

21: Virtual Substitution & Real Arithmetic

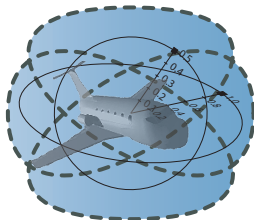
15-424: Foundations of Cyber-Physical Systems

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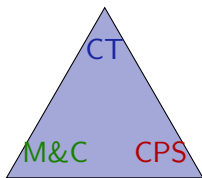
- 1 Learning Objectives
 - Recap: Quadratic Equations
- 2 Real Arithmetic
 - Quadratic Weak Inequalities
 - Virtual Substitution of Infinities
 - Expedition: Infinities
 - Quadratic Strict Inequalities
 - Infinitesimals
 - Virtual Substitution of Infinitesimals
- 3 Quantifier Elimination by Virtual Substitution
- 4 Summary

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Learning Objectives

Virtual Substitution & Real Equations

rigorous arithmetical reasoning
miracle of quantifier elimination
logical trinity for reals
switch between syntax & semantics at will
virtual substitution lemma
bridge gap between semantics and inexpressibles



analytic complexity
modeling tradeoffs

verifying CPS at scale

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$a \neq 0 \vee b \neq 0 \vee c \neq 0 \rightarrow$$

$$\left(\exists x (ax^2 + bx + c = 0 \wedge F) \leftrightarrow \right.$$

$$a = 0 \wedge b \neq 0 \wedge F_{\bar{x}}^{-c/b}$$

$$\vee a \neq 0 \wedge b^2 - 4ac \geq 0 \wedge \left(F_{\bar{x}}^{(-b + \sqrt{b^2 - 4ac})/(2a)} \vee F_{\bar{x}}^{(-b - \sqrt{b^2 - 4ac})/(2a)} \right)$$

Lemma (Virtual Substitution Lemma for $\sqrt{\cdot}$)

Extended logic

$$F_x^{(a+b\sqrt{c})/d} \equiv F_{\bar{x}}^{(a+b\sqrt{c})/d}$$

FOL $_{\mathbb{R}}$

$$\omega_x^r \in \llbracket F \rrbracket \text{ iff } \omega \in \llbracket F_{\bar{x}}^{(a+b\sqrt{c})/d} \rrbracket \text{ where } r = (\llbracket a \rrbracket \omega + \llbracket b \rrbracket \omega \sqrt{\llbracket c \rrbracket \omega}) / \llbracket d \rrbracket \omega \in \mathbb{R}$$

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Theorem (Virtual Substitution: Quadratic Inequality $x \notin a, b, c$)

$$\exists x (ax^2 + bx + c \leq 0 \wedge F) \leftrightarrow$$

Quadratic Inequality Virtual Substitution

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$$\vee F_{\bar{x}}^{\text{small}} \Big)$$

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$-\infty$ the rubber band number that's smaller on any comparison

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$-\infty$ needs to satisfy the quadratic inequality (obvious for roots)

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$$\omega_x^r \in \llbracket F \rrbracket \text{ iff } \omega \in \llbracket F_{\bar{x}}^{-\infty} \rrbracket \text{ where } r \rightarrow -\infty$$

Virtual Substitution of $-\infty$ into Comparisons

$$p = \sum_{i=0}^n a_i x^i$$

$$(p = 0)_{\bar{x}}^{-\infty} \equiv$$

$$(p \leq 0)_{\bar{x}}^{-\infty} \equiv$$

$$(p < 0)_{\bar{x}}^{-\infty} \equiv$$

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$$(p < 0)_{\bar{x}}^{-\infty} \equiv p(-\infty) < 0$$

$$(p \neq 0)_{\bar{x}}^{-\infty} \equiv$$

Ultimately negative at $-\infty$

$$\lim_{x \rightarrow -\infty} p(x) < 0$$

$$p(-\infty) < 0 \stackrel{\text{def}}{\equiv} \left\{ \begin{array}{l} \text{if} \\ \text{if} \end{array} \right.$$

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if $\deg(p) \leq 0$
if

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Example: Virtual Substitution of Infinities

Virtual Substitution of $-\infty$ into Comparisons

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$$(p < 0)_{\bar{x}}^{-\infty} \equiv (p < 0)_{\bar{x}}^{-\infty}$$

