17: Winning Strategies & Regions

15-424: Foundations of Cyber-Physical Systems

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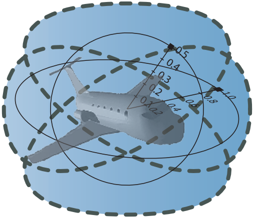

Outline

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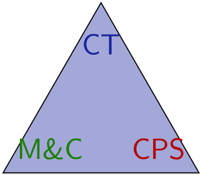
fundamental principles of computational thinking
logical extensions
PL modularity principles
compositional extensions
differential game logic
denotational vs. operational semantics



adversarial dynamics
adversarial semantics

CPS semantics
multi-agent operational-effects
mutual reactions
complementary hybrid systems

Differential Game Logic: Syntax

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game | Dual Game

Definition (Hybrid game α)

$$x := e \mid ?Q \mid x' = f(x) \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^* \mid \alpha^d$$

Definition (dG$\mathcal{L}$ Formula P)

$$p(e_1,\dots,e_n) \mid e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \langle\alpha\rangle P \mid [\alpha]P$$

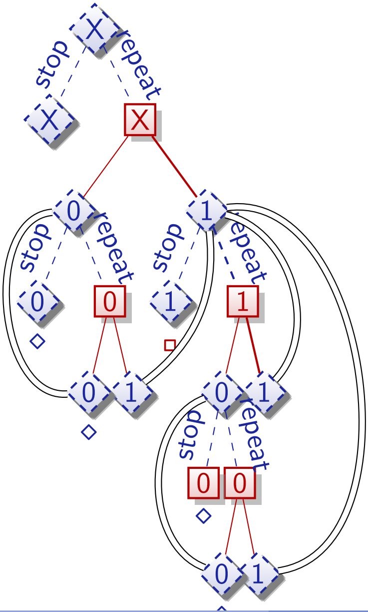
All Reals | Some Reals | Angel Wins | Demon Wins

TOCL'15

Outline

$$\langle (x := 0 \cap x := 1)^* \rangle x = 0$$

$\overset{\text{wfd}}{\rightsquigarrow}$ false unless $x = 0$



Differential Game Logic: Denotational Semantics

Definition (dG$\mathcal{L}$ Formula P) $[\![\cdot]\!] : \text{Fml} \to \wp(\mathcal{S})$

$$
\begin{aligned}
[\![e_1 \geq e_2]\!] &= \{\omega \in \mathcal{S} \ : \ [\![e_1]\!]\omega \geq [\![e_2]\!]\omega\} \\
[\![\neg P]\!] &= ([\![P]\!])^{\complement} \\
[\![P \land Q]\!] &= [\![P]\!] \cap [\![Q]\!] \\
[\![\langle\alpha\rangle P]\!] &= \varsigma_\alpha([\![P]\!]) \quad \{\omega \ : \ \nu \in [\![P]\!] \text{ for some } \nu \text{ with } (\omega,\nu) \in [\![\alpha]\!]\} \text{ ???} \\
[\![[\alpha]P]\!] &= \delta_\alpha([\![P]\!])
\end{aligned}
$$

Differential Game Logic: Denotational Semantics

Definition (Hybrid game α: denotational semantics)

$\varsigma_{x:=e}(X) =$

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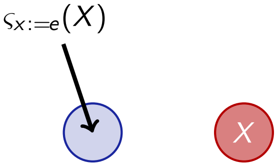

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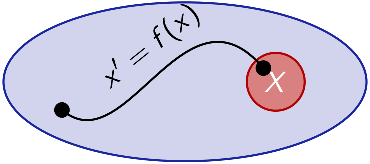

