# 07: Control Loops & Invariants

## 15-424: Foundations of Cyber-Physical Systems

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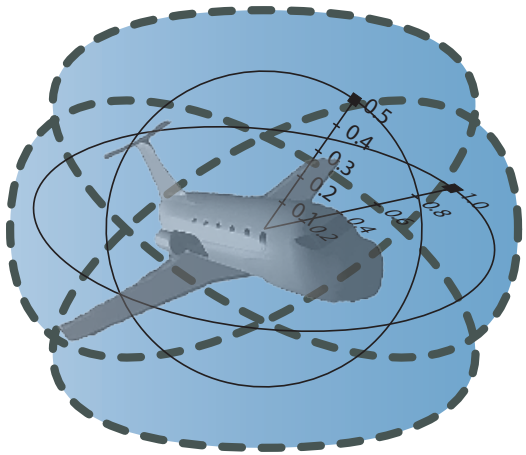

# Outline

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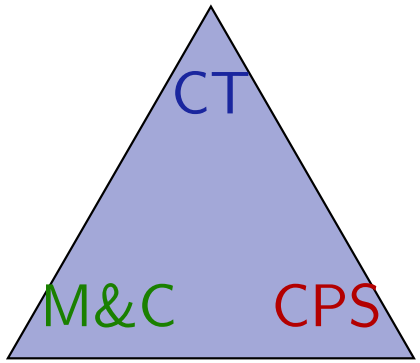
# Learning Objectives

## Control Loops & Invariants

rigorous reasoning for repetitions
identifying and expressing invariants
global vs. local reasoning
relating iterations to invariants
finitely accessible infinities
operationalize invariant construction
splitting & generalizations



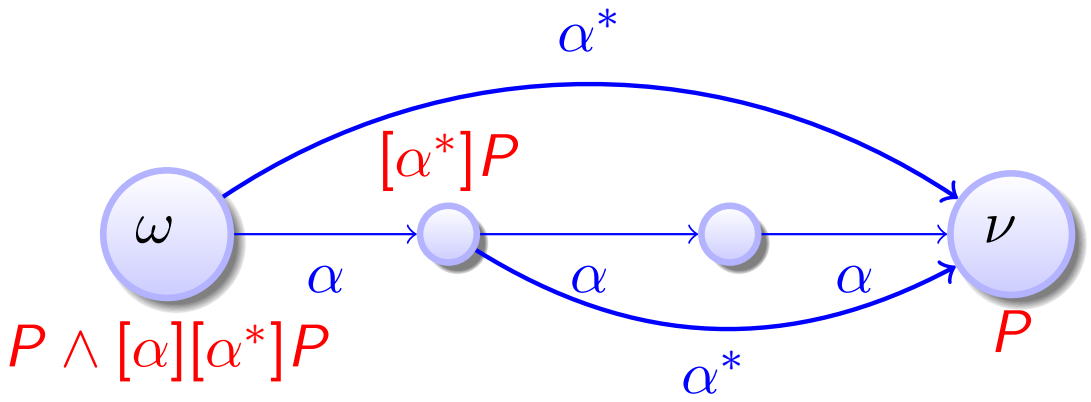

control loops
feedback mechanisms
dynamics of iteration

semantics of control loops
operational effects of control

# Outline

# Proofs for Loops

$[^*]$ $[\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$



# Loops of Proofs: Iterations

$$[^*]\ [\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$$

$$\frac{}{A \vdash [\alpha^*]B}$$

# Loops of Proofs: Iterations

$$[^*]\ [\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$$

$$[^*]\frac{\dfrac{}{A \vdash B \wedge [\alpha][\alpha^*]B}}{A \vdash [\alpha^*]B}$$

# Loops of Proofs: Iterations

$$[^*]\ [\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$$

$$\dfrac{\dfrac{\dfrac{}{A \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}}{A \vdash B \wedge [\alpha][\alpha^*]B}\,[^*]}{A \vdash [\alpha^*]B}\,[^*]$$

# Loops of Proofs: Iterations

$$[^*]\ [\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$$

$$\dfrac{\dfrac{\dfrac{}{A \vdash B \wedge [\alpha](B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B))}}{A \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}[^*]}{A \vdash B \wedge [\alpha][\alpha^*]B}[^*]}{A \vdash [\alpha^*]B}[^*]$$