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$$\begin{aligned} (ax^2 + bx + c \leq 0)_{\bar{x}}^{-\infty} &\equiv (-1)^2 a < 0 \vee a = 0 \wedge ((-1)b < 0 \vee b = 0 \wedge c < 0) \\ &\equiv a < 0 \vee a = 0 \wedge (b > 0 \vee b = 0 \wedge c < 0) \end{aligned}$$

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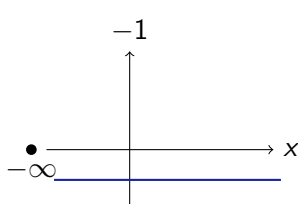
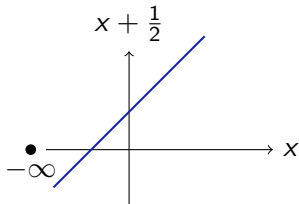
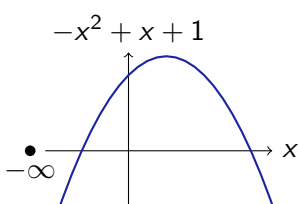
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Expedition: Infinite Challenges with Infinities

$\mathbb{R} \cup \{-\infty, \infty\}$ extended reals

- Order: $\forall x (-\infty \leq x \leq \infty)$
- Complete lattice since every subset has a supremum and infimum
- Arithmetic? $\infty + 1$?

$$\infty + x =$$

$$-\infty + x =$$

$$\infty \cdot x =$$

$$-\infty \cdot x =$$

$$-\infty \cdot x =$$

$$-\infty \cdot x =$$

$$\infty - \infty =$$

$$0 \cdot \infty =$$

$$\pm\infty / \pm\infty =$$

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$\mathbb{R} \cup \{-\infty, \infty\}$ extended reals

- Order: $\forall x (-\infty \leq x \leq \infty)$
- Complete lattice since every subset has a supremum and infimum
- Arithmetic? $\infty + 1?$ $\infty \leq \infty + x$ but $\infty + x \leq \infty$

$$\infty + x = \infty \quad \text{for all } x \neq -\infty$$

$$-\infty + x = -\infty \quad \text{for all } x \neq \infty$$

$$\infty \cdot x = \infty \quad \text{for all } x > 0$$

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$$\infty - \infty = \text{undefined} \quad \infty + (-\infty) = \infty + (-\infty + 1) = (\infty - \infty) + 1$$

$$0 \cdot \infty =$$

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Theorem (Virtual Substitution: Quadratic Inequality $x \notin a, b, c$)

$$\exists x (ax^2 + bx + c < 0 \wedge F) \leftrightarrow$$

Quadratic Strict Inequality Virtual Substitution

Theorem (Virtual Substitution: Quadratic Inequality $x \notin a, b, c$)

$$a \neq 0 \vee b \neq 0 \vee c \neq 0 \rightarrow$$

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$$a = 0 \wedge b \neq 0 \wedge F_{\bar{x}}^{-c/b}$$

$$\vee a \neq 0 \wedge b^2 - 4ac \geq 0 \wedge \left(F_{\bar{x}}^{(-b + \sqrt{b^2 - 4ac})/(2a)} \vee F_{\bar{x}}^{(-b - \sqrt{b^2 - 4ac})/(2a)} \right)$$

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strict inequality can't be satisfied at the roots but slightly off

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ε the rubber band number that's smaller in magnitude on any comparison

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Lemma (Virtual Substitution Lemma for ε)

$$F_x^{e+\varepsilon} \equiv F_{\bar{x}}^{e+\varepsilon}$$

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Nonstandard analysis $\text{FOL}_{\mathbb{R}[\varepsilon]}$

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$\text{FOL}_{\mathbb{R}}$

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$$\omega_x^r \in \llbracket F \rrbracket \text{ iff } \omega \in \llbracket F_{\bar{x}}^{-\infty} \rrbracket \text{ where } r \searrow e$$

Expedition: Infinitesimals

ε in nonstandard field extension $\mathbb{R}[\varepsilon]$ is “always as small as needed”

