

# 13: Differential Invariants & Proof Theory

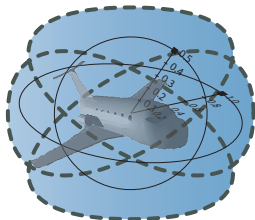
15-424: Foundations of Cyber-Physical Systems

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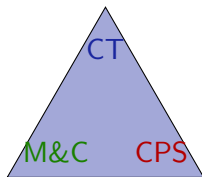
- 1 Learning Objectives
- 2 Recap: Proofs for Differential Equations
- 3 Differential Equation Proof Theory
  - Propositional Equivalences
  - Differential Invariants & Arithmetic
  - Differential Structure
  - Differential Invariant Equations
  - Equational Incompleteness
  - Strict Differential Invariant Inequalities
  - Differential Invariant Equations to Differential Invariant Inequalities
  - Differential Invariant Atoms
- 4 Differential Cut Power & Differential Ghost Power
- 5 Curves Playing with Norms and Degrees
- 6 Summary

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# Learning Objectives

## Differential Invariants & Proof Theory

limits of computation  
proof theory for differential equations  
provability of differential equations  
proofs about proofs  
relativity theory of proofs  
inform differential invariant search  
intuition for differential equation proofs



core argumentative principles  
tame analytic complexity

none

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# Differential Invariants for Differential Equations

## Differential Weakening

$$\frac{Q \vdash F}{P \vdash [x' = f(x) \& Q]F}$$

## Differential Invariant

$$\frac{Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x) \& Q]F}$$

## Differential Cut

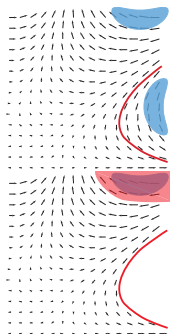
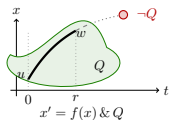
$$\frac{F \vdash [x' = f(x) \& Q]C \quad F \vdash [x' = f(x) \& Q \wedge C]F}{F \vdash [x' = f(x) \& Q]F}$$

$$\text{DW } [x' = f(x) \& Q]F \leftrightarrow [x' = f(x) \& Q](Q \rightarrow F)$$

$$\text{DI } [x' = f(x) \& Q]F \leftarrow (Q \rightarrow F \wedge [x' = f(x) \& Q])(F)'$$

$$\text{DC } ([x' = f(x) \& Q]F \leftrightarrow [x' = f(x) \& Q \wedge C]F) \leftarrow [x' = f(x) \& Q]C$$

$$\text{DE } [x' = f(x) \& Q]F \leftrightarrow [x' = f(x) \& Q][x' := f(x)]F$$



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- 3 **Differential Equation Proof Theory**
  - Propositional Equivalences
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  - Differential Structure
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# Relativity Theory of Proofs

## Differential Invariant

$$\frac{Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x) \& Q]F}$$

But generalizations are helpful to find the right  $F$  in the first place:

$$\text{cut,MR} \frac{A \vdash F \quad F \vdash [x' = f(x) \& Q]F \quad F \vdash B}{A \vdash [x' = f(x) \& Q]B}$$

## Compare Provability with Classes $\Omega$ of Differential Invariants

$\mathcal{DI}_\Omega$  : properties provable with differential invariants in  $\Omega \subseteq \{\geq, >, =, \wedge, \vee\}$

$\mathcal{A} \leq \mathcal{B}$  iff **all** properties provable with  $\mathcal{A}$  are also provable somehow with  $\mathcal{B}$

$\mathcal{A} \not\leq \mathcal{B}$  otherwise i.e. **some** property can be proved with  $\mathcal{A}$  but not with  $\mathcal{B}$

$\mathcal{A} \equiv \mathcal{B}$  iff  $\mathcal{A} \leq \mathcal{B}$  and  $\mathcal{B} \leq \mathcal{A}$  so **same** deductive power

$\mathcal{A} < \mathcal{B}$  iff  $\mathcal{A} \leq \mathcal{B}$  and  $\mathcal{B} \not\leq \mathcal{A}$  so  $\mathcal{A}$  has strictly **less** deductive power



# Relativity Theory of Proofs

## Differential Invariant

$$\frac{Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x) \& Q]F}$$

$\mathcal{DI}_{e=k} \equiv \mathcal{DI}_{e=0}$  by considering  $(e - k) = 0$

But generalizations are helpful to find the right  $F$  in the first place:

$$\text{cut,MR} \frac{A \vdash F \quad F \vdash [x' = f(x) \& Q]F \quad F \vdash B}{A \vdash [x' = f(x) \& Q]B}$$

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# Propositional Equivalences of Differential Invariants

Lemma (Differential invariants and propositional logic)

*If  $F \leftrightarrow G$  is a propositional tautology then*

*$F$  differential invariant of  $x' = f(x) \ \& \ Q$   
iff  $G$  differential invariant of  $x' = f(x) \ \& \ Q$*

Proof.



Can use any propositional normal form

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Proof.

$\text{MR, cut} \frac{}{F \vdash [x' = f(x) \& Q]F}$



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Proof.

$$\text{dl} \frac{}{G \vdash [x' = f(x) \& Q]G}$$
$$\text{MR, cut} \frac{}{F \vdash [x' = f(x) \& Q]F}$$



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Proof.

$$\begin{array}{l} \frac{}{Q \vdash [x' := f(x)](G)'} \\ \text{dl} \frac{}{G \vdash [x' = f(x) \ \& \ Q]G} \\ \text{MR, cut} \frac{}{F \vdash [x' = f(x) \ \& \ Q]F} \end{array}$$

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□

Can use any propositional normal form



# Arithmetic Equivalences of Differential Invariants

## Lemma (Differential invariants and propositional logic)

If  $F \leftrightarrow G$  is *real-arithmetic* equivalence then

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$$\text{dl } \frac{}{-5 \leq x \wedge x \leq 5 \vdash [x' = -x](-5 \leq x \wedge x \leq 5)}$$



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Proof.

$$\begin{array}{c} \text{[':=]} \\ \hline \vdash [x' := -x](0 \leq x' \wedge x' \leq 0) \\ \text{dl} \\ \hline -5 \leq x \wedge x \leq 5 \vdash [x' = -x](-5 \leq x \wedge x \leq 5) \end{array}$$

□

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## Proof.

not valid

---

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---

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arithmetic equivalence  $-5 \leq x \wedge x \leq 5 \leftrightarrow x^2 \leq 5^2$

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Proof.