# Loops of Proofs: Iterations & Splitting the Box

$$[^*]\ [\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$$

$$[]\wedge\ [\alpha](P \wedge Q) \leftrightarrow [\alpha]P \wedge [\alpha]Q$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash B \wedge [\alpha]B \wedge [\alpha][\alpha](B \wedge [\alpha][\alpha^*]B)}}{A \vdash B \wedge [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}\,[]\wedge}{A \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}\,[^*]}{A \vdash B \wedge [\alpha][\alpha^*]B}\,[^*]}{A \vdash [\alpha^*]B}\,[^*]$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash B \wedge [\alpha]B \wedge [\alpha]([\alpha]B \wedge [\alpha][\alpha][\alpha^*]B)}}{A \vdash B \wedge [\alpha]B \wedge [\alpha][\alpha](B \wedge [\alpha][\alpha^*]B)}\,[]\wedge}{A \vdash B \wedge [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}\,[]\wedge}{A \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}\,[^*]}{A \vdash B \wedge [\alpha][\alpha^*]B}\,[^*]}{A \vdash [\alpha^*]B}\,[^*]$$

# Loops of Proofs: Iterations & Splitting the Box

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$$[]\wedge\ [\alpha](P \wedge Q) \leftrightarrow [\alpha]P \wedge [\alpha]Q$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash B \wedge [\alpha]B \wedge [\alpha][\alpha]B \wedge [\alpha][\alpha][\alpha][\alpha^*]B}\,[]\wedge}{A \vdash B \wedge [\alpha]B \wedge [\alpha]([\alpha]B \wedge [\alpha][\alpha][\alpha^*]B)}\,[]\wedge}{A \vdash B \wedge [\alpha]B \wedge [\alpha][\alpha](B \wedge [\alpha][\alpha^*]B)}\,[]\wedge}{A \vdash B \wedge [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}\,[^*]}{A \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}\,[^*]}{A \vdash B \wedge [\alpha][\alpha^*]B}\,[^*]}{A \vdash [\alpha^*]B}$$

# Loops of Proofs: Iterations & Splitting the Box

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$$[]\wedge\ [\alpha](P \wedge Q) \leftrightarrow [\alpha]P \wedge [\alpha]Q$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash B \quad A \vdash [\alpha]B \quad A \vdash [\alpha][\alpha]B \quad A \vdash [\alpha][\alpha][\alpha][\alpha^*]B}{A \vdash B \wedge [\alpha]B \wedge [\alpha][\alpha]B \wedge [\alpha][\alpha][\alpha][\alpha^*]B}\wedge R}{A \vdash B \wedge [\alpha]B \wedge [\alpha]([\alpha]B \wedge [\alpha][\alpha][\alpha^*]B)}[]\wedge}{A \vdash B \wedge [\alpha]B \wedge [\alpha][\alpha](B \wedge [\alpha][\alpha^*]B)}[]\wedge}{A \vdash B \wedge [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}[]\wedge}{A \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}[^*]}{A \vdash B \wedge [\alpha][\alpha^*]B}[^*]}{A \vdash [\alpha^*]B}[^*]$$

# Loops of Proofs: Iterations & Splitting the Box

$$[^*]\ [\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$$

$$[]\wedge\ [\alpha](P \wedge Q) \leftrightarrow [\alpha]P \wedge [\alpha]Q$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash B \quad A \vdash [\alpha]B \quad A \vdash [\alpha][\alpha]B \quad A \vdash [\alpha][\alpha][\alpha][\alpha^*]B}{A \vdash B \wedge [\alpha]B \wedge [\alpha][\alpha]B \wedge [\alpha][\alpha][\alpha][\alpha^*]B}\wedge R}{A \vdash B \wedge [\alpha]B \wedge [\alpha]([\alpha]B \wedge [\alpha][\alpha][\alpha^*]B)}[]\wedge}{A \vdash B \wedge [\alpha]B \wedge [\alpha][\alpha](B \wedge [\alpha][\alpha^*]B)}[]\wedge}{A \vdash B \wedge [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}[]\wedge}{A \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}[^*]}{A \vdash B \wedge [\alpha][\alpha^*]B}[^*]}{A \vdash [\alpha^*]B}[^*]$$

1. Simple approach . . . if we don't mind unrolling until the end of time
2. Useful for finding counterexamples

# Loops of Proofs: Iterations & Generalizations

$$[^*]\ [\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash B \wedge [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}}{A \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}[^*]}{A \vdash B \wedge [\alpha][\alpha^*]B}[^*]}{A \vdash [\alpha^*]B}[^*]$$

# Loops of Proofs: Iterations & Generalizations

$$[^*]\ [\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$$

$$\text{MR}\ \frac{\Gamma \vdash [\alpha]Q, \Delta \qquad Q \vdash P}{\Gamma \vdash [\alpha]P, \Delta}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{A \vdash B}{A \vdash [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}}{A \vdash B \wedge [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}\wedge R}{A \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}[^*]}{A \vdash B \wedge [\alpha][\alpha^*]B}[^*]}{A \vdash [\alpha^*]B}[^*]$$