- Positive: $\varepsilon > 0$

- Smaller: $\forall x \in \mathbb{R} (x > 0 \rightarrow \varepsilon < x)$

- Standard \mathbb{R} are Archimedean:

$$\forall x \in \mathbb{R} \setminus \{0\} \exists n \in \mathbb{N} \underbrace{|x + x + \dots + x|}_{n \text{ times}} > 1$$

- $\mathbb{R}[\varepsilon]$ are non-Archimedean: $\underbrace{\varepsilon + \varepsilon + \dots + \varepsilon}_{\text{any } n \in \mathbb{N} \text{ times}} < 1$

- Infinitesimals as inverses of infinities?

$$\varepsilon \cdot \infty = 1? \quad -\varepsilon \cdot -\infty = 1? \quad (\varepsilon + 1) \cdot \underbrace{(\infty + 2)}_{\infty?} = \dots$$

- How to order?

$$\varepsilon^2 \quad \varepsilon \quad x^2 + \varepsilon \quad (x + \varepsilon)^2 \quad x^2 + 2\varepsilon x + 5\varepsilon + \varepsilon^2$$

Virtual Substitution of $e + \varepsilon$ into Comparisons

$$p = \sum_{i=0}^n a_i x^i$$

$$(p = 0)_{\bar{x}}^{e+\varepsilon} \equiv$$

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Immediately negative at e

$$\lim_{x \searrow e} p(x) < 0$$

$$p^+ < 0 \stackrel{\text{def}}{\equiv} \left\{ \begin{array}{l} \text{if} \\ \text{if} \end{array} \right.$$

ordinary virtual substitution into immediate negativity

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Successive derivative p' immediately negative at root of p to break ties

Example: Virtual Substitution of Infinitesimals

$$(ax^2+bx+c)^+ < 0 \equiv ax^2+bx+c < 0 \\ \vee ax^2+bx+c = 0 \wedge (2ax+b < 0 \vee 2ax+b = 0 \wedge 2a < 0)$$

$$(ax^2+bx+c < 0)_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)+\epsilon} \equiv ((ax^2+bx+c)^+ < 0)_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \equiv$$

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$$\equiv 0 \cdot 1 < 0 \vee 0 = 0 \wedge \underbrace{(0 < 0 \vee 4a^2 \leq 0 \wedge (0 < 0 \vee -4a^2(b^2-4ac) < 0))}_{(2ax+b < 0)_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)}}$$

$$(2ax+b < 0)_{\bar{x}}^{(-b+\sqrt{b^2-4ac})/(2a)}$$

$$\vee \underbrace{0=0}_{(2ax+b=0)_{\bar{x}}^{\dots}} \wedge \underbrace{2a1 < 0}_{(2a < 0)_{\bar{x}}^{\dots}}$$

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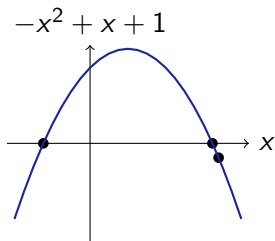
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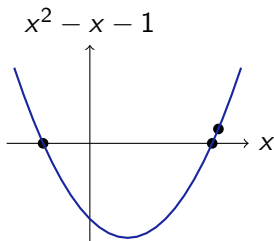
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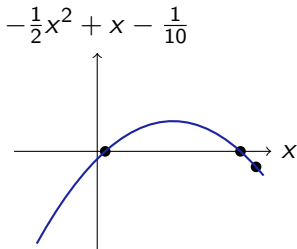
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case $a < 0$



case $a > 0$



case $a < 0$

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- 1 Learning Objectives
 - Recap: Quadratic Equations
- 2 Real Arithmetic
 - Quadratic Weak Inequalities
 - Virtual Substitution of Infinities
 - Expedition: Infinities
 - Quadratic Strict Inequalities
 - Infinitesimals
 - Virtual Substitution of Infinitesimals
- 3 Quantifier Elimination by Virtual Substitution
- 4 Summary

Quantifier Elimination by Virtual Substitution

Theorem (Virtual Substitution: Quadratics)

(Weispfenning'97)

Let all atomic formulas in F be of the form $ax^2 + bx + c \sim 0$ with $x \notin a, b, c$ and $\sim \in \{=, \leq, <, \neq\}$ and $d \stackrel{\text{def}}{=} b^2 - 4ac$.