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$\frac{}{\vdash 0 \leq -x \wedge -x \leq 0}$

$\frac{[\prime :=]}{\vdash [x' := -x](0 \leq x' \wedge x' \leq 0)}$   $\frac{[\prime :=]}{\vdash [x' := -x]2xx' \leq 0}$   
 $\text{dl } \frac{}{-5 \leq x \wedge x \leq 5 \vdash [x' = -x](-5 \leq x \wedge x \leq 5)}$   $\text{dl } \frac{}{x^2 \leq 5^2 \vdash [x' = -x]x^2 \leq 5^2}$

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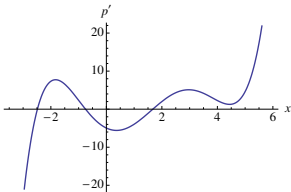
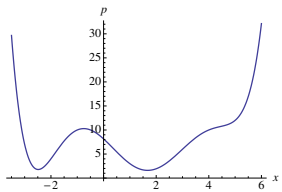
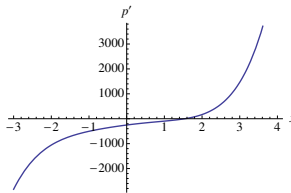
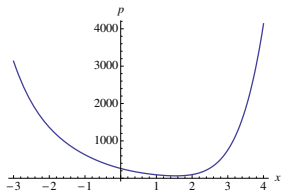
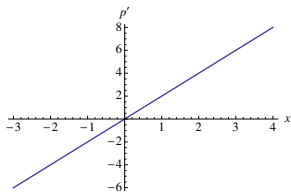
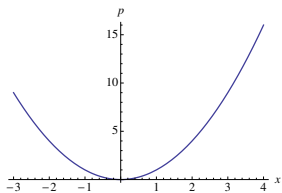
Proof.

not valid	*
$\frac{}{\vdash 0 \leq -x \wedge -x \leq 0}$	$\frac{}{\vdash -2x^2 \leq 0}$
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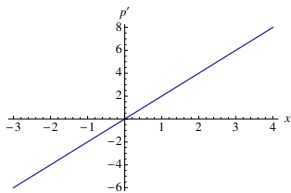
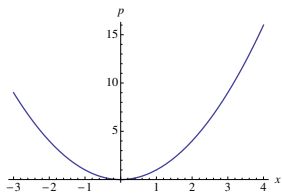
Despite arithmetic equivalence  $-5 \leq x \wedge x \leq 5 \leftrightarrow x^2 \leq 5^2$  □

Differential structure matters! Higher degree helps here

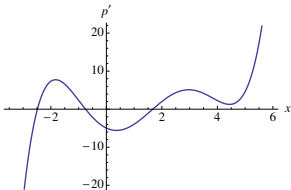
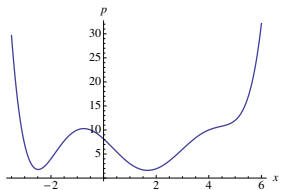
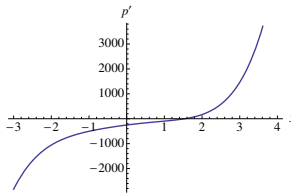
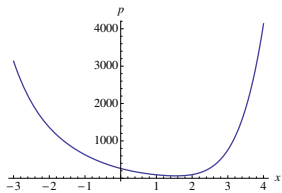
# Different Differential Structure for Equivalent Solutions $\geq 0$



# Different Differential Structure for Equivalent Solutions $\geq 0$



Can still normalize  
atomic formulas to  
 $e = 0, e \geq 0, e > 0$



# Differential Invariant Equations

Proposition (Equational deductive power [5, 1])

$$\mathcal{DI}_= \quad \mathcal{DI}_{=,\wedge,\vee}$$

Proof core.



Generalizations see [5, 1]



# Differential Invariant Equations

Proposition (Equational deductive power [5, 1])

*atomic equations are enough:*  $\mathcal{DI}_= \equiv \mathcal{DI}_{=,\wedge,\vee}$

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# Differential Invariant Equations

Proposition (Equational deductive power [5, 1])

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Proof core.

- $e_1 = e_2 \vee k_1 = k_2$

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Generalizations see [5, 1]



# Differential Invariant Equations

Proposition (Equational deductive power [5, 1])

*atomic equations are enough:*  $\mathcal{DI}_= \equiv \mathcal{DI}_{=,\wedge,\vee}$

Proof core.

- $e_1 = e_2 \vee k_1 = k_2 \leftrightarrow (e_1 - e_2)(k_1 - k_2) = 0$
  
  
  
  
  
  
  
  
  
  
- $e_1 = e_2 \wedge k_1 = k_2 \leftrightarrow (e_1 - e_2)^2 + (k_1 - k_2)^2 = 0$

Generalizations see [5, 1]





# Differential Invariant Equations

Proposition (Equational deductive power [5, 1])

*atomic equations are enough:*  $\mathcal{DI}_= \equiv \mathcal{DI}_{=,\wedge,\vee}$

Proof core.