# Loops of Proofs: Iterations & Generalizations

$$[^*]\ [\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$$

$$\text{MR}\ \frac{\Gamma \vdash [\alpha]Q, \Delta \qquad Q \vdash P}{\Gamma \vdash [\alpha]P, \Delta}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash [\alpha]J_1 \qquad J_1 \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}{A \vdash B}\,\text{MR}}{A \vdash [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}\,\wedge\text{R}}{A \vdash B \wedge [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}\,[^*]}{A \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}\,[^*]}{A \vdash B \wedge [\alpha][\alpha^*]B}\,[^*]}{A \vdash [\alpha^*]B}$$

# Loops of Proofs: Iterations & Generalizations

$$[^*]\ [\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$$

$$\text{MR}\ \frac{\Gamma \vdash [\alpha]Q, \Delta \qquad Q \vdash P}{\Gamma \vdash [\alpha]P, \Delta}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{J_1 \vdash B}}{J_1 \vdash [\alpha](B \wedge [\alpha][\alpha^*]B)}}{A \vdash [\alpha]J_1}\,\wedge\text{R}}{J_1 \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}}{A \vdash B}\,\text{MR}}{A \vdash [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}}{A \vdash B \wedge [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}\,\wedge\text{R}}{A \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}\,[^*]}{A \vdash B \wedge [\alpha][\alpha^*]B}\,[^*]}{A \vdash [\alpha^*]B}\,[^*]$$

# Loops of Proofs: Iterations & Generalizations

$$[^*]\ [\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$$

$$\text{MR}\ \frac{\Gamma \vdash [\alpha]Q, \Delta \qquad Q \vdash P}{\Gamma \vdash [\alpha]P, \Delta}$$

$$[^*]\ \frac{[^*]\ \frac{\wedge\text{R}\ \frac{A \vdash B \;\text{MR}\; \frac{A \vdash [\alpha]J_1 \;\wedge\text{R}\; \frac{J_1 \vdash B \;\text{MR}\; \frac{J_1 \vdash [\alpha]J_2 \qquad J_2 \vdash B \wedge [\alpha][\alpha^*]B}{J_1 \vdash [\alpha](B \wedge [\alpha][\alpha^*]B)}}{J_1 \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}}{A \vdash [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}}{A \vdash B \wedge [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}}{A \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}}{A \vdash B \wedge [\alpha][\alpha^*]B}}{A \vdash [\alpha^*]B}$$

# Loops of Proofs: Iterations & Generalizations

$$[^*]\ [\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$$

$$\text{MR}\ \frac{\Gamma \vdash [\alpha]Q, \Delta \qquad Q \vdash P}{\Gamma \vdash [\alpha]P, \Delta}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash B}\,\text{MR}}{A \vdash [\alpha]J_1}\,\wedge\text{R} \quad \dfrac{}{J_1 \vdash B}\,\text{MR}}{J_1 \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}}{A \vdash [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}\,\text{MR}}{A \vdash B \wedge [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}\,\wedge\text{R}}{A \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}\,[^*]}{A \vdash B \wedge [\alpha][\alpha^*]B}\,[^*]}{A \vdash [\alpha^*]B}\,[^*]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{J_1 \vdash [\alpha]J_2}\,\wedge\text{R} \quad \dfrac{}{J_2 \vdash B}}{J_2 \vdash B \wedge [\alpha][\alpha^*]B}}{J_1 \vdash [\alpha](B \wedge [\alpha][\alpha^*]B)}\,\text{MR} \quad \dfrac{}{J_2 \vdash [\alpha][\alpha^*]B}}{J_1 \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}\,\wedge\text{R}$$

# Loops of Proofs: Iterations & Generalizations

$$[^*]\ [\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$$

$$\text{MR}\ \frac{\Gamma \vdash [\alpha]Q, \Delta \qquad Q \vdash P}{\Gamma \vdash [\alpha]P, \Delta}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{A \vdash B}\,\text{MR}}{A \vdash [\alpha]J_1}\,\wedge\text{R} \qquad \cfrac{}{J_1 \vdash B}\,\text{MR}}{J_1 \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}}{A \vdash [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}\,\text{MR}}{A \vdash B \wedge [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}\,\wedge\text{R}}{A \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}\,[^*]}{A \vdash B \wedge [\alpha][\alpha^*]B}\,[^*]}{A \vdash [\alpha^*]B}\,[^*]$$

$$\cfrac{\cfrac{\cfrac{\cfrac{}{J_2 \vdash B} \qquad \cfrac{}{J_2 \vdash [\alpha]J_3} \quad \ldots}{J_2 \vdash [\alpha][\alpha^*]B}}{J_1 \vdash [\alpha]J_2}\,\wedge\text{R}}{J_2 \vdash B \wedge [\alpha][\alpha^*]B}$$