$\exists x F \leftrightarrow$

$F_x^{-\infty}$

$$\bigvee_{(ax^2+bx+c \{ \leq \} 0) \in F} \left(a=0 \wedge b \neq 0 \wedge F_x^{-c/b} \vee a \neq 0 \wedge d \geq 0 \wedge (F_x^{(-b+\sqrt{d})/(2a)} \vee F_x^{(-b-\sqrt{d})/(2a)}) \right)$$

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Need roots and off-roots from all atomic formulas in F

Quantifier Elimination by Virtual Substitution

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$$F_x^{-\infty}$$

$$\bigvee_{(ax^2+bx+c \{ \leq \} 0) \in F} \left(a=0 \wedge b \neq 0 \wedge F_x^{-c/b} \vee a \neq 0 \wedge d \geq 0 \wedge (F_x^{(-b+\sqrt{d})/(2a)} \vee F_x^{(-b-\sqrt{d})/(2a)}) \right)$$

$$\bigvee_{(ax^2+bx+c \{ \neq \} 0) \in F} \left(a=0 \wedge b \neq 0 \wedge F_x^{-c/b+\varepsilon} \vee a \neq 0 \wedge d \geq 0 \wedge (F_x^{(-b+\sqrt{d})/(2a)+\varepsilon} \vee F_x^{(-b-\sqrt{d})/(2a)+\varepsilon}) \right)$$

Need roots and off-roots from all atomic formulas in F

Quantifier Elimination by Virtual Substitution

Theorem (Virtual Substitution: Quadratics)

(Weispfenning'97)

Let all atomic formulas in F be of the form $ax^2 + bx + c \sim 0$ with $x \notin a, b, c$ and $\sim \in \{=, \leq, <, \neq\}$ and $d \stackrel{\text{def}}{=} b^2 - 4ac$.

$\exists x F \leftrightarrow$

$F_{\bar{x}}^{-\infty}$

$\vee \bigvee_{(ax^2+bx+c \{ \leq \} 0) \in F} (a=0 \wedge b \neq 0 \wedge F_{\bar{x}}^{-c/b} \vee a \neq 0 \wedge d \geq 0 \wedge (F_{\bar{x}}^{(-b+\sqrt{d})/(2a)} \vee F_{\bar{x}}^{(-b-\sqrt{d})/(2a)}))$

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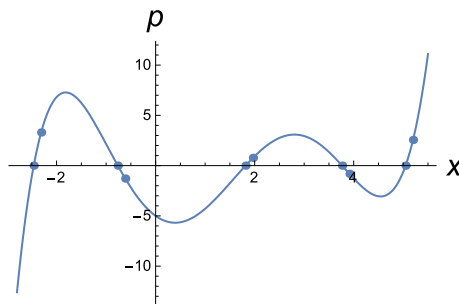
Lemma (Virtual Substitution Lemmas)

$$F_x^{(a+b\sqrt{c})/d} \equiv F_{\bar{x}}^{(a+b\sqrt{c})/d}$$

$$F_x^{-\infty} \equiv F_{\bar{x}}^{-\infty}$$

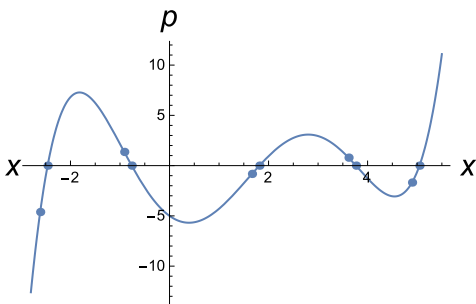
$$F_x^{e+\epsilon} \equiv F_{\bar{x}}^{e+\epsilon}$$

Alternative Formulations



$-\infty$ and roots e with offsets $e + \varepsilon$

No rejection without mention



roots e with offsets $e - \varepsilon$ and ∞

Other parts of F not satisfied by the points of p have their own polynomial q that contributes different roots \tilde{e} and off-roots $\tilde{e} + \varepsilon$.

Generalizations of quantifier elimination to higher degrees also place a representative point into every region of interest, but derivatives and relationships of derivatives become relevant.