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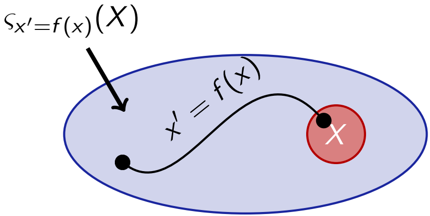
$$\varsigma_{x'=f(x)}(X) = \{\varphi(0) \in \mathcal{S} : \varphi(\zeta) \in X, \frac{\mathrm{d}\,\varphi(t)(x)}{\mathrm{d}t}(\zeta) = [\![f(x)]\!]_{\varphi(\zeta)} \text{ for all } \zeta\}$$



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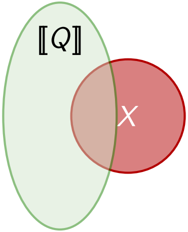

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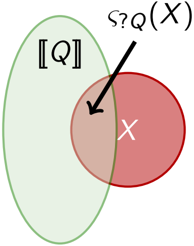
$$\varsigma_{?Q}(X) = [\![Q]\!] \cap X$$



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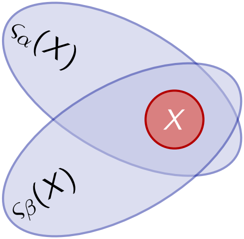

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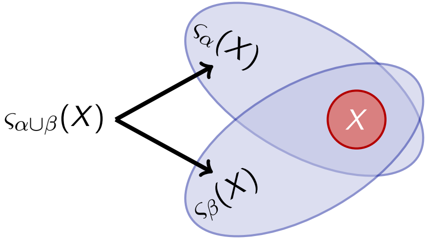

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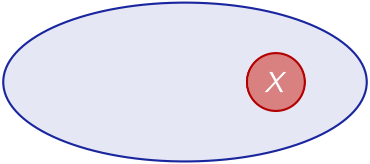

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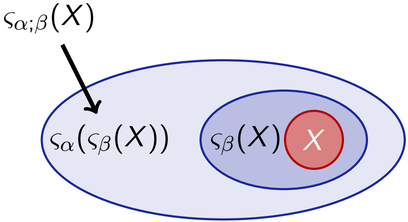

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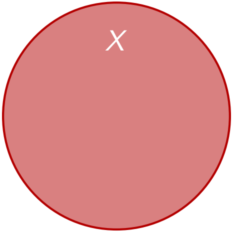

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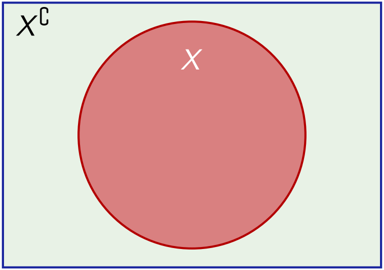

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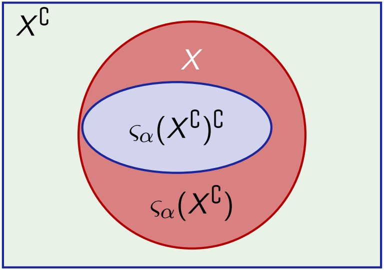

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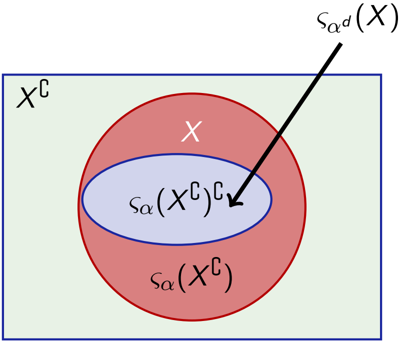

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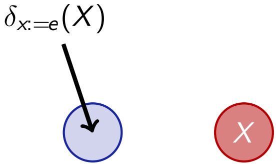

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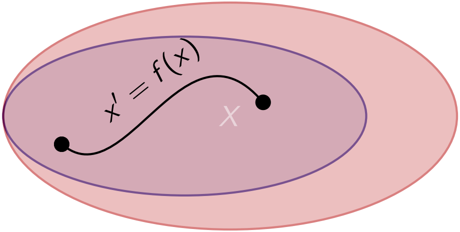