- $e_1 = e_2 \vee k_1 = k_2 \leftrightarrow (e_1 - e_2)(k_1 - k_2) = 0$   
 $[x' := f(x)]((e_1)' = (e_2)' \wedge (k_1)' = (k_2)')$
  
- $e_1 = e_2 \wedge k_1 = k_2 \leftrightarrow (e_1 - e_2)^2 + (k_1 - k_2)^2 = 0$

Generalizations see [5, 1]



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*atomic equations are enough:*  $\mathcal{DI}_= \equiv \mathcal{DI}_{=,\wedge,\vee}$

Proof core.

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$$[x' := f(x)]((e_1)' = (e_2)' \wedge (k_1)' = (k_2)')$$

$$\text{So } [x' := f(x)]((e_1 - e_2)(k_1 - k_2))' = 0$$

$$\equiv [x' := f(x)](((e_1)' - (e_2)')(k_1 - k_2) + (e_1 - e_2)((k_1)' - (k_2)')) = 0$$

- $e_1 = e_2 \wedge k_1 = k_2 \leftrightarrow (e_1 - e_2)^2 + (k_1 - k_2)^2 = 0$

Generalizations see [5, 1]



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Proof core.

- $e_1 = e_2 \vee k_1 = k_2 \leftrightarrow (e_1 - e_2)(k_1 - k_2) = 0$   
     $[x':=f(x)]((e_1)' = (e_2)' \wedge (k_1)' = (k_2)')$   
    So  $[x':=f(x)]((e_1 - e_2)(k_1 - k_2))' = 0$   
     $\equiv [x':=f(x)](((e_1)' - (e_2)')(k_1 - k_2) + (e_1 - e_2)((k_1)' - (k_2)')) = 0$
- $e_1 = e_2 \wedge k_1 = k_2 \leftrightarrow (e_1 - e_2)^2 + (k_1 - k_2)^2 = 0$

Generalizations see [5, 1]



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*atomic equations are enough:*  $\mathcal{DI}_= \equiv \mathcal{DI}_{=,\wedge,\vee}$

Proof core.

- $e_1 = e_2 \vee k_1 = k_2 \leftrightarrow (e_1 - e_2)(k_1 - k_2) = 0$

$$[x':=f(x)]((e_1)' = (e_2)' \wedge (k_1)' = (k_2)')$$

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- $e_1 = e_2 \wedge k_1 = k_2 \leftrightarrow (e_1 - e_2)^2 + (k_1 - k_2)^2 = 0$

Generalizations see [5, 1]



# Differential Invariant Equations

Proposition (Equational deductive power [5, 1])

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Generalizations see [5, 1]





Proposition (Equational [1])

$$\mathcal{DI}_= \equiv \mathcal{DI}_{=,\wedge,\vee} \quad \mathcal{DI} \quad \mathcal{DI}_\geq \quad \mathcal{DI}_=$$

Proof core.



# Equational Incompleteness

Proposition (Equational incompleteness [1])

*Equations are not enough:  $\mathcal{DI}_= \equiv \mathcal{DI}_{=,\wedge,\vee} < \mathcal{DI}$  because  $\mathcal{DI}_\geq \not\equiv \mathcal{DI}_=$*

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$$\text{dl } \overline{x \geq 0 \vdash [x' = 5]x \geq 0}$$



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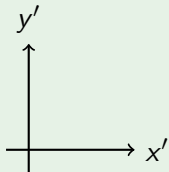
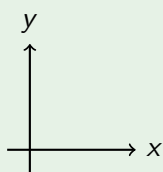


## Example (Sets Bijective or Not)

$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow 6$

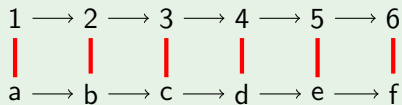
$a \longrightarrow b \longrightarrow c \longrightarrow d \longrightarrow e \longrightarrow f$

## Example (Vector Spaces Isomorphic or Not)

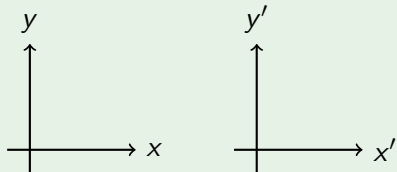




## Example (Sets Bijective or Not)



## Example (Vector Spaces Isomorphic or Not)



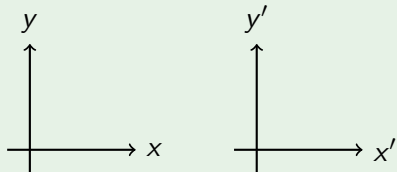
# Proving Differences in Set Theory & Linear Algebra

## Example (Sets Bijective or Not)

1 → 2 → 3 → 4 → 5 → 6  
|    |    |    |    |  
a → b → c → d → e → f

1 → 2 → 3 → 4 → 5 → 6  
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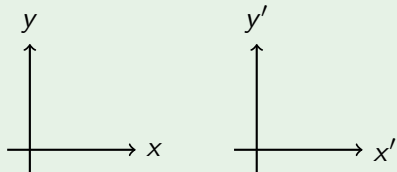
1 → 2 → 3 → 4 → 5 → 6  
| | | | |  
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cardinality  $|\{1, \dots, 6\}| = 6 \neq |\{a, b, c, d, e\}| = 5$

