09: Reactions & Delays

15-424: Foundations of Cyber-Physical Systems

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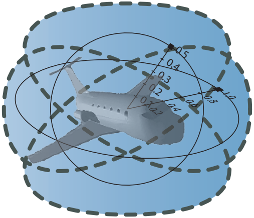

Outline

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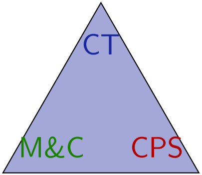
1. Learning Objectives
2. Delays in Control
   - Back to the Drawing Desk: Quantum the Ping Pong Ball
   - Quantum the Time-triggered Ping Pong Ball
   - The Impact of Delays on Events
   - Cartesian Demon
   - Predictive Control
   - Design-by-Invariant
   - Controlling the Control Points
   - Short Invariants
3. Proof
4. Summary
   - Zeno's Quantum Turtles
   - A Note on Assignments

Learning Objectives

using loop invariants
design time-triggered control
design-by-invariant



modeling CPS
designing controls
time-triggered control
reaction delays
discrete sensing

semantics of time-triggered control
operational effect
finding control constraints
model-predictive control

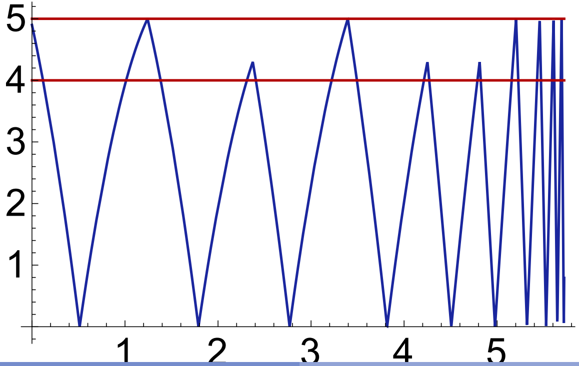
Outline

Quantum's Ping Pong Proof Invariants

Proposition (Quantum can play ping pong safely)

$$0 \le x \land x \le 5 \land v \le 0 \land g > 0 \land 1 \ge c \ge 0 \land f \ge 0 \rightarrow$$
$$\big[\big(((x' = v, v' = -g \,\&\, x \ge 0 \land x \le 5) \cup (x' = v, v' = -g \,\&\, x \ge 5));$$
$$\texttt{if}(x=0)\, v := -cv \;\texttt{else if}(4 \le x \le 5 \land v \ge 0)\, v := -fv\big)^*\big](0 \le x \le 5)$$

Proof `@invariant`$(0 \le x \le 5 \land (x = 5 \rightarrow v \le 0))$



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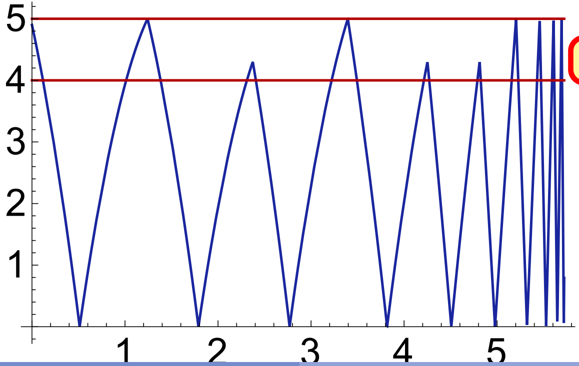

Just can't implement . . .

Physical vs. Controller Events

Physical vs. Controller Events

1. Justifiable: Physical events (on ground $x = 0$)
2. Justifiable: Physical evolution domains (above ground $x \geq 0$)
3. Questionable: Controller evolution domain ($x \leq 5$)
4. Unlike physics, controllers won't run *all* the time. Just often.

Back to the Drawing Desk: Quantum the Ping Pong Ball

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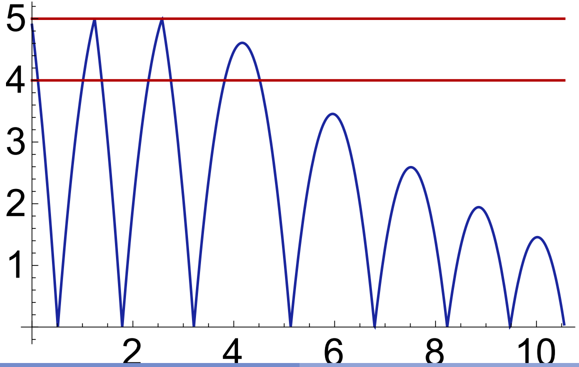
Proof? Ask René Descartes



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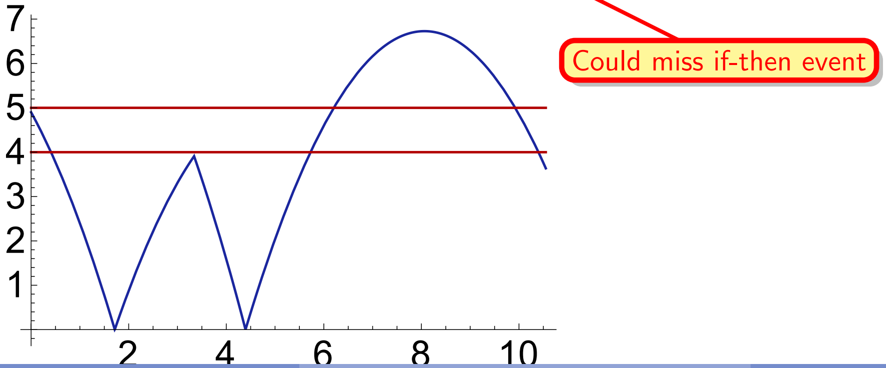