# Loops of Proofs: Common Generalizations

$$[^*]\ [\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$$

$$\text{MR}\ \frac{\Gamma \vdash [\alpha]Q, \Delta \qquad Q \vdash P}{\Gamma \vdash [\alpha]P, \Delta}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash B}\text{MR}}{A \vdash [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}}{A \vdash B \wedge [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}\wedge\text{R}}{A \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}[^*]}{A \vdash B \wedge [\alpha][\alpha^*]B}[^*]}{A \vdash [\alpha^*]B}[^*]$$

$$\dfrac{\dfrac{}{A \vdash [\alpha]J}\qquad \dfrac{\dfrac{}{J \vdash B}\text{MR}}{J \vdash [\alpha](B \wedge [\alpha][\alpha^*]B)}}{J \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}\wedge\text{R}$$

$$\dfrac{\dfrac{}{J \vdash B}\qquad \dfrac{\dfrac{}{J \vdash [\alpha]J}\qquad \dots}{J \vdash [\alpha][\alpha^*]B}}{J \vdash B \wedge [\alpha][\alpha^*]B}\wedge\text{R}$$

# Loops of Proofs: Extracting a Proof Rule

$$\frac{J \vdash B}{A \vdash [\alpha^*]B}$$

$$[^*]\ [\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$$

$$\text{MR}\ \frac{\Gamma \vdash [\alpha]Q, \Delta \qquad Q \vdash P}{\Gamma \vdash [\alpha]P, \Delta}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{J \vdash B}\quad \dfrac{\dfrac{}{J \vdash [\alpha]J}\quad \dots}{J \vdash [\alpha][\alpha^*]B}}{J \vdash B \wedge [\alpha][\alpha^*]B}\wedge\text{R}}{J \vdash [\alpha](B \wedge [\alpha][\alpha^*]B)}\text{MR}}{J \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}\wedge\text{R}}{A \vdash [\alpha]J}}{A \vdash [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}\text{MR}}{A \vdash B \wedge [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}\wedge\text{R}}{A \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}[^*]}{A \vdash B \wedge [\alpha][\alpha^*]B}[^*]}{A \vdash [\alpha^*]B}[^*]$$

# Loops of Proofs: Extracting a Proof Rule

$$\frac{J \vdash [\alpha]J \qquad J \vdash B}{A \vdash [\alpha^*]B}$$

$$[^*]\ [\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$$

$$\text{MR}\ \frac{\Gamma \vdash [\alpha]Q, \Delta \qquad Q \vdash P}{\Gamma \vdash [\alpha]P, \Delta}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash B}\,\text{MR}}{A \vdash [\alpha]J}\,\wedge\text{R}}{J \vdash B}\,\text{MR}}{J \vdash [\alpha]J}\,\wedge\text{R}}{J \vdash B \wedge [\alpha][\alpha^*]B}}{J \vdash [\alpha](B \wedge [\alpha][\alpha^*]B)}}{J \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}}{A \vdash [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}}{A \vdash B \wedge [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}$$

$$\frac{A \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}{A \vdash B \wedge [\alpha][\alpha^*]B}$$

$$\frac{A \vdash B \wedge [\alpha][\alpha^*]B}{A \vdash [\alpha^*]B}$$

# Loops of Proofs: Extracting a Proof Rule

$$\frac{A \vdash J \qquad J \vdash [\alpha]J \qquad J \vdash B}{A \vdash [\alpha^*]B}$$

$$[^*]\ [\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$$

$$\text{MR}\ \frac{\Gamma \vdash [\alpha]Q, \Delta \qquad Q \vdash P}{\Gamma \vdash [\alpha]P, \Delta}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash J}\quad \dfrac{\dfrac{}{J \vdash [\alpha]J}\quad \dfrac{\dfrac{}{J \vdash B}\quad \dfrac{\dfrac{}{J \vdash [\alpha]J}\quad \dots}{J \vdash [\alpha][\alpha^*]B}}{J \vdash B \wedge [\alpha][\alpha^*]B}\wedge\text{R}}{J \vdash [\alpha](B \wedge [\alpha][\alpha^*]B)}\text{MR}}{J \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}\wedge\text{R}}{A \vdash [\alpha](B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B))}\text{MR}}{A \vdash B \wedge [\alpha](B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B))}\wedge\text{R}}{A \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}[^*]}{A \vdash B \wedge [\alpha][\alpha^*]B}[^*]}{A \vdash [\alpha^*]B}[^*]$$