Quantifier Elimination by Virtual Substitution

Theorem (Virtual Substitution: Quadratics)

(Weispfenning'97)

$$\exists x F \leftrightarrow F_{\bar{x}}^{-\infty}$$

$$\bigvee_{(ax^2+bx+c \begin{cases} = \\ \leq \end{cases} 0) \in F} (a=0 \wedge b \neq 0 \wedge F_{\bar{x}}^{-c/b} \vee a \neq 0 \wedge d \geq 0 \wedge (F_{\bar{x}}^{(-b+\sqrt{d})/(2a)} \vee F_{\bar{x}}^{(-b-\sqrt{d})/(2a)}))$$

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“Proof” Sketch.

“ \leftarrow ” simple from virtual substitution lemma with (extended) term witness

“ \rightarrow ” Valid iff true in every state, so all variables have real numeric value

- o-minimal: solutions form finite union of disjoint intervals (univariate)
- WLOG endpoints are the roots since all polynomials quadratic
- All side conditions have to be met otherwise can't be solution
- Non-point intervals contain ε offset since that's smaller than endpoint

- 1 Learning Objectives
 - Recap: Quadratic Equations
- 2 Real Arithmetic
 - Quadratic Weak Inequalities
 - Virtual Substitution of Infinities
 - Expedition: Infinities
 - Quadratic Strict Inequalities
 - Infinitesimals
 - Virtual Substitution of Infinitesimals
- 3 Quantifier Elimination by Virtual Substitution
- 4 Summary

Quadratic Virtual Substitution

Theorem (Virtual Substitution: Quadratic Equation $x \notin a, b, c$)

$$a \neq 0 \vee b \neq 0 \vee c \neq 0 \rightarrow$$

$$\left(\exists x (ax^2 + bx + c = 0 \wedge F) \leftrightarrow \right.$$

$$a = 0 \wedge b \neq 0 \wedge F_{\bar{x}}^{-c/b}$$

$$\vee a \neq 0 \wedge b^2 - 4ac \geq 0 \wedge \left(F_{\bar{x}}^{(-b + \sqrt{b^2 - 4ac})/(2a)} \vee F_{\bar{x}}^{(-b - \sqrt{b^2 - 4ac})/(2a)} \right)$$

Lemma (Virtual Substitution Lemma for $\sqrt{\cdot}$)

Extended logic

$$F_x^{(a+b\sqrt{c})/d} \equiv F_{\bar{x}}^{(a+b\sqrt{c})/d}$$

FOL $_{\mathbb{R}}$

$$\omega_x^r \in \llbracket F \rrbracket \text{ iff } \omega \in \llbracket F_{\bar{x}}^{(a+b\sqrt{c})/d} \rrbracket \text{ where } r = (\llbracket a \rrbracket \omega + \llbracket b \rrbracket \omega \sqrt{\llbracket c \rrbracket \omega}) / \llbracket d \rrbracket \omega \in \mathbb{R}$$

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$$F_x^{(a+b\sqrt{c})/d} \equiv F_{\bar{x}}^{(a+b\sqrt{c})/d}$$

$$F_x^{-\infty} \equiv F_{\bar{x}}^{-\infty}$$

$$F_x^{e+\varepsilon} \equiv F_{\bar{x}}^{e+\varepsilon}$$

- Miracle: $\text{FOL}_{\mathbb{R}}$ is decidable: Tarski'31
(Complex) Algorithm says whether (closed) formula valid or not.
- Quantifier elimination effectively associates quantifier-free equivalent
- Successive quantifier elimination decides $\text{FOL}_{\mathbb{R}}$ (universal closure)
- QE accepts free variables, giving equivalent that identifies the requirements for truth
- Virtual substitution does QE for degree ≤ 3 by equivalent syntactic rephrasing of semantics Weispfenning'97
- QE proceeds inside out, so degree ≤ 3 needed on each iteration
- Important fragments permit many optimizations your research?
- Universally quantified weak inequalities / existentially quantified strict inequalities are easier since infinitesimals/infinities rarely satisfy =
- Cylindrical algebraic decomposition (CAD) any degree Collins'75
- Simplify arithmetic to relevant parts, transform to fit together



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