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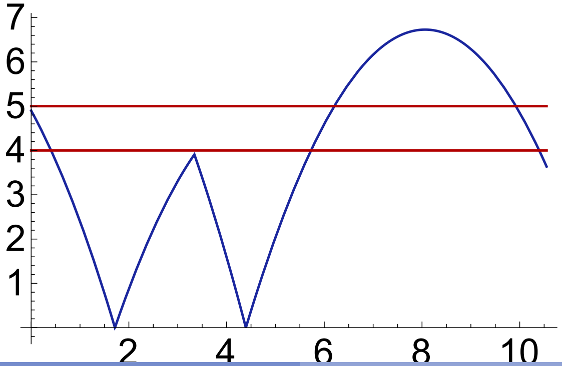

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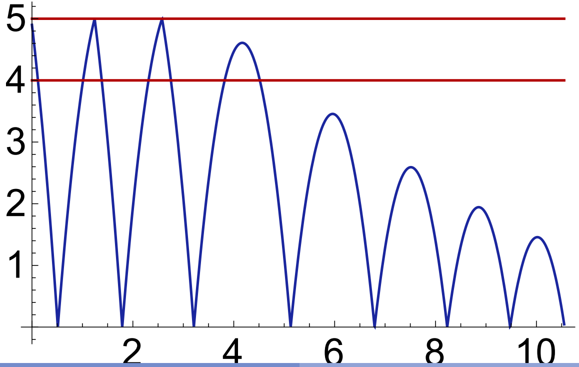

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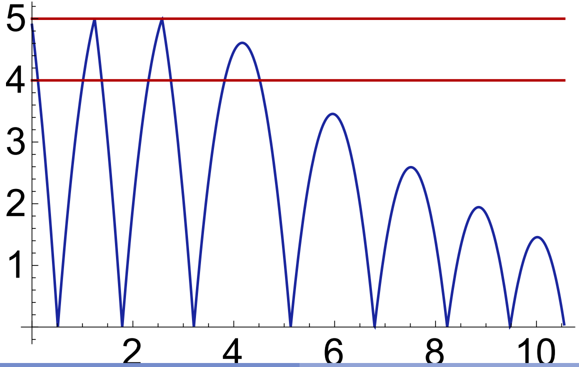

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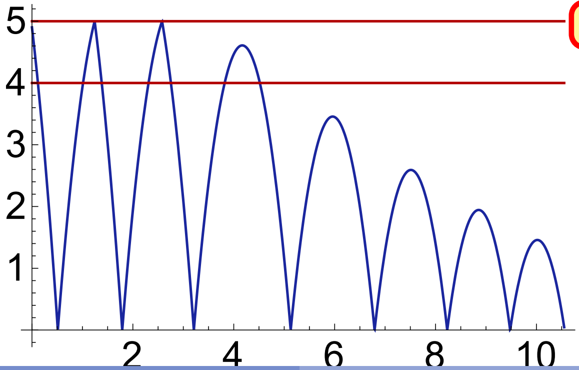

Control before physics

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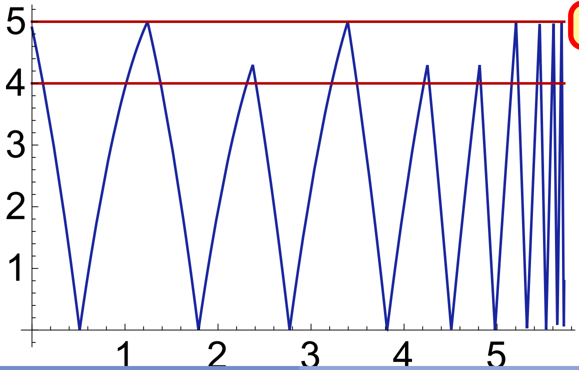

Could act early or late

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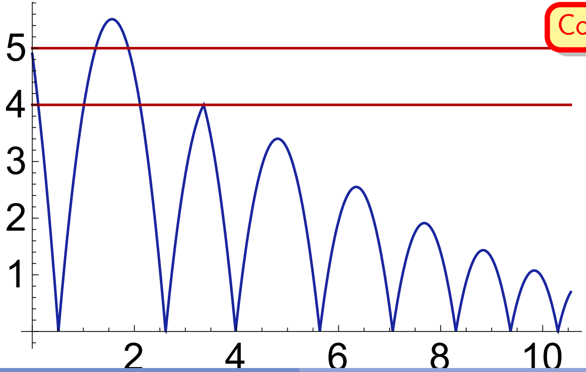
Could miss event off cycle



Delays May Miss Events

Delays vs. Events

1. Periodically/frequently monitoring for an event with a polling frequency / reaction time
2. Delays may make the controller miss events.
3. Discrepancy event-driven idea vs. real time-triggered implementation.
4. Slow controllers monitoring small regions of a fast moving system.
5. Issues indicate poor event abstraction
6. Controller need to be aware of its own delay

Cartesian Doubt: René Descartes's Cartesian Demon 1641

Outwit the Cartesian Demon

Skeptical about the truth of all beliefs until justification has been found.

Quantum the Time-triggered Ping Pong Ball

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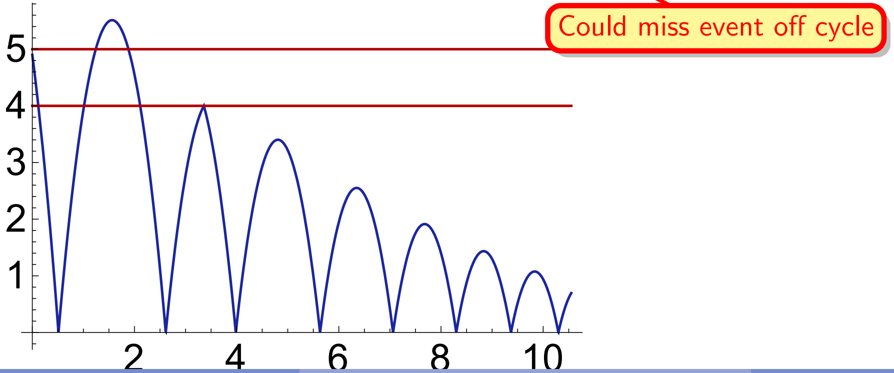

