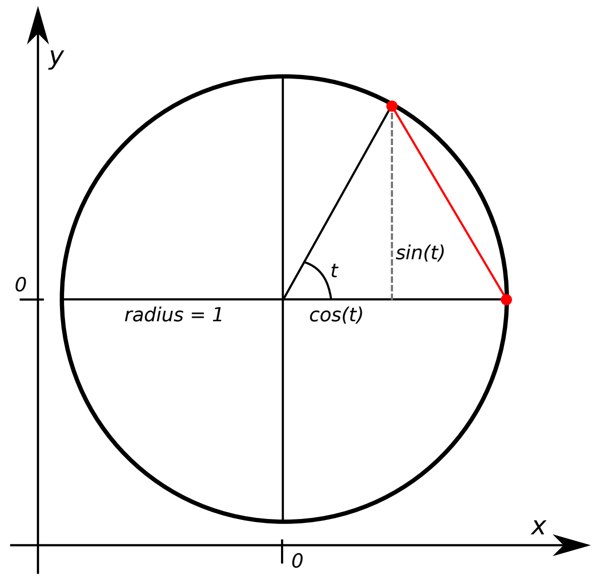


Figure 1: A beautiful picture, meticulously & lovingly hand-crafted by your local TAs

The following formula states that chord length (in red) is always smaller than arc length in a circle of radius 1.

$$x = 1 \land y = 0 \land t = 0 \rightarrow [x' = -y, y' = x, t' = 1]\ 2(1 - x) \leq t^2$$

It will be your job to figure out why that's the case! Fortunately, your dearest TAs are here to help you figure out how the heck $2(1 - x) \leq t^2$ means "the chord length is less than the arc length".

Despite its misleading name, $t$ is not time (sneaky, sneaky $t$!). As you can see above, it's the angle. Moreover, we know that $t = \frac{\text{arc length}}{r}$, i.e. arc length $= tr$. One famous version of this formula is the arc of 360° degrees, i.e. $c = 2\pi r$. Because we are assuming the unit radius $r = 1$, then the arc is simply $t$!

Using the quantities in the figure and the initial values of $x = 1, y = 0$, explain how $2(1 - x) \leq t^2$ means that the chord length is smaller than the arc length.

5. **Chord length vs. arc length (proof edition)!** We know you like circular motion. But we're on to your dark secret! More than circular motion, we know you love *proving things*! What can we do but oblige?

Please prove the above formula in $\mathsf{d}\mathcal{L}$.

*Hint:* use DC and DI, a dash of DW, and a sprinkle of circular motion properties. One very useful insight is that applying DI will generally yield formulas that you need in order to prove the original property. So try to find a way to include those formulas in the proof!