# Loops of Proofs: Loop Invariants

$$\text{loop}\ \frac{A \vdash J \quad J \vdash [\alpha]J \quad J \vdash B}{A \vdash [\alpha^*]B}$$

$$[^*]\ [\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$$

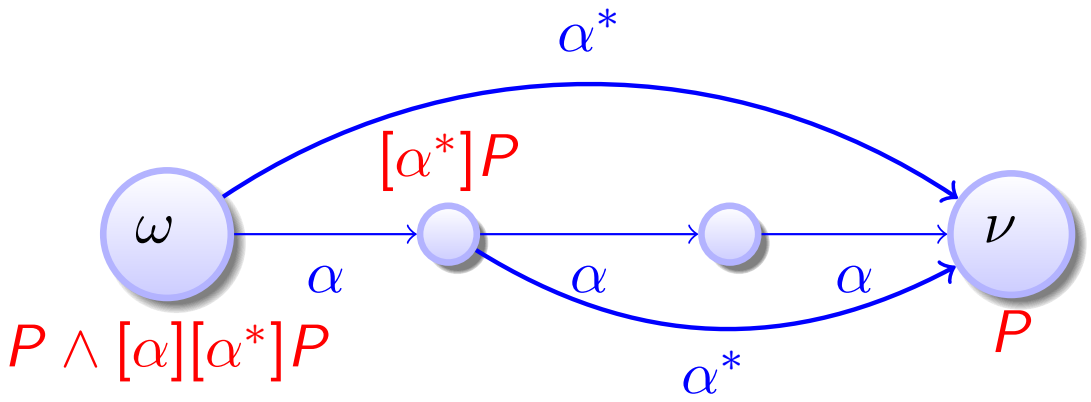
$$\text{MR}\ \frac{\Gamma \vdash [\alpha]Q, \Delta \quad Q \vdash P}{\Gamma \vdash [\alpha]P, \Delta}$$

Invariant $J$ generalized intermediate condition

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{J \vdash B} \quad \dfrac{\dfrac{}{J \vdash [\alpha]J} \quad \dots}{J \vdash [\alpha][\alpha^*]B}\,\wedge\text{R}}{J \vdash B \wedge [\alpha][\alpha^*]B}}{J \vdash [\alpha](B \wedge [\alpha][\alpha^*]B)}\,\text{MR}}{J \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}\,\wedge\text{R}}{A \vdash [\alpha]J}}{A \vdash [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}\,\text{MR}}{A \vdash B}}{A \vdash B \wedge [\alpha]\big(B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)\big)}\,\wedge\text{R}}{A \vdash B \wedge [\alpha](B \wedge [\alpha][\alpha^*]B)}\,[^*]}{A \vdash B \wedge [\alpha][\alpha^*]B}\,[^*]}{A \vdash [\alpha^*]B}\,[^*]$$