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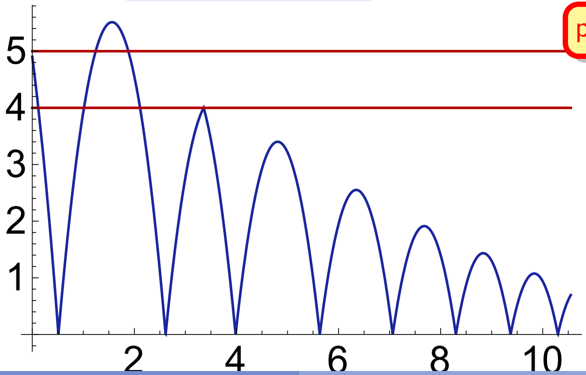
predict: $x + v - \frac{g}{2} > 5$



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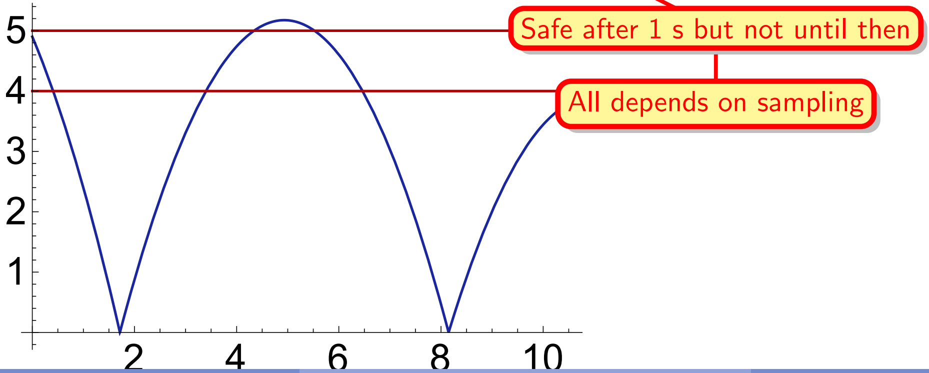

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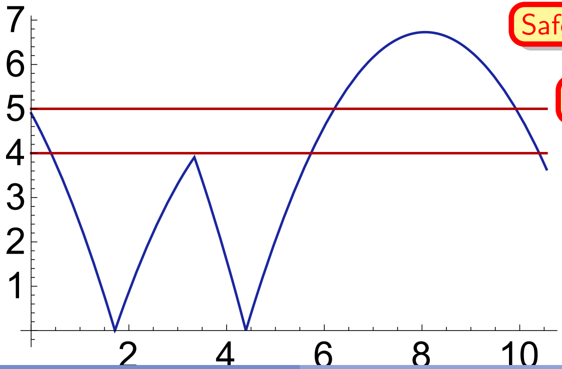


Safe after 1 s but not until then

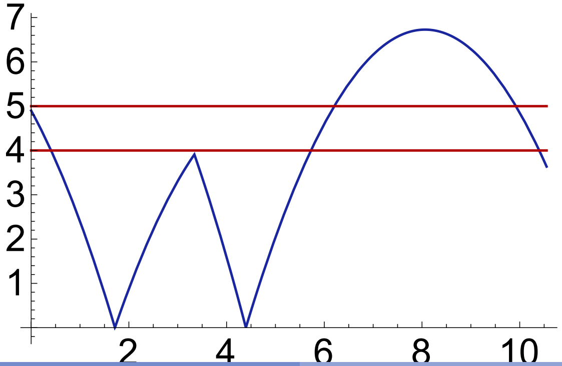
All depends on sampling

Quantum Discovers Design-by-Invariant

Design-by-Invariant

$2gx = 2gH - v^2 \wedge x \geq 0 \; \wedge c = 1 \wedge g > 0$

bouncing ball invariant

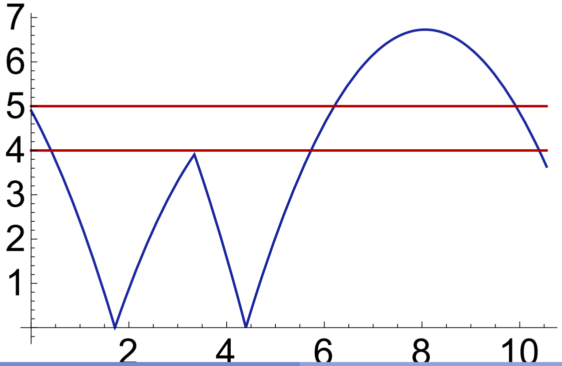


Quantum Discovers Design-by-Invariant

Design-by-Invariant

$2gx = 2gH - v^2 \land x \geq 0 \land c = 1 \land g = 1$

simplify arithmetic



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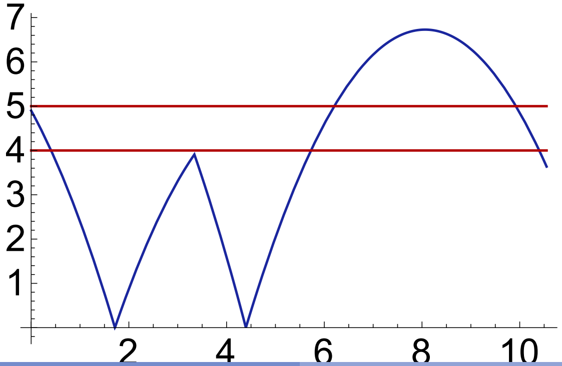

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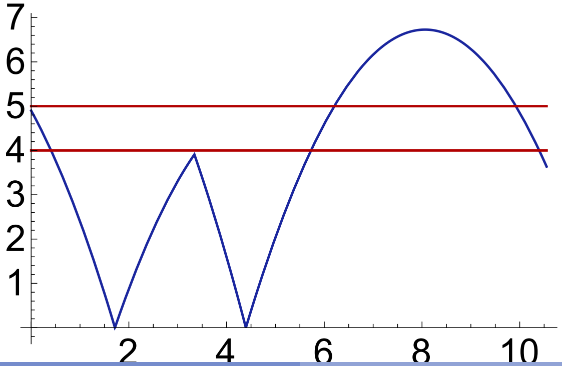
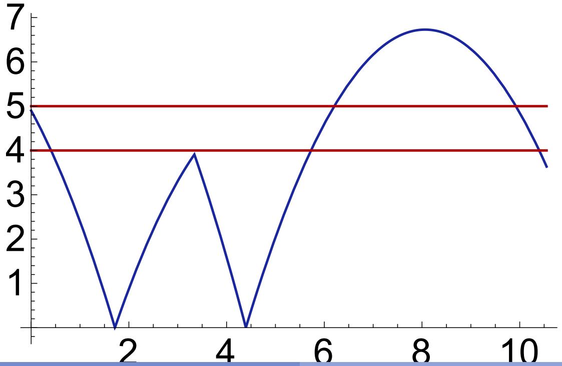


Quantum Discovers Design-by-Invariant

Design-by-Invariant

$2x = 2 \cdot 5 - v^2 \land x \geq 0$

critical height

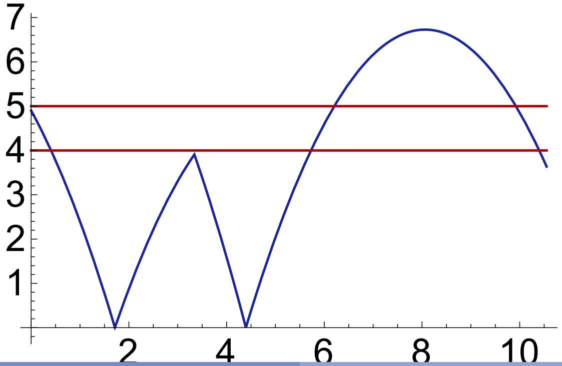


Quantum Discovers Design-by-Invariant

Design-by-Invariant

$2x > 2 \cdot 5 - v^2 \land x \geq 0$

potential exceeds safe height

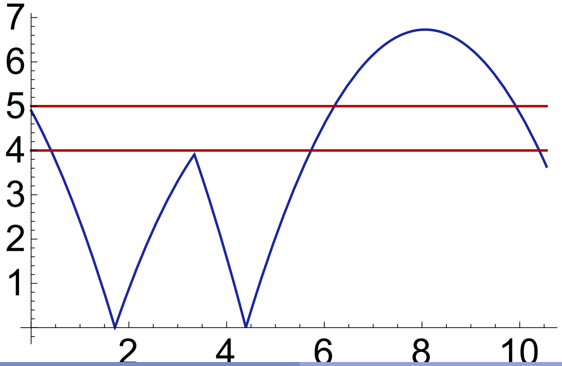


Quantum Discovers Design-by-Invariant

Design-by-Invariant

$2x > 2 \cdot 5 - v^2 \land x \geq 0$

use invariant for control



Quantum the Time-triggered Ping Pong Ball

Conjecture (Quantum can play ping pong safely)

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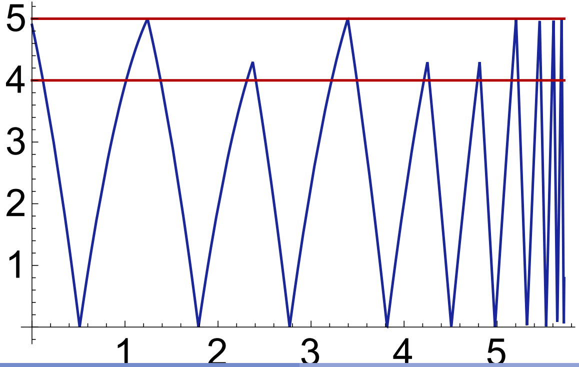

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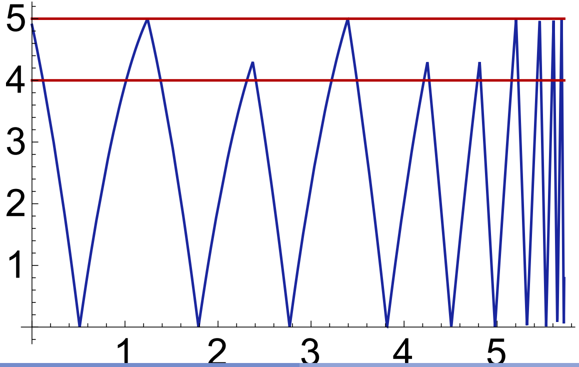

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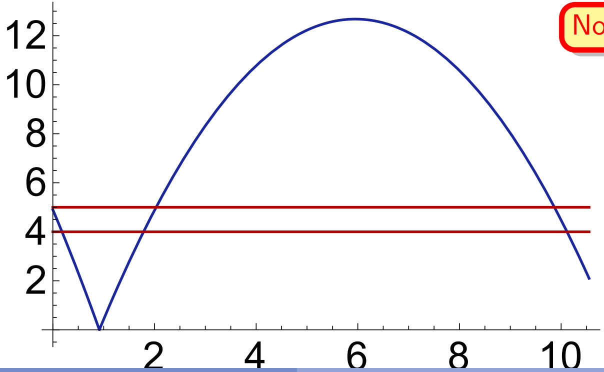
No control near ground



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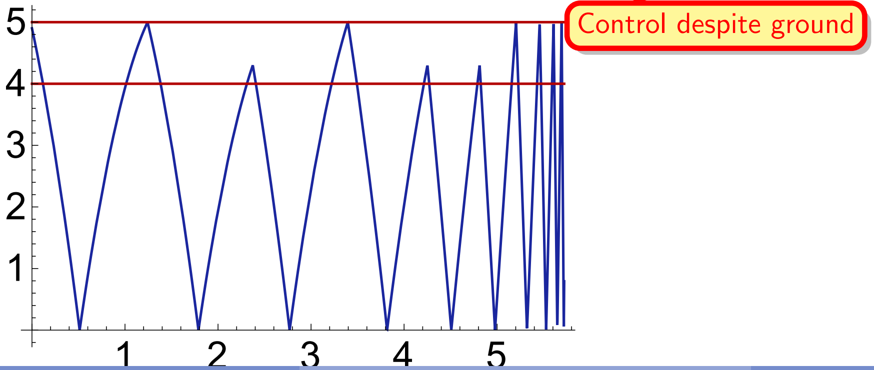

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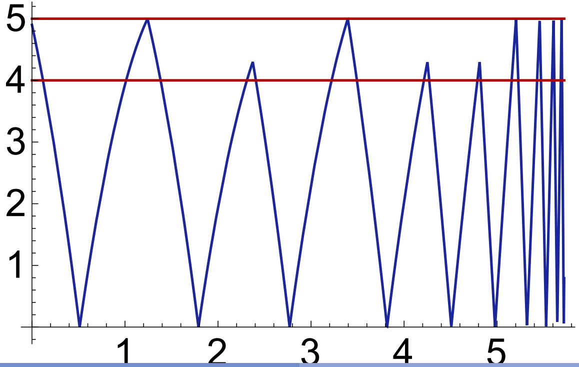

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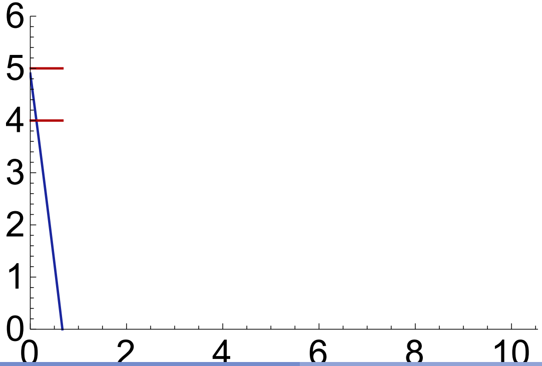

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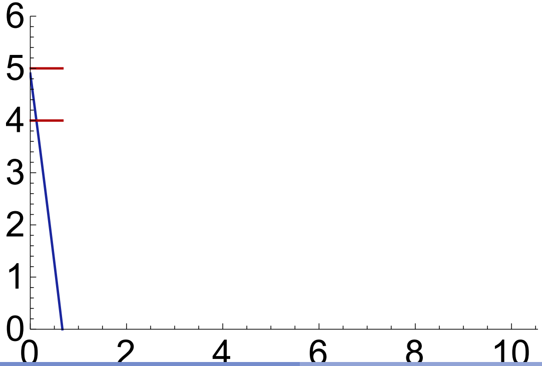
Invariants are **invariants**!

True ever $\rightsquigarrow$ true initially



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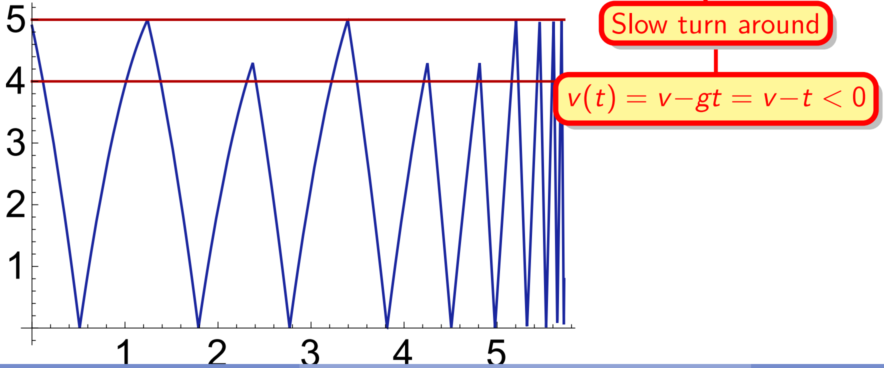
$\big[\big(\mathtt{if}(x{=}0)\ v := -cv;\ \mathtt{if}((x{>}5\tfrac{1}{2}{-}v \lor 2x{>}2{\cdot}5{-}v^2 \land v{<}1) \land v{\ge}0)\ v := -fv;$
$t := 0; \{x' = v, v' = -g, t' = 1\ \&\ x \ge 0 \land t{\le}1\}\big)^*\big](0{\le}x{\le}5)$

Proof? Ask René Descartes



Quantum the Time-triggered Ping Pong Ball

Conjecture (Quantum can play ping pong safely)

$$0 \le x \land x \le 5 \land v \le 0 \land g = 1 \land 1 = c \land f = 1 \rightarrow$$
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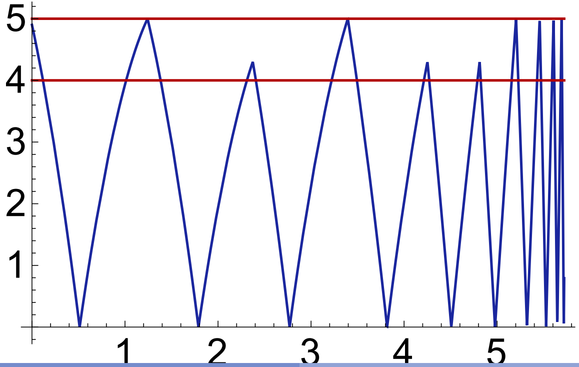
$$t := 0; \{x' = v, v' = -g, t' = 1 \,\&\, x \ge 0 \land t{\le}1\}\big)^*\Big](0{\le}x{\le}5)$$

Proof? Ask René Descartes who says yes



Outline

Quantum's Time-triggered Ping Pong Proof Invariants

Proposition (Quantum can play ping pong safely in real-time)

$$0 \leq x \land x \leq 5 \land v \leq 0 \land g{=}1{>}0 \land 1{=}c{\geq}0 \land 1{=}f{\geq}0 \rightarrow$$
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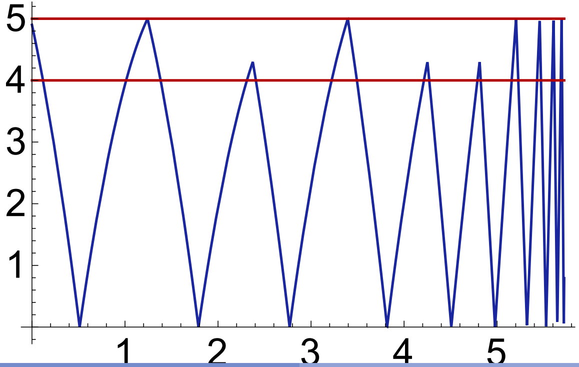

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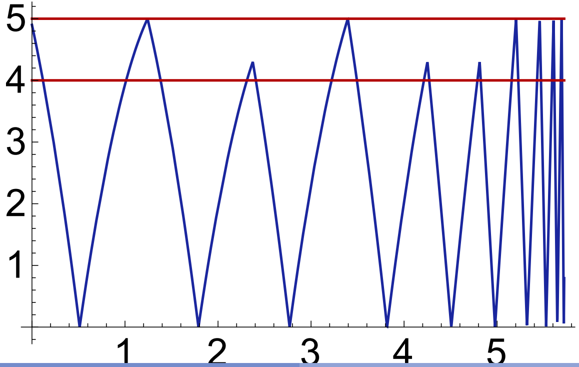
Proof $\texttt{@invariant}(2x = 2H - v^2 \land x \ge 0 \land x{\le}5)$



Quantum's Time-triggered Ping Pong Proof Invariants

Proposition (Quantum can play ping pong safely in real-time)

$$2x = 2H - v^2 \land 0 \le x \land x \le 5 \land v \le 0 \land g{=}1{>}0 \land 1{=}c{\ge}0 \land 1{=}f{\ge}0 \rightarrow$$
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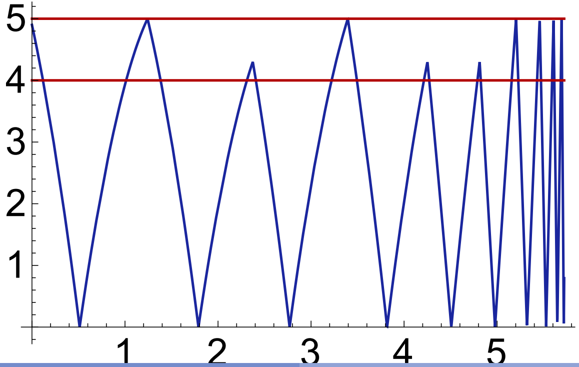
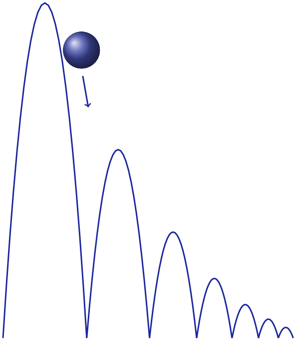


Outline

Summary: Time-triggered Control

1. Common paradigm for designing real controllers
2. Periodical or pseudo-periodical control (jitter)
3. Expects delays, expects inertia
4. Implementation: discrete-time sensing
5. Predict events, not just `if`$(\mathit{eventnow}(x))$ ...
6. Safe controllers know their own reaction delays
7. Burden of event detection brought to attention of CPS programmer
8. Time-triggered controls are implementable and more robust, but make design and verification more challenging!
9. Use knowledge gained from verified event-triggered model as a basis for designing a time-triggered controller

How Quantum Met Achilles and His Tortoise



Example (Quantum the Bouncing Ball)

$$\big(x' = v, v' = -g \,\&\, x \geq 0;$$
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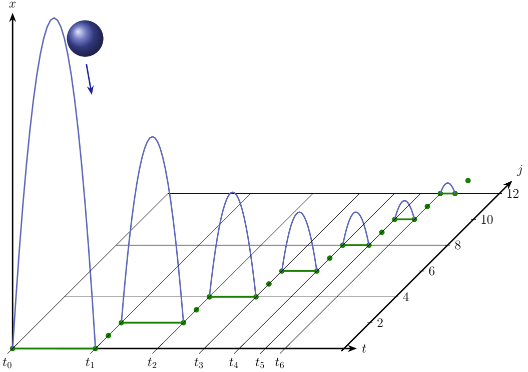

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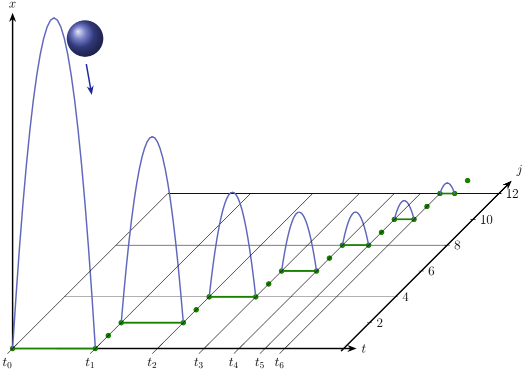

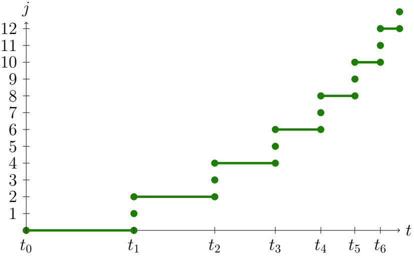

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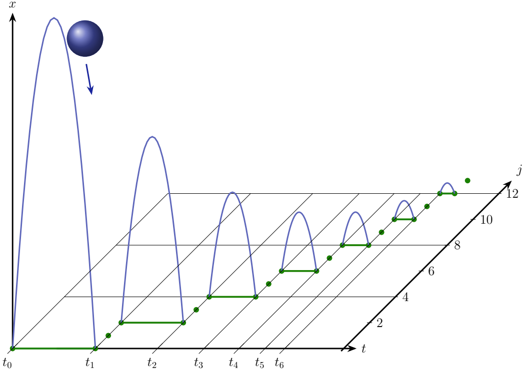

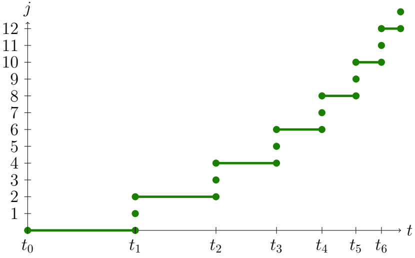

Example (Quantum the Bouncing Ball experiences time)

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

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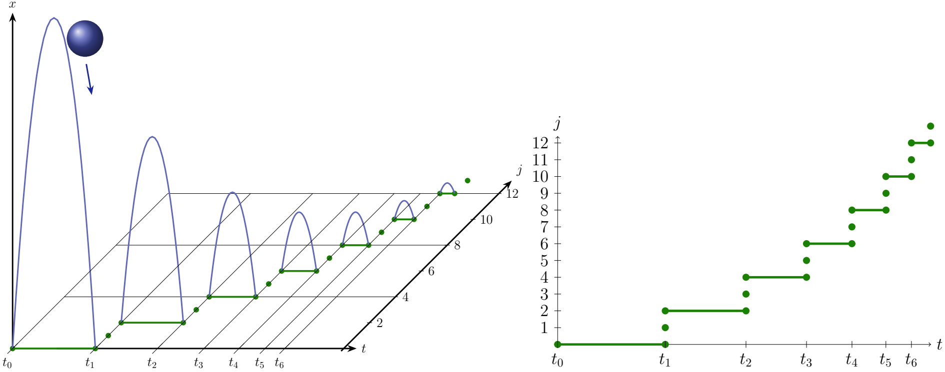

Example (Quantum the Bouncing Ball experiences time)

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i}$$

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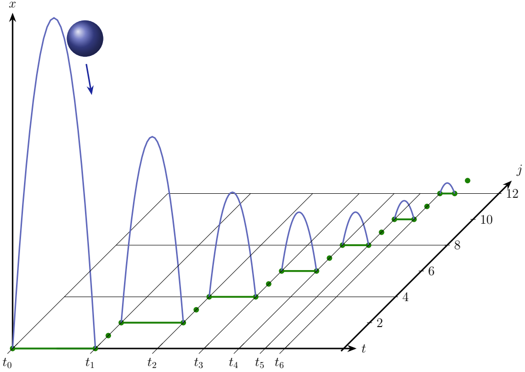

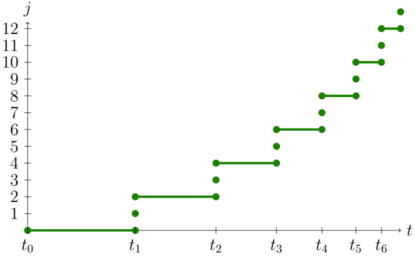

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$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1 - \frac{1}{2}}$$

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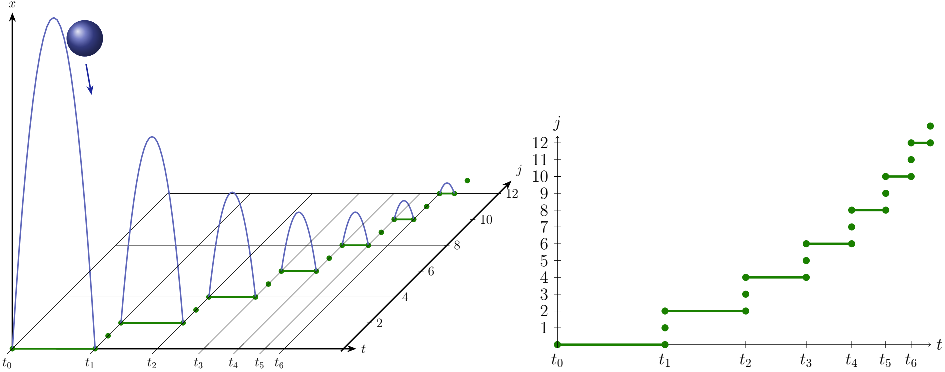

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$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1 - \frac{1}{2}} = 2$$

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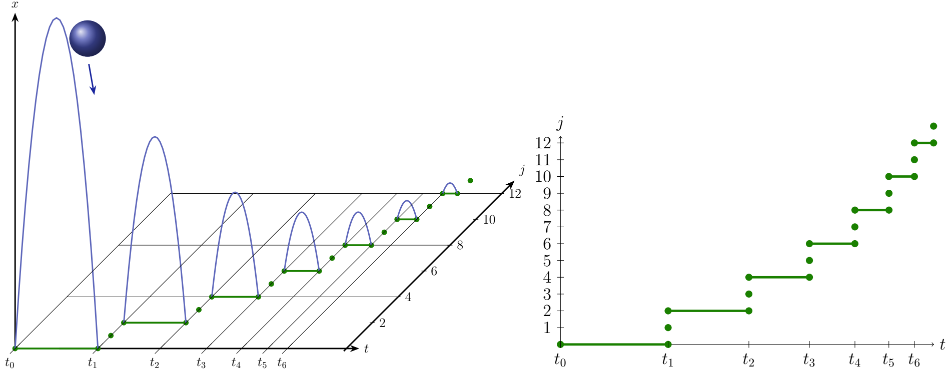

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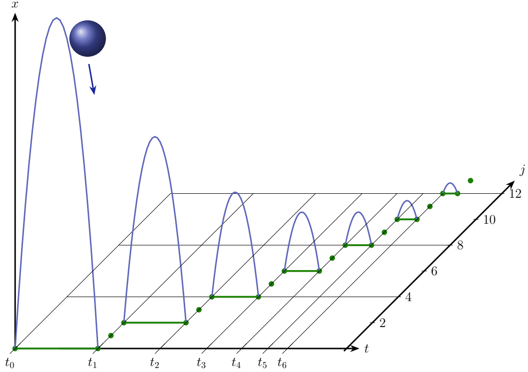

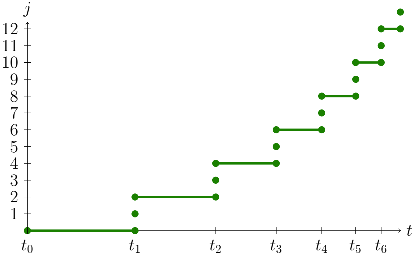

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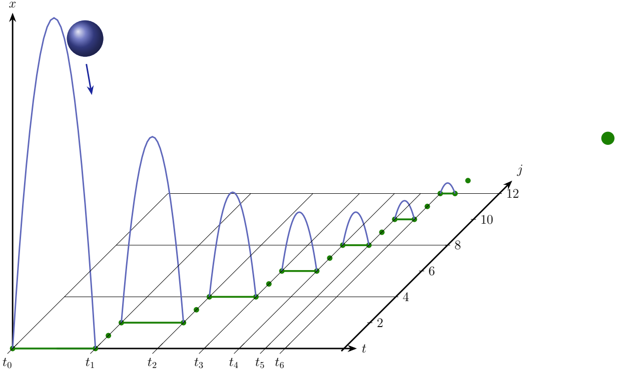

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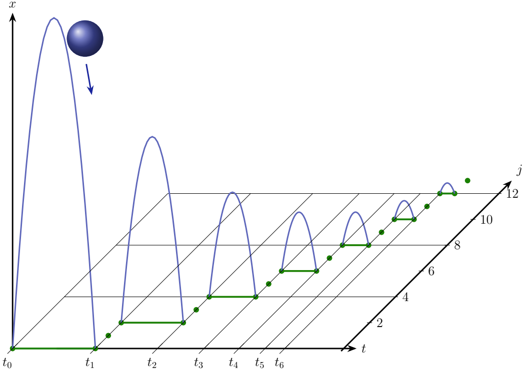

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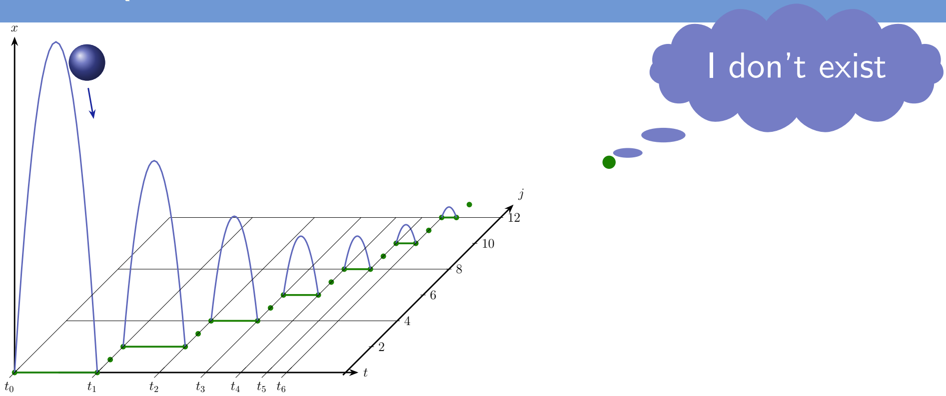

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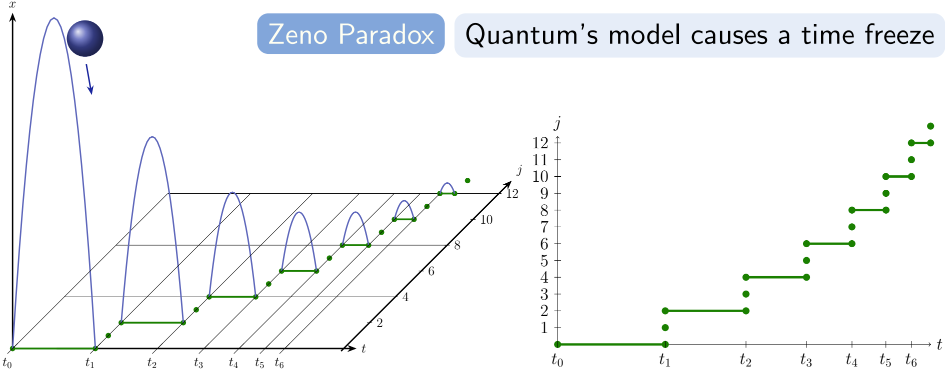

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What to do with assignments

$$\frac{[x:=e]p(x) \leftrightarrow p(e)}{[x:=x^2]x \neq 0 \leftrightarrow x^2 \neq 0} \qquad e \rightsquigarrow x^2,\ p(\cdot) \rightsquigarrow \cdot \neq 0$$

$$\frac{[x:=e]p(x) \leftrightarrow p(e)}{[x:=x^2][y:=2x]x>0 \leftrightarrow [y:=2x^2]x^2>0} \qquad e \rightsquigarrow x^2,\ p(\cdot) \rightsquigarrow [y:=2\cdot](\cdot>0)$$

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$$\frac{[x:=e]p(x) \leftrightarrow p(e)}{[x:=x^2][x' = 2x]x>0 \leftrightarrow [x' = 2x^2]x^2>0} \qquad e \rightsquigarrow x^2,\ p(\cdot) \rightsquigarrow [\cdot' = 2\cdot]\cdot>0$$

$$\frac{[x:=e]p(x) \leftrightarrow p(e)}{[x:=x^2][(x:=x+1)^*]x\geq 0 \leftrightarrow [(x:=x^2+1)^*]x^2\geq 0} \qquad e \rightsquigarrow x^2,\ p(\cdot) \rightsquigarrow [(x:=\cdot+1)^*]\cdot$$

What to do with assignments and what not to do!

$$\frac{[x:=e]p(x) \leftrightarrow p(e)}{[x:=x^2]x \neq 0 \leftrightarrow x^2 \neq 0} \qquad e \rightsquigarrow x^2,\ p(\cdot) \rightsquigarrow \cdot \neq 0$$

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What else to do with assignments

$$[:=]\quad [x := e]p(x) \leftrightarrow p(e)$$

$$\frac{\Gamma, x = e \vdash P, \Delta}{\Gamma \vdash [x{:=}e]P, \Delta}$$

What else to do with assignments and what not to do!

$$[:=] \quad [x := e]p(x) \leftrightarrow p(e)$$

$$\frac{\Gamma, x = e \vdash P, \Delta}{\Gamma \vdash [x{:=}e]P, \Delta} \quad \text{if } x \notin \Gamma, \Delta$$

André Platzer.
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