Need an indirect criterion especially if these sets are infinite

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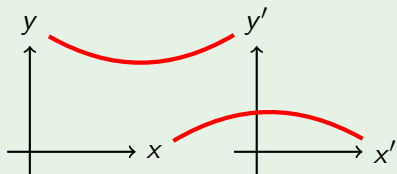
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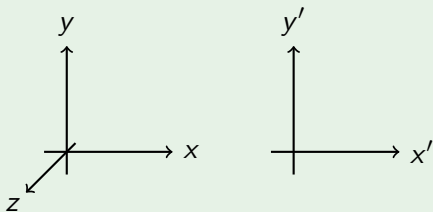
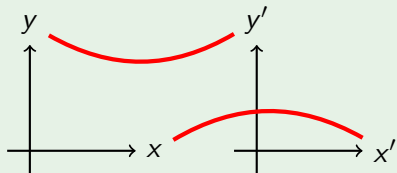
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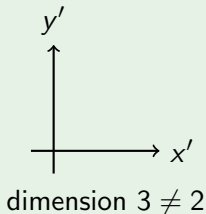
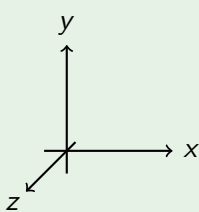
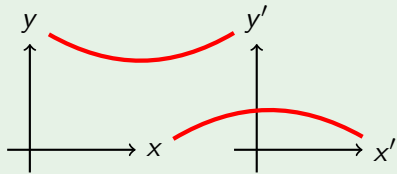
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# Equational Incompleteness

Proposition (Equational incompleteness [1])

*Equations are not enough:  $\mathcal{DI}_= \equiv \mathcal{DI}_{=,\wedge,\vee} < \mathcal{DI}$  because  $\mathcal{DI}_\geq \not\leq \mathcal{DI}_=$*

Proof core.

Provable with  $\mathcal{DI}_\geq$

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$$\begin{array}{c} \mathbb{R} \frac{*}{\vdash 5 \geq 0} \\ \text{[':=]} \frac{\vdash [x':=5]x' \geq 0}{x \geq 0 \vdash [x' = 5]x \geq 0} \\ \text{dl} \end{array}$$



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$$\text{dl} \quad \frac{p(x) = 0 \vdash [x' = 5]p(x) = 0}{x \geq 0 \vdash [x' = 5]x \geq 0}$$

cut,MR



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Likewise for indirect proofs [1].



# Strict Inequality

Proposition (Strict barrier )

$$\mathcal{DI}_{>} \quad \mathcal{DI} \quad \mathcal{DI}_{=} \quad \mathcal{DI}_{>}$$

Proof core.



# Strict Inequality Incompleteness

## Proposition (Strict barrier incompleteness)

*Strict inequalities are not enough:  $\mathcal{DI}_> < \mathcal{DI}$  because  $\mathcal{DI}_= \not\subseteq \mathcal{DI}_>$*

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$$\text{dl } \overline{v^2 + w^2 = c^2 \vdash [v' = w, w' = -v] v^2 + w^2 = c^2}$$



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Unprovable with  $\mathcal{DI}_>$

$$\frac{[':=] \quad \vdash [v':=w][w':=-v]2vv' + 2ww' = 0}{dl \quad \frac{v^2+w^2=c^2 \quad \vdash [v'=w, w'=-v]v^2+w^2=c^2}}{}}$$



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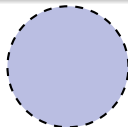
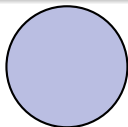
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Unprovable with  $\mathcal{DI}_>$   
 $e > 0$  is open set.

$v^2 + w^2 = c^2$  is a closed set

□

closed  $v^2 + w^2 \leq 1$   
with boundary



open  $v^2 + w^2 < 1$   
without boundary

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\*

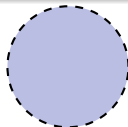
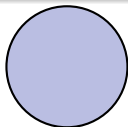
$$\begin{array}{l} \mathbb{R} \text{ -----} \\ \vdash 2vw + 2w(-v) = 0 \\ \text{[':=]} \text{ -----} \\ \vdash [v':=w][w':=-v]2vv' + 2ww' = 0 \\ \text{dl} \text{ -----} \\ v^2 + w^2 = c^2 \vdash [v' = w, w' = -v]v^2 + w^2 = c^2 \end{array}$$

Unprovable with  $\mathcal{DI}_>$   
 $e > 0$  is open set.  
Only *true* and *false*  
are both

$v^2 + w^2 = c^2$  is a closed set

□

closed  $v^2 + w^2 \leq 1$   
with boundary



open  $v^2 + w^2 < 1$   
without boundary

# Strict Inequality Incompleteness

## Proposition (Strict barrier incompleteness)

*Strict inequalities are not enough:  $\mathcal{DI}_> < \mathcal{DI}$  because  $\mathcal{DI}_= \not\subseteq \mathcal{DI}_>$*

Proof core.

Provable with  $\mathcal{DI}_=$   
\*

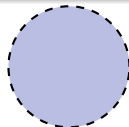
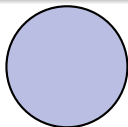
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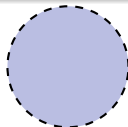
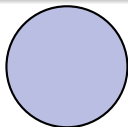
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$v^2+w^2=c^2$  is a closed set

Likewise for indirect proofs [1].



closed  $v^2+w^2 \leq 1$   
with boundary



open  $v^2+w^2 < 1$   
without boundary



# Differential Invariant Equations to Inequalities

Proposition (Equational )

$$\mathcal{DI}_{=,\wedge,\vee} \quad \mathcal{DI}_{\geq}$$

Proof core.



## Proposition (Equational definability)

*Equations are definable by weak inequalities:*  $\mathcal{DI}_{=,\wedge,\vee} \leq \mathcal{DI}_{\geq}$

Proof core.



# Differential Invariant Equations to Inequalities

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Proof core.

Provable with  $\mathcal{DI}_{=}$

Provable with  $\mathcal{DI}_{\geq}$



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Proof core.

Provable with  $\mathcal{DI}_{=}$

Provable with  $\mathcal{DI}_{\geq}$

$$\overline{e = 0 \vdash [x' = f(x) \ \& \ Q]e = 0}^{\text{dl}}$$



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Proof core.

Provable with  $\mathcal{DI}_{=}$

Provable with  $\mathcal{DI}_{\geq}$

$$\frac{Q \vdash [x' := f(x)](e)' = 0}{\text{dl } e = 0 \vdash [x' = f(x) \ \& \ Q]e = 0}$$



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Provable with  $\mathcal{DI}_{\geq}$

$$\frac{\begin{array}{c} \hline Q \vdash [x' := f(x)] - 2e(e)' \geq 0 \\ \hline \end{array}}{\text{dl } -e^2 \geq 0 \vdash [x' = f(x) \ \& \ Q](-e^2 \geq 0)}$$

□



# Differential Invariant Equations to Inequalities

## Proposition (Equational definability)

Equations are definable by weak inequalities:  $\mathcal{DI}_{=,\wedge,\vee} \leq \mathcal{DI}_{\geq}$

Proof core.

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Likewise for indirect proofs [1].



Local view of logic is crucial for this proof.

Degree increases

# Differential Invariant Atoms

Theorem (Atomic )

$$\mathcal{DI}_{\geq} = \mathcal{DI}_{\geq, \wedge, \vee} \text{ and } \mathcal{DI}_{>} = \mathcal{DI}_{>, \wedge, \vee}$$

Proof idea.



## Theorem (Atomic incompleteness)

*Atomic inequalities not enough:  $\mathcal{DI}_{\geq} < \mathcal{DI}_{\geq, \wedge, \vee}$  and  $\mathcal{DI}_{>} < \mathcal{DI}_{>, \wedge, \vee}$*

Proof idea.



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Proof idea.

Provable with  $\mathcal{DI}_{\geq, \wedge, \vee}$

Unprovable with  $\mathcal{DI}_{\geq}$



# Differential Invariant Atoms

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### Proof idea.

Provable with  $\mathcal{DI}_{\geq, \wedge, \vee}$

Unprovable with  $\mathcal{DI}_{\geq}$

\*

$$\mathbb{R} \frac{}{\vdash 5 \geq 0 \wedge y^2 \geq 0}$$

$$[':=] \frac{}{\vdash [x':=5][y':=y^2](x' \geq 0 \wedge y' \geq 0)}$$

$$dI \frac{}{x \geq 0 \wedge y \geq 0 \vdash [x' = 5, y' = y^2](x \geq 0 \wedge y \geq 0)}$$



# Differential Invariant Atoms

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Unprovable with  $\mathcal{DI}_{\geq}$   
 $p(x, y) \geq 0 \leftrightarrow x \geq 0 \wedge y \geq 0$   
impossible since this implies  
 $p(x, 0) \geq 0 \leftrightarrow x \geq 0$   
so  $p(x, 0)$  is 0



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### Proof idea.

Provable with  $\mathcal{DI}_{\geq, \wedge, \vee}$

\*

$\mathbb{R}$

$$\vdash 5 \geq 0 \wedge y^2 \geq 0$$

[':=]

$$\vdash [x' := 5][y' := y^2](x' \geq 0 \wedge y' \geq 0)$$

$$\text{dl } x \geq 0 \wedge y \geq 0 \vdash [x' = 5, y' = y^2](x \geq 0 \wedge y \geq 0)$$

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# Differential Invariant Atoms

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Substantial remaining parts of the proof shown elsewhere [1]. □

DC still possible here but more involved argument separates.

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# Deductive Power of Differential Cuts & Differential Ghosts

Theorem (Gentzen's Cut Elimination)

(1935)

$$\frac{A \vdash B \vee C \quad A \wedge C \vdash B}{A \vdash B}$$

*cut can be eliminated*

Theorem (No Differential Cut Elimination)

(LMCS 2012)

*Deductive power with differential cuts exceeds deductive power without.*

$$\mathcal{DI} + \mathbf{DC} > \mathcal{DI}$$

Theorem (Auxiliary Differential Variables)

(LMCS 2012)

*Deductive power with differential ghosts exceeds power without.*

$$\mathcal{DI} + \mathbf{DC} + \mathbf{DG} > \mathcal{DI} + \mathbf{DC}$$

$$\text{dl} \frac{x^3 \geq -1 \wedge y^5 \geq 0}{x' = (x - 2)^4 + y^5, y' = y^2} x^3 \geq -1$$

$$\frac{[':=] \quad \vdash [x':=(x-2)^4 + y^5][y':=y^2]2x^2x' \geq 0}{\text{dl } x^3 \geq -1 \wedge y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\frac{}{\vdash 2x^2((x-2)^4 + y^5) \geq 0}$$

$$\frac{[':=]}{\vdash [x':=(x-2)^4 + y^5][y':=y^2]2x^2x' \geq 0}$$

$$\text{dl } x^3 \geq -1 \wedge y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1$$

not valid

---


$$\vdash 2x^2((x-2)^4 + y^5) \geq 0$$


---

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---

$$dI \quad x^3 \geq -1 \wedge y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1$$

Have to know something about  $y^5$



---

$$\text{DC } x^3 \geq -1 \wedge y^5 \geq 0 \vdash [x' = (x - 2)^4 + y^5, y' = y^2] x^3 \geq -1$$

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---


$$[\text{':=}] \quad \vdash [x' := (x - 2)^4 + y^5][y' := y^2] 5y^4 y' \geq 0$$

---


$$\text{dl} \quad y^5 \geq 0 \vdash [x' = (x - 2)^4 + y^5, y' = y^2] y^5 \geq 0$$

---


$$\text{DC } x^3 \geq -1 \wedge y^5 \geq 0 \vdash [x' = (x - 2)^4 + y^5, y' = y^2] x^3 \geq -1$$

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\*

---


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$$\text{dl } y^5 \geq 0 \vdash [x' = (x - 2)^4 + y^5, y' = y^2] y^5 \geq 0$$

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$$\text{DC} \ x^3 \geq -1 \wedge y^5 \geq 0 \vdash [x' = (x - 2)^4 + y^5, y' = y^2] x^3 \geq -1$$

$$*$$

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$$\begin{array}{c}
 \hline
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 \\
 * \\
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 \end{array}$$

$$\mathbb{R} \quad \frac{}{y^5 \geq 0 \vdash 2x^2((x-2)^4 + y^5) \geq 0}$$

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\*

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# Arithmetic Equivalences of Differential Invariants

## Lemma (Differential invariants and propositional logic)

If  $F \leftrightarrow G$  is *real-arithmetic* equivalence then

$F$  differential invariant of  $x' = f(x) \ \& \ Q$   
iff  $G$  differential invariant of  $x' = f(x) \ \& \ Q$

Proof.

	not valid		*
	$\frac{}{\vdash 0 \leq -x \wedge -x \leq 0}$	$\mathbb{R}$	$\frac{}{\vdash -2x^2 \leq 0}$
$[\prime :=]$	$\frac{}{\vdash [x' := -x](0 \leq x' \wedge x' \leq 0)}$	$[\prime :=]$	$\frac{}{\vdash [x' := -x]2xx' \leq 0}$
dl	$\frac{}{-5 \leq x \wedge x \leq 5 \vdash [x' = -x](-5 \leq x \wedge x \leq 5)}$	dl	$\frac{}{x^2 \leq 5^2 \vdash [x' = -x]x^2 \leq 5^2}$

Despite arithmetic equivalence  $-5 \leq x \wedge x \leq 5 \leftrightarrow x^2 \leq 5^2$  □

Differential structure matters! Higher degree helps here

# Curves Playing with Norms and Degrees

$$\text{DC} \frac{}{A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1] \|(x, y)\|_\infty \leq t}$$

$$A \stackrel{\text{def}}{\equiv} v^2 + w^2 \leq 1 \wedge x = y = t = 0$$

$$\|(x, y)\|_\infty \leq t \stackrel{\text{def}}{\equiv} -t \leq x \leq t \wedge -t \leq y \leq t$$

Supremum norm

$$\|(x, y)\|_2 \leq t \stackrel{\text{def}}{\equiv} x^2 + y^2 \leq t^2$$

Euclidean norm

# Curves Playing with Norms and Degrees

$$\frac{\text{dl} \quad \triangleleft \quad A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1 \ \& \ v^2 + w^2 \leq 1] \|(x, y)\|_\infty \leq t}{\text{DC} \quad A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1] \|(x, y)\|_\infty \leq t}$$

$$A \stackrel{\text{def}}{\equiv} v^2 + w^2 \leq 1 \wedge x = y = t = 0$$

$$\|(x, y)\|_\infty \leq t \stackrel{\text{def}}{\equiv} -t \leq x \leq t \wedge -t \leq y \leq t \quad \text{Supremum norm}$$

$$\|(x, y)\|_2 \leq t \stackrel{\text{def}}{\equiv} x^2 + y^2 \leq t^2 \quad \text{Euclidean norm}$$

# Curves Playing with Norms and Degrees

$$\begin{array}{l}
 \text{[':=]} \frac{\sqrt{v^2+w^2} \leq 1 \vdash [x':=v][y':=w][v':=\omega w][w':=-\omega v][t':=1](-t' \leq x' \leq t' \wedge -t' \leq y' \leq t')}{\text{dl}} \\
 \text{dl} \frac{\text{A} \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1 \& v^2 + w^2 \leq 1] \|(x, y)\|_\infty \leq t}{\text{DC}} \\
 \text{DC} \frac{\text{A} \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1] \|(x, y)\|_\infty \leq t}{}
 \end{array}$$

$$A \stackrel{\text{def}}{=} v^2 + w^2 \leq 1 \wedge x = y = t = 0$$

$$\|(x, y)\|_\infty \leq t \stackrel{\text{def}}{=} -t \leq x \leq t \wedge -t \leq y \leq t$$

Supremum norm

$$\|(x, y)\|_2 \leq t \stackrel{\text{def}}{=} x^2 + y^2 \leq t^2$$

Euclidean norm

# Curves Playing with Norms and Degrees

$$\begin{array}{c}
 \mathbb{R} \\
 \hline
 \mathbb{R} \quad v^2 + w^2 \leq 1 \vdash -1 \leq v \leq 1 \wedge -1 \leq w \leq 1 \\
 \hline
 [':=] \quad v^2 + w^2 \leq 1 \vdash [x':=v][y':=w][v':=\omega w][w':=-\omega v][t':=1](-t' \leq x' \leq t' \wedge -t' \leq y' \leq t') \\
 \hline
 \text{dI} \quad \triangleleft \quad A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1 \& v^2 + w^2 \leq 1] \|(x, y)\|_\infty \leq t \\
 \hline
 \text{DC} \quad A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1] \|(x, y)\|_\infty \leq t
 \end{array}$$

$$A \stackrel{\text{def}}{=} v^2 + w^2 \leq 1 \wedge x = y = t = 0$$

$$\|(x, y)\|_\infty \leq t \stackrel{\text{def}}{=} -t \leq x \leq t \wedge -t \leq y \leq t$$

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# Curves Playing with Norms and Degrees

$$\begin{array}{c}
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 \mathbb{R} \frac{v^2 + w^2 \leq 1 \vdash -1 \leq v \leq 1 \wedge -1 \leq w \leq 1}{v^2 + w^2 \leq 1 \vdash [x' := v][y' := w][v' := \omega w][w' := -\omega v][t' := 1](-t' \leq x' \leq t' \wedge -t' \leq y' \leq t')} \\
 [':=] \frac{v^2 + w^2 \leq 1 \vdash -1 \leq v \leq 1 \wedge -1 \leq w \leq 1}{v^2 + w^2 \leq 1 \vdash [x' := v][y' := w][v' := \omega w][w' := -\omega v][t' := 1](-t' \leq x' \leq t' \wedge -t' \leq y' \leq t')} \\
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$$\|(x, y)\|_\infty \leq t \stackrel{\text{def}}{=} -t \leq x \leq t \wedge -t \leq y \leq t$$