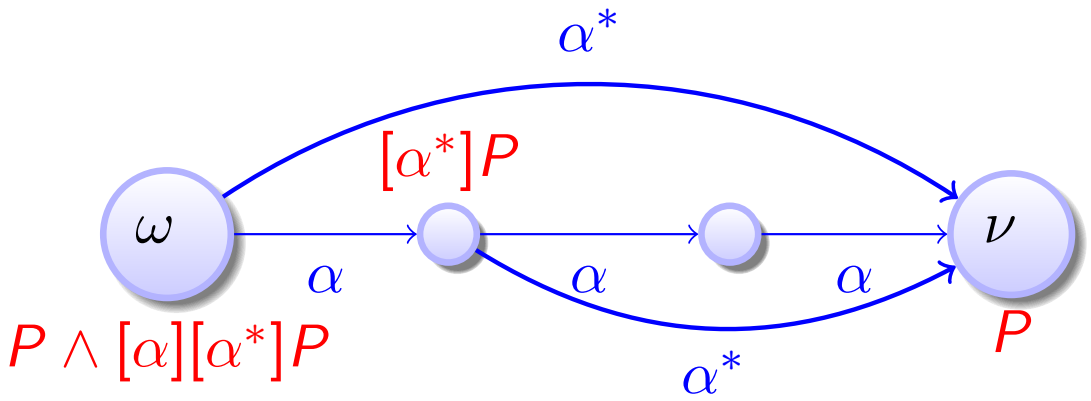
# Proofs for Loops

$$[^*]\quad [\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$$

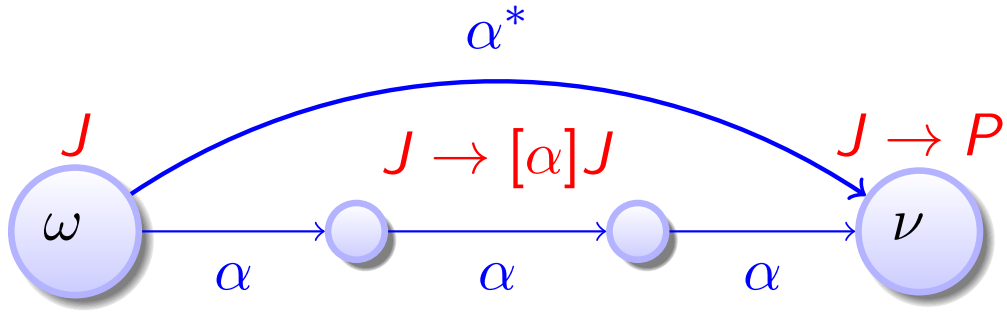


# Proofs for Loops

$[^*]$ $[\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$




loop $\dfrac{\Gamma \vdash J, \Delta \quad J \vdash [\alpha]J \quad J \vdash P}{\Gamma \vdash [\alpha^*]P, \Delta}$



# Outline

# Proving Quantum the Acrophobic Bouncing Ball

$$A \vdash [(x''=; (?x=0; v:=-cv \cup ?x\neq 0))^*]B(x,v)$$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$

$$x''=.. \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

# Proving Quantum the Acrophobic Bouncing Ball

$$\text{loop}\ \frac{A \vdash j(x,v) \qquad \dfrac{}{j(x,v) \vdash [x''=;(?x=0; v:=-cv \cup ?x\neq 0]j(x,v)} \qquad j(x,v) \vdash B(x,v)}{A \vdash [(x''=;(?x=0; v:=-cv \cup ?x\neq 0))^*]B(x,v)}$$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$

$$x''=.. \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

# Proving Quantum the Acrophobic Bouncing Ball

$$\dfrac{}{j(x,v) \vdash [x''=;(?x=0; v:=-cv \cup ?x\neq 0]j(x,v)}$$

$$\text{loop}\dfrac{A \vdash j(x,v) \quad [;]\dfrac{}{j(x,v) \vdash [x''=;(?x=0; v:=-cv \cup ?x\neq 0]j(x,v)} \qquad j(x,v) \vdash B(x,v)}{A \vdash [(x''=;(?x=0; v:=-cv \cup ?x\neq 0))^*]B(x,v)}$$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$

$$x''=.. \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

# Proving Quantum the Acrophobic Bouncing Ball

$$[;]\ \frac{\dfrac{}{j(x,v) \vdash [x''=][?x=0;\, v:=-cv \cup ?x\neq 0]j(x,v)}}{j(x,v) \vdash [x''=;\,(?x=0;\, v:=-cv \cup ?x\neq 0]j(x,v)}$$

$$\text{loop}\ \frac{A \vdash j(x,v) \quad [;]\ \dfrac{}{j(x,v) \vdash [x''=;\,(?x=0;\, v:=-cv \cup ?x\neq 0]j(x,v)} \quad j(x,v) \vdash B(x,v)}{A \vdash [\left(x''=;\,(?x=0;\, v:=-cv \cup ?x\neq 0)\right)^*]B(x,v)}$$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$

$$x''=.. \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

# Proving Quantum the Acrophobic Bouncing Ball

$$\text{MR}\,\dfrac{\text{[;]}\,\dfrac{j(x,v) \vdash [x''=]j(x,v) \qquad j(x,v) \vdash [?x=0;\, v:=-cv \cup ?x\neq 0]j(x,v)}{j(x,v) \vdash [x''=][?x=0;\, v:=-cv \cup ?x\neq 0]j(x,v)}}{j(x,v) \vdash [x''=;(?x=0;\, v:=-cv \cup ?x\neq 0]j(x,v)}$$

$$\text{loop}\,\dfrac{A \vdash j(x,v) \quad \text{[;]}\,\dfrac{}{j(x,v) \vdash [x''=;(?x=0;\, v:=-cv \cup ?x\neq 0]j(x,v)} \quad j(x,v) \vdash B(x,v)}{A \vdash [\big(x''=;(?x=0;\, v:=-cv \cup ?x\neq 0)\big)^*]B(x,v)}$$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$

$$x''=.. \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

# Proving Quantum the Acrophobic Bouncing Ball

$$\text{MR}\,\frac{\text{[;]}\,\dfrac{j(x,v) \vdash [x''=]j(x,v) \quad [\cup]\,\dfrac{j(x,v) \vdash [?x=0;\, v:=-cv]j(x,v) \wedge [?x\neq 0]j(x,v)}{j(x,v) \vdash [?x=0;\, v:=-cv \cup ?x\neq 0]j(x,v)}}{j(x,v) \vdash [x''=][?x=0;\, v:=-cv \cup ?x\neq 0]j(x,v)}}{j(x,v) \vdash [x''=;(?x=0;\, v:=-cv \cup ?x\neq 0]j(x,v)}$$