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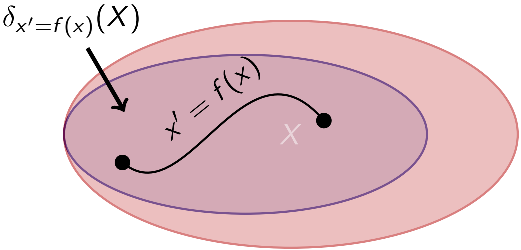

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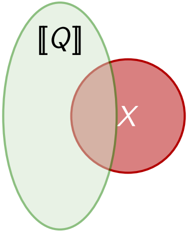

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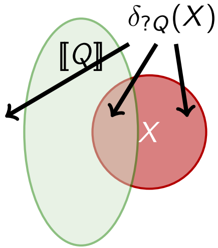

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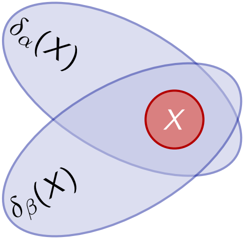

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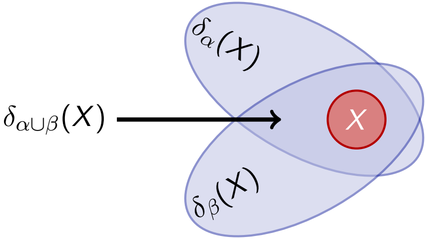

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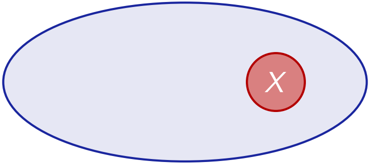

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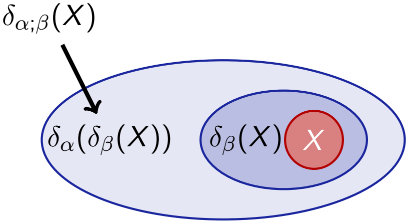

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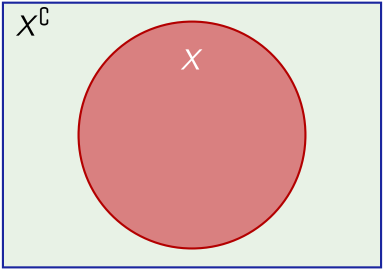

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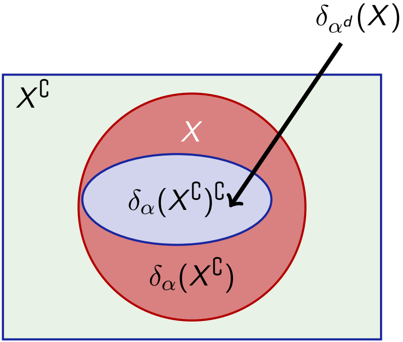

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\varsigma_{?Q}(X) &= [\![Q]\!] \cap X \\
\varsigma_{\alpha\cup\beta}(X) &= \varsigma_\alpha(X) \cup \varsigma_\beta(X) \\
\varsigma_{\alpha;\beta}(X) &= \varsigma_\alpha(\varsigma_\beta(X)) \\
\varsigma_{\alpha^*}(X) &= \\
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[\![\langle\alpha\rangle P]\!] &= \varsigma_\alpha([\![P]\!]) \\
[\![[\alpha]P]\!] &= \delta_\alpha([\![P]\!])
\end{aligned}$$

Filibusters & The Significance of Finitude

$\overset{<\infty}{\rightsquigarrow}$ true

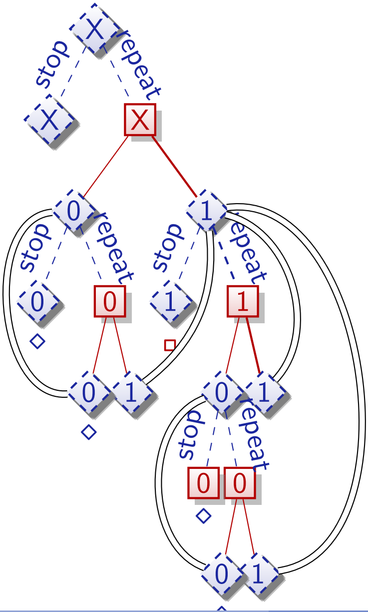
$\langle (x' = 1^d; x := 0)^* \rangle x = 0$

$\langle (x := 0; x' = 1^d)^* \rangle x = 0$

$\langle (x := 0 \cap x := 1)^* \rangle x = 0$

$\overset{\text{wfd}}{\rightsquigarrow}$ false unless $x = 0$

Well-defined games
can't be postponed forever



Consistency & Determinacy

Theorem (Consistency & determinacy)

Hybrid games are consistent and determined, i.e. $\vDash \neg\langle\alpha\rangle\neg\phi \leftrightarrow [\alpha]\phi$.

Corollary (Determinacy: At least one player wins)

$\vDash \neg\langle\alpha\rangle\neg\phi \rightarrow [\alpha]\phi$, *thus* $\vDash \langle\alpha\rangle\neg\phi \lor [\alpha]\phi$.

Corollary (Consistency: At most one player wins)

$\vDash [\alpha]\phi \rightarrow \neg\langle\alpha\rangle\neg\phi$, *thus* $\vDash \neg([\alpha]\phi \land \langle\alpha\rangle\neg\phi)$

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Corollary (Consistency: At most one player wins)

$\vDash [\alpha]\phi \rightarrow \neg\langle\alpha\rangle\neg\phi$, *thus* $\vDash \neg([\alpha]\phi \wedge \langle\alpha\rangle\neg\phi)$

Proof Sketch.

$$\varsigma_{\alpha\cup\beta}(X^{\complement})^{\complement} = (\varsigma_\alpha(X^{\complement}) \cup \varsigma_\beta(X^{\complement}))^{\complement} = \varsigma_\alpha(X^{\complement})^{\complement} \cap \varsigma_\beta(X^{\complement})^{\complement} = \delta_\alpha(X) \cap \delta_\beta(X) = \delta_{\alpha\cup\beta}(X)$$ □

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Semantics of Repetition

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) =$

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$\varsigma_{\alpha^*}(X) = \bigcup_{n\in\mathbb{N}} \varsigma_{\alpha^n}(X)$

$$[\![\alpha^*]\!] = \bigcup_{n\in\mathbb{N}} [\![\alpha^n]\!] \quad \text{where } \alpha^{n+1} \equiv \alpha^n;\alpha \quad \alpha^0 \equiv ?\mathit{true}$$

Semantics of Repetition

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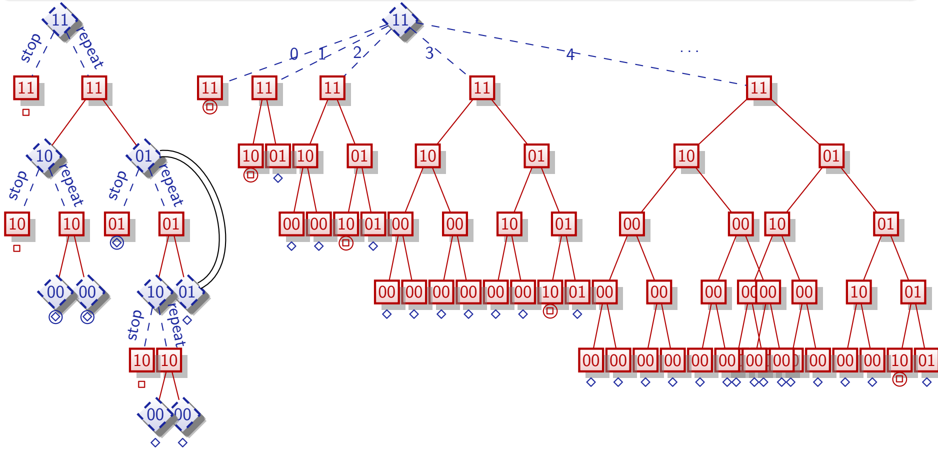
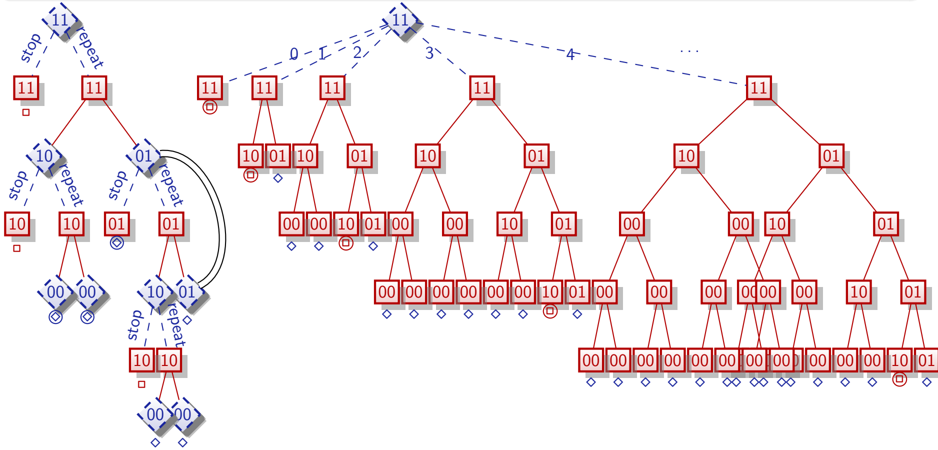


Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcup_{n\in\mathbb{N}} \varsigma_{\alpha^n}(X)$ — advance notice semantics



+1 Argument

Note (+1 argument)

$$Y \subseteq \varsigma_{\alpha^*}(X) \text{ then } \varsigma_\alpha(Y) \subseteq \varsigma_{\alpha^*}(X)$$

Since $\varsigma_\alpha(Y)$ is just one round away from Y.

Semantics of Repetition

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcup_{n\in\mathbb{N}} \varsigma_\alpha^n(X)$

$$\varsigma_\alpha^0(X) \overset{\text{def}}{=} X$$

$$\varsigma_\alpha^{\kappa+1}(X) \overset{\text{def}}{=} X \cup \varsigma_\alpha(\varsigma_\alpha^\kappa(X))$$

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Example

$\langle (x := 1; x' = 1^d \cup x := x - 1)^* \rangle\, (0 \le x < 1)$

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcup_{n\in\mathbb{N}} \varsigma_\alpha^n(X)$ ω-semantics

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$$\varsigma_\alpha^\lambda(X) \stackrel{\text{def}}{=} \bigcup_{\kappa<\lambda} \varsigma_\alpha^\kappa(X) \qquad \lambda \neq 0 \text{ a limit ordinal}$$

Example

$\langle (x := 1; x' = 1^d \cup x := x - 1)^* \rangle\, (0 \leq x < 1)$ $\quad \varsigma_\alpha^n([0,1)) = [0,n) \neq \mathbb{R}$

$\varsigma_\alpha^\omega([0,1)) = \bigcup_{n\in\mathbb{N}} \varsigma_\alpha^n([0,1)) = [0,\infty) \neq \mathbb{R}$

Semantics of Repetition

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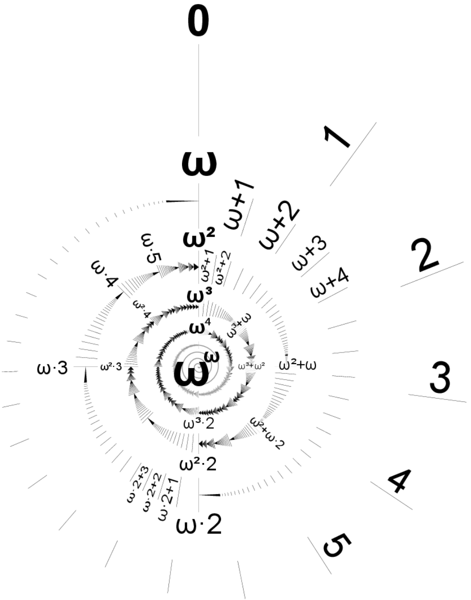
$\varsigma_\alpha^{\omega+1}([0,1)) = \varsigma_\alpha([0,\infty)) = \mathbb{R}$ $\qquad \varsigma_\alpha^\omega([0,1)) = \bigcup_{n\in\mathbb{N}} \varsigma_\alpha^n([0,1)) = [0,\infty) \neq \mathbb{R}$

Strategic Closure Ordinal

Theorem

Hybrid game closure ordinal $\geq \omega_1^{CK}$



Expedition: Ordinal Arithmetic

$$\iota + 0 = \iota$$

$$\iota + (\kappa{+}1) = (\iota + \kappa) + 1 \quad \text{successor } \kappa{+}1$$

$$\iota + \lambda = \bigsqcup_{\kappa<\lambda} \iota + \kappa \quad \text{limit } \lambda$$

$$\iota \cdot 0 = 0$$

$$\iota \cdot (\kappa{+}1) = (\iota \cdot \kappa) + \iota \quad \text{successor } \kappa{+}1$$

$$\iota \cdot \lambda = \bigsqcup_{\kappa<\lambda} \iota \cdot \kappa \quad \text{limit } \lambda$$

$$\iota^0 = 1$$

$$\iota^{\kappa+1} = \iota^\kappa \cdot \iota \quad \text{successor } \kappa{+}1$$

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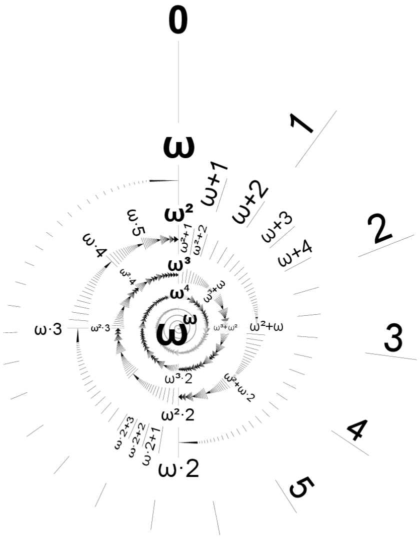

$$2 \cdot \omega = 4 \cdot \omega \neq \omega \cdot 2 < \omega \cdot 4$$

Monotonicity

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\varsigma_{x'=f(x)}(X) &= \{\varphi(0) \in \mathcal{S} \ :\ \varphi(r) \in X, \frac{\mathrm{d}\,\varphi(t)(x)}{\mathrm{d}t}(\zeta) = [\![f(x)]\!]\varphi(\zeta) \text{ for all } \zeta\} \\
\varsigma_{?Q}(X) &= [\![Q]\!] \cap X \\
\varsigma_{\alpha\cup\beta}(X) &= \varsigma_\alpha(X) \cup \varsigma_\beta(X) \\
\varsigma_{\alpha;\beta}(X) &= \varsigma_\alpha(\varsigma_\beta(X)) \\
\varsigma_{\alpha^*}(X) &= \\
\varsigma_{\alpha^d}(X) &= (\varsigma_\alpha(X^\complement))^\complement
\end{aligned}
$$

Lemma (Monotonicity)

$\varsigma_\alpha(X) \subseteq \varsigma_\alpha(Y)$ *and* $\delta_\alpha(X) \subseteq \delta_\alpha(Y)$ *for all* $X \subseteq Y$

Proof Sketch

- $X \subseteq Y$ so $X^\complement \supseteq Y^\complement$ so $\varsigma_\alpha(X^\complement) \supseteq \varsigma_\alpha(Y^\complement)$ so $\varsigma_{\alpha^d}(X) = (\varsigma_\alpha(X^\complement))^\complement \subseteq (\varsigma_\alpha(Y^\complement))^\complement = \varsigma_{\alpha^d}(Y)$.

Outline

Differential Game Logic: Denotational Semantics

Definition (Hybrid game α) $[\![\cdot]\!] : \text{HG} \to (\wp(\mathcal{S}) \to \wp(\mathcal{S}))$

$$
\begin{aligned}
\varsigma_{x:=e}(X) &= \{\omega \in \mathcal{S} \,:\, \omega_x^{[\![e]\!]\omega} \in X\} \\
\varsigma_{x'=f(x)}(X) &= \{\varphi(0) \in \mathcal{S} \,:\, \varphi(r) \in X, \frac{d\,\varphi(t)(x)}{dt}(\zeta) = [\![f(x)]\!]\varphi(\zeta) \text{ for all } \zeta\} \\
\varsigma_{?Q}(X) &= [\![Q]\!] \cap X \\
\varsigma_{\alpha\cup\beta}(X) &= \varsigma_\alpha(X) \cup \varsigma_\beta(X) \\
\varsigma_{\alpha;\beta}(X) &= \varsigma_\alpha(\varsigma_\beta(X)) \\
\varsigma_{\alpha^*}(X) &= \\
\varsigma_{\alpha^d}(X) &= (\varsigma_\alpha(X^\complement))^\complement
\end{aligned}
$$

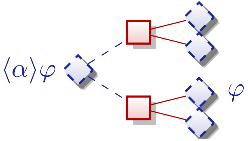
Definition (dG$\mathcal{L}$ Formula P) $[\![\cdot]\!] : \text{Fml} \to \wp(\mathcal{S})$

$$
\begin{aligned}
[\![e_1 \geq e_2]\!] &= \{\omega \in \mathcal{S} \,:\, [\![e_1]\!]\omega \geq [\![e_2]\!]\omega\} \\
[\![\neg P]\!] &= ([\![P]\!])^\complement \\
[\![P \land Q]\!] &= [\![P]\!] \cap [\![Q]\!] \\
[\![\langle\alpha\rangle P]\!] &= \varsigma_\alpha([\![P]\!]) \\
[\![[\alpha]P]\!] &= \delta_\alpha([\![P]\!])
\end{aligned}
$$

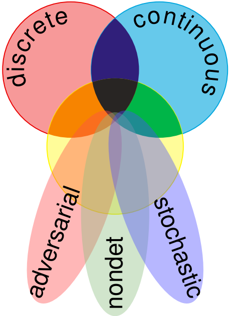
Summary

differential game logic

$\mathsf{dG}\mathcal{L} = \mathsf{GL} + \mathsf{HG} = \mathsf{d}\mathcal{L} + {}^{d}$



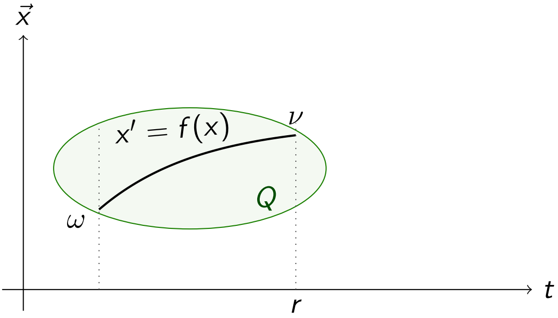
- Semantics for differential game logic
- Simple compositional denotational semantics
- Meaning is a simple function of its pieces
- Outlier: repetition is subtle higher-ordinal iteration
- Monotonicity
- Determinacy



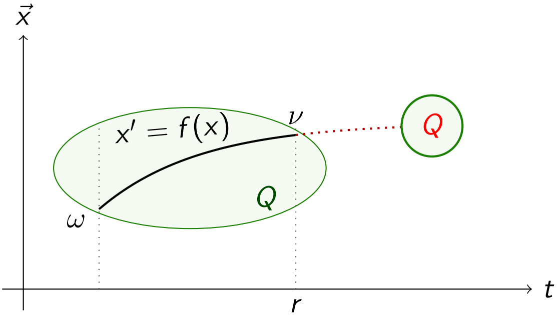
Next lecture

1. Revisit repetition
2. Axiomatics

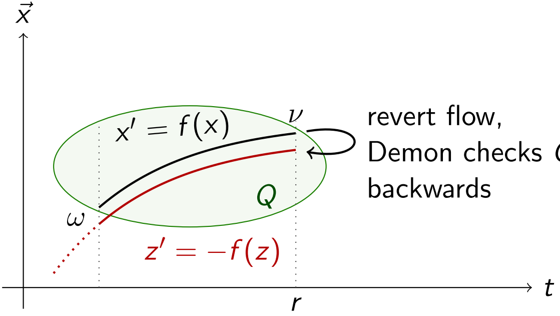
$x' = f(x) \,\&\, Q$ $\qquad$ $x' = f(x); ?(Q)$



$x' = f(x) \,\&\, Q$ $\qquad$ $x' = f(x); ?(Q)$

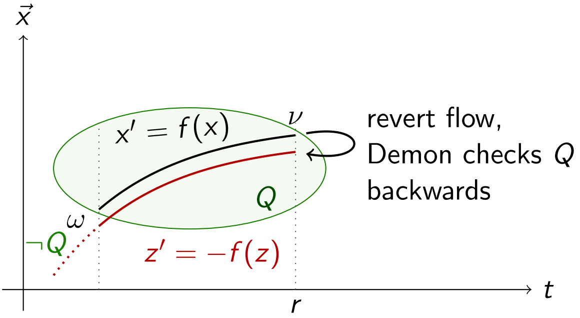


$x' = f(x) \,\&\, Q$ $\qquad$ $x' = f(x); (z := x; z' = -f(z))^d; ?(Q(z))$

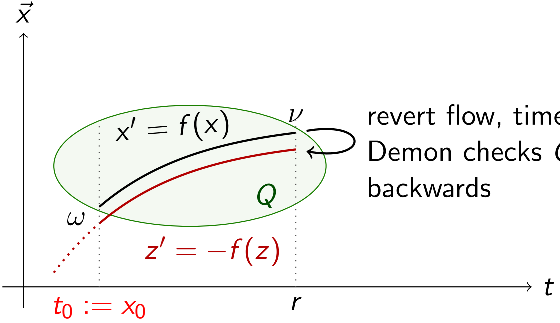


revert flow,
Demon checks Q
backwards

$x' = f(x) \,\&\, Q$ $\qquad$ $x' = f(x); (z := x; z' = -f(z))^d; ?(Q(z))$



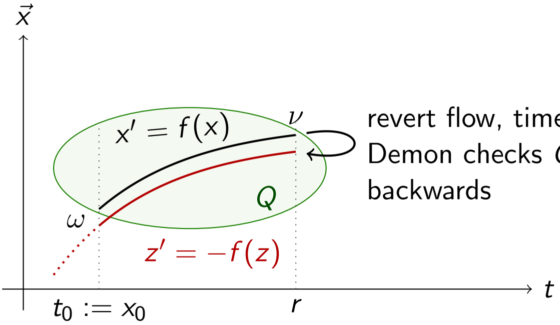
$$x' = f(x) \,\&\, Q \equiv t_0 := x_0; x' = f(x); (z := x; z' = -f(z))^d; ?(z_0 \geq t_0 \rightarrow Q(z))$$



revert flow, time x_0;
Demon checks Q
backwards

"There and Back Again" Game

$$x' = f(x) \,\&\, Q \equiv t_0 := x_0; x' = f(x); (z := x; z' = -f(z))^d; ?(z_0 \geq t_0 \rightarrow Q(z))$$



revert flow, time x_0;
Demon checks Q
backwards

Lemma

Evolution domains definable by games

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