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$$\begin{array}{c}
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 \mathbb{R} \frac{v^2 + w^2 \leq 1 \vdash -1 \leq v \leq 1 \wedge -1 \leq w \leq 1}{v^2 + w^2 \leq 1 \vdash [x' := v][y' := w][v' := \omega w][w' := -\omega v][t' := 1](-t' \leq x' \leq t' \wedge -t' \leq y' \leq t')} \\
 [':=] \frac{v^2 + w^2 \leq 1 \vdash [x' := v][y' := w][v' := \omega w][w' := -\omega v][t' := 1](-t' \leq x' \leq t' \wedge -t' \leq y' \leq t')}{\triangleleft A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1 \& v^2 + w^2 \leq 1] \|(x, y)\|_\infty \leq t} \\
 \text{dl} \\
 \text{DC} \frac{A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1] \|(x, y)\|_\infty \leq t}{A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1] \|(x, y)\|_2 \leq t}
 \end{array}$$

$$\text{DC} \frac{A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1] \|(x, y)\|_2 \leq t}{A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1] \|(x, y)\|_2 \leq t}$$

$$A \stackrel{\text{def}}{=} v^2 + w^2 \leq 1 \wedge x = y = t = 0$$

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# Curves Playing with Norms and Degrees

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 \mathbb{R} \frac{v^2 + w^2 \leq 1 \vdash -1 \leq v \leq 1 \wedge -1 \leq w \leq 1}{v^2 + w^2 \leq 1 \vdash [x' := v][y' := w][v' := \omega w][w' := -\omega v][t' := 1](-t' \leq x' \leq t' \wedge -t' \leq y' \leq t')} \\
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 \text{DC} \frac{}{A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1] \|(x, y)\|_\infty \leq t}
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$$A \stackrel{\text{def}}{=} v^2 + w^2 \leq 1 \wedge x = y = t = 0$$

$$\|(x, y)\|_\infty \leq t \stackrel{\text{def}}{=} -t \leq x \leq t \wedge -t \leq y \leq t \quad \text{Supremum norm}$$

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# Curves Playing with Norms and Degrees

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$$\begin{array}{l} \mathbb{R} \frac{v^2 + w^2 \leq 1 \vdash -1 \leq v \leq 1 \wedge -1 \leq w \leq 1}{v^2 + w^2 \leq 1 \vdash [x' := v][y' := w][v' := \omega w][w' := -\omega v][t' := 1](-t' \leq x' \leq t' \wedge -t' \leq y' \leq t')} \\ \text{dl} \frac{\Delta}{A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1] \& v^2 + w^2 \leq 1} \|(x, y)\|_\infty \leq t \\ \text{DC} \frac{}{A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1] \|(x, y)\|_\infty \leq t} \end{array}$$

$$\begin{array}{l} \text{dl} \frac{v^2 + w^2 \leq 1 \vdash [x' := v][y' := w][v' := \omega w][w' := -\omega v][t' := 1](2xx' + 2yy' \leq 2tt')}{\Delta} \\ \text{DC} \frac{}{A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1] \|(x, y)\|_2 \leq t} \end{array}$$

$$A \stackrel{\text{def}}{=} v^2 + w^2 \leq 1 \wedge x = y = t = 0$$

$$\|(x, y)\|_\infty \leq t \stackrel{\text{def}}{=} -t \leq x \leq t \wedge -t \leq y \leq t \quad \text{Supremum norm}$$

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 \text{DC} \frac{}{A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1] \|(x, y)\|_\infty \leq t}
 \end{array}$$

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 \frac{v^2 + w^2 \leq 1 \vdash 2xv + 2yw \leq 2t}{v^2 + w^2 \leq 1 \vdash [x' := v][y' := w][v' := \omega w][w' := -\omega v][t' := 1](2xx' + 2yy' \leq 2tt')} \\
 \text{dl} \frac{\triangleleft}{A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1 \& v^2 + w^2 \leq 1] \|(x, y)\|_2 \leq t} \\
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# Curves Playing with Norms and Degrees

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$$\begin{array}{c} \mathbb{R} \\ \hline v^2 + w^2 \leq 1 \vdash -1 \leq v \leq 1 \wedge -1 \leq w \leq 1 \\ \hline [':=] v^2 + w^2 \leq 1 \vdash [x':=v][y':=w][v':=\omega w][w':=-\omega v][t':=1](-t' \leq x' \leq t' \wedge -t' \leq y' \leq t') \\ \hline \text{dl} \quad \triangleleft \quad A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1 \& v^2 + w^2 \leq 1] \|(x, y)\|_\infty \leq t \\ \hline \text{DC} \quad A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1] \|(x, y)\|_\infty \leq t \end{array}$$

not valid

$$\begin{array}{c} v^2 + w^2 \leq 1 \vdash 2xv + 2yw \leq 2t \\ \hline [':=] v^2 + w^2 \leq 1 \vdash [x':=v][y':=w][v':=\omega w][w':=-\omega v][t':=1](2xx' + 2yy' \leq 2tt') \\ \hline \text{dl} \quad \triangleleft \quad A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1 \& v^2 + w^2 \leq 1] \|(x, y)\|_2 \leq t \\ \hline \text{DC} \quad A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1] \|(x, y)\|_2 \leq t \end{array}$$

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Lower degree helps here

# Interreducing Norms in Dimension $n$

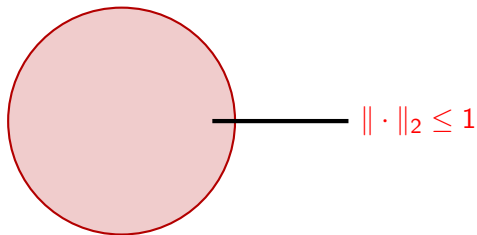
$$\forall x \forall y (\|(x, y)\|_\infty \leq \|(x, y)\|_2 \leq \sqrt{n} \|(x, y)\|_\infty)$$

$$\forall x \forall y \left( \frac{1}{\sqrt{n}} \|(x, y)\|_2 \leq \|(x, y)\|_\infty \leq \|(x, y)\|_2 \right)$$

# Interreducing Norms in Dimension $n$

$$\forall x \forall y (\|(x, y)\|_\infty \leq \|(x, y)\|_2 \leq \sqrt{n} \|(x, y)\|_\infty)$$

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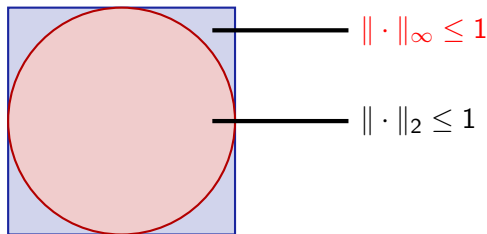




# Interreducing Norms in Dimension $n$

$$\forall x \forall y (\|(x, y)\|_\infty \leq \|(x, y)\|_2 \leq \sqrt{n} \|(x, y)\|_\infty)$$

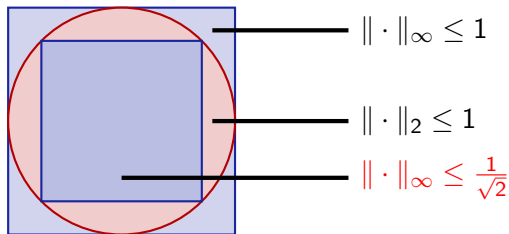
$$\forall x \forall y (\frac{1}{\sqrt{n}} \|(x, y)\|_2 \leq \|(x, y)\|_\infty \leq \|(x, y)\|_2)$$



# Interreducing Norms in Dimension $n$

$$\forall x \forall y (\|(x, y)\|_\infty \leq \|(x, y)\|_2 \leq \sqrt{n} \|(x, y)\|_\infty)$$

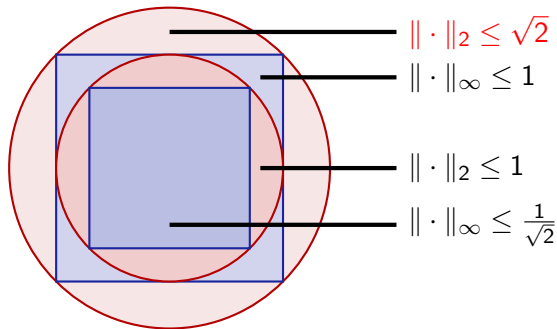
$$\forall x \forall y (\frac{1}{\sqrt{n}} \|(x, y)\|_2 \leq \|(x, y)\|_\infty \leq \|(x, y)\|_2)$$



# Interreducing Norms in Dimension $n$

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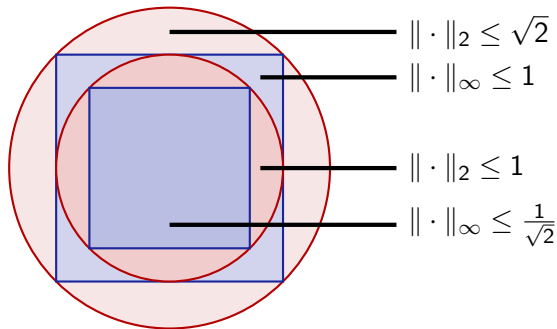
$$\forall x \forall y (\frac{1}{\sqrt{n}} \|(x, y)\|_2 \leq \|(x, y)\|_\infty \leq \|(x, y)\|_2)$$



# Interreducing Norms in Dimension $n$

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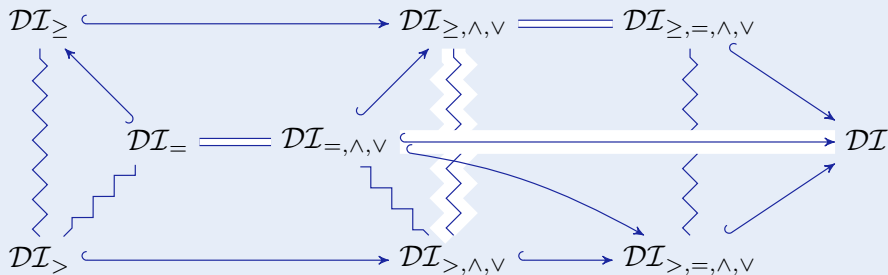


Benefit from norm relations but be mindful of approximation error factors

- 1 Learning Objectives
- 2 Recap: Proofs for Differential Equations
- 3 Differential Equation Proof Theory
  - Propositional Equivalences
  - Differential Invariants & Arithmetic
  - Differential Structure
  - Differential Invariant Equations
  - Equational Incompleteness
  - Strict Differential Invariant Inequalities
  - Differential Invariant Equations to Differential Invariant Inequalities
  - Differential Invariant Atoms
- 4 Differential Cut Power & Differential Ghost Power
- 5 Curves Playing with Norms and Degrees
- 6 Summary

# Differential Invariance Chart

## Theorem (Differential Invariance Chart)



- Rich theory and structure behind differential invariants
- Scrutinize what property can be proved with what invariant
- Use provability sanity checks like open/closed/univariate
- Real differential semialgebraic geometry
- Exploit differential cuts to obtain more knowledge



André Platzer.

The structure of differential invariants and differential cut elimination.

*Log. Meth. Comput. Sci.*, 8(4):1–38, 2012.

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*J. Log. Comput.*, 20(1):309–352, 2010.

[doi:10.1093/logcom/exn070](https://doi.org/10.1093/logcom/exn070).