$$\text{loop}\,\frac{A \vdash j(x,v) \quad \text{[;]}\,\dfrac{}{j(x,v) \vdash [x''=;(?x=0;\, v:=-cv \cup ?x\neq 0]j(x,v)} \qquad j(x,v) \vdash B(x,v)}{A \vdash [\big(x''=;(?x=0;\, v:=-cv \cup ?x\neq 0)\big)^*]B(x,v)}$$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$

$$x''=.. \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

# Proving Quantum the Acrophobic Bouncing Ball

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{j(x,v) \vdash [?x{=}0;\, v{:=}{-}cv]j(x,v)} \quad \dfrac{}{j(x,v) \vdash [?x{\neq}0]j(x,v)}}{j(x,v) \vdash [?x{=}0;\, v{:=}{-}cv]j(x,v) \land [?x{\neq}0]j(x,v)}\land R}{j(x,v) \vdash [?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}[\cup]}{j(x,v) \vdash [x''{=}][?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}MR}{j(x,v) \vdash [x''{=};(?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}[;]$$

$$\dfrac{A \vdash j(x,v) \quad \dfrac{}{j(x,v) \vdash [x''{=};(?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}[;] \quad j(x,v) \vdash B(x,v)}{A \vdash [(x''{=};(?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0))^*]B(x,v)}\text{loop}$$

$$A \equiv 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \land x \leq H$$

$$x''{=}.. \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

# Proving Quantum the Acrophobic Bouncing Ball

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{j(x,v) \vdash [?x{=}0][v{:=}{-}cv]j(x,v)}}{j(x,v) \vdash [?x{=}0; v{:=}{-}cv]j(x,v)}[;] \quad \dfrac{}{j(x,v) \vdash [?x{\neq}0]j(x,v)}}{j(x,v) \vdash [?x{=}0; v{:=}{-}cv]j(x,v) \wedge [?x{\neq}0]j(x,v)}\wedge R}{j(x,v) \vdash [?x{=}0; v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}[\cup]}{j(x,v) \vdash [x''{=}][?x{=}0; v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}MR}{j(x,v) \vdash [x''{=}; (?x{=}0; v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}[;]$$

$$\dfrac{A \vdash j(x,v) \quad \dfrac{}{j(x,v) \vdash [x''{=}; (?x{=}0; v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}[;] \quad j(x,v) \vdash B(x,v)}{A \vdash [\big(x''{=}; (?x{=}0; v{:=}{-}cv \cup ?x{\neq}0)\big)^*]B(x,v)}\text{loop}$$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$

$$x''{=}.. \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

# Proving Quantum the Acrophobic Bouncing Ball

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{j(x,v),\, x{=}0 \vdash [v{:=}{-}cv]j(x,v)}[?]}{j(x,v) \vdash [?x{=}0][v{:=}{-}cv]j(x,v)}[;]}{j(x,v) \vdash [?x{=}0;\, v{:=}{-}cv]j(x,v)} \quad \dfrac{}{j(x,v) \vdash [?x{\neq}0]j(x,v)}\wedge R}{j(x,v) \vdash [?x{=}0;\, v{:=}{-}cv]j(x,v) \wedge [?x{\neq}0]j(x,v)}}{j(x,v) \vdash [?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}[\cup] \quad j(x,v) \vdash [x''{=}]j(x,v)}{j(x,v) \vdash [x''{=}][?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}MR}{j(x,v) \vdash [x''{=};\, (?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}[;]$$

$$\dfrac{A \vdash j(x,v) \quad \dfrac{}{j(x,v) \vdash [x''{=};\, (?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}[;] \quad j(x,v) \vdash B(x,v)}{A \vdash [\big(x''{=};\, (?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0)\big)^*]B(x,v)}\text{loop}$$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$

$$x''{=}.. \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

# Proving Quantum the Acrophobic Bouncing Ball

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{j(x,v),\, x{=}0 \vdash j(x,{-}cv)}{j(x,v),\, x{=}0 \vdash [v{:=}{-}cv]j(x,v)}[:=]}{j(x,v) \vdash [?x{=}0][v{:=}{-}cv]j(x,v)}[?]}{j(x,v) \vdash [?x{=}0;\, v{:=}{-}cv]j(x,v)}[;] \quad \dfrac{}{j(x,v) \vdash [?x{\neq}0]j(x,v)}}{j(x,v) \vdash [?x{=}0;\, v{:=}{-}cv]j(x,v) \wedge [?x{\neq}0]j(x,v)}\wedge R}{j(x,v) \vdash [?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}[\cup] \quad j(x,v) \vdash [x''{=}]j(x,v)}{j(x,v) \vdash [x''{=}][?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}MR}{j(x,v) \vdash [x''{=};(?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}[;]$$

$$\dfrac{A \vdash j(x,v) \quad \dfrac{}{j(x,v) \vdash [x''{=};(?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}[;] \quad j(x,v) \vdash B(x,v)}{A \vdash [(x''{=};(?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0))^*]B(x,v)}\text{loop}$$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$

$$x''{=}.. \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

# Proving Quantum the Acrophobic Bouncing Ball

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{j(x,v),\, x{=}0 \vdash j(x,{-}cv)}{j(x,v),\, x{=}0 \vdash [v{:=}{-}cv]j(x,v)}[:=]}{j(x,v) \vdash [?x{=}0][v{:=}{-}cv]j(x,v)}[?]}{j(x,v) \vdash [?x{=}0;\, v{:=}{-}cv]j(x,v)}[;] \quad \dfrac{j(x,v),\, x{\neq}0 \vdash j(x,v)}{j(x,v) \vdash [?x{\neq}0]j(x,v)}[?]}{j(x,v) \vdash [?x{=}0;\, v{:=}{-}cv]j(x,v) \wedge [?x{\neq}0]j(x,v)}\wedge R}{j(x,v) \vdash [?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}[\cup] \quad j(x,v) \vdash [x''{=}]j(x,v)}{j(x,v) \vdash [x''{=}][?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}MR}{j(x,v) \vdash [x''{=};\, (?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}[;]$$

$$\dfrac{A \vdash j(x,v) \quad \dfrac{}{j(x,v) \vdash [x''{=};\, (?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}[;] \quad j(x,v) \vdash B(x,v)}{A \vdash [\big(x''{=};\, (?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0)\big)^*]B(x,v)}\text{loop}$$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$

$$x''{=}.. \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

# Proving Quantum the Acrophobic Bouncing Ball

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{j(x,v),\, x{=}0 \vdash j(x,{-}cv)}{j(x,v),\, x{=}0 \vdash [v{:=}{-}cv]j(x,v)}[:=]}{j(x,v) \vdash [?x{=}0][v{:=}{-}cv]j(x,v)}[?]}{j(x,v) \vdash [?x{=}0;\, v{:=}{-}cv]j(x,v)}[;] \quad \dfrac{j(x,v),\, x{\neq}0 \vdash j(x,v)}{j(x,v) \vdash [?x{\neq}0]j(x,v)}[?]}{j(x,v) \vdash [?x{=}0;\, v{:=}{-}cv]j(x,v) \wedge [?x{\neq}0]j(x,v)}\wedge R}{j(x,v) \vdash [?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}[\cup] \quad j(x,v) \vdash [x''{=}]j(x,v)}{j(x,v) \vdash [x''{=}][?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}MR}{j(x,v) \vdash [x''{=};(?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}[;]$$

$$\dfrac{A \vdash j(x,v) \quad \dfrac{}{j(x,v) \vdash [x''{=};(?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0]j(x,v)}[;] \quad j(x,v) \vdash B(x,v)}{A \vdash [(x''{=};(?x{=}0;\, v{:=}{-}cv \cup ?x{\neq}0))^*]B(x,v)}\text{loop}$$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$

$$x''{=}.. \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

# Proving Quantum the Acrophobic Bouncing Ball

$A \vdash j(x,v)$

$j(x,v) \vdash [x''=](j(x,v))$

$j(x,v), x{=}0 \vdash j(x,(-cv))$

$j(x,v), x{\neq}0 \vdash j(x,v)$

$j(x,v) \vdash B(x,v)$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$

$$x''=.. \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

# Proving Quantum the Acrophobic Bouncing Ball

$0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \vdash j(x,v)$

$j(x,v) \vdash [\{x'=v, v'=-g \,\&\, x \geq 0\}](j(x,v))$

$j(x,v), x=0 \vdash j(x,(-cv))$

$j(x,v), x \neq 0 \vdash j(x,v)$

$j(x,v) \vdash 0 \leq x \wedge x \leq H$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$

$$x''=.. \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

# Proving Quantum the Acrophobic Bouncing Ball

$0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \vdash j(x,v)$
$j(x,v) \vdash [\{x'=v, v'=-g \,\&\, x \geq 0\}](j(x,v))$
$j(x,v), x=0 \vdash j(x,(-cv))$
$j(x,v), x \neq 0 \vdash j(x,v)$
$j(x,v) \vdash 0 \leq x \wedge x \leq H$

2. $j(x,v) \equiv 0 \leq x \wedge x \leq H$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$
$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$
$$x''=.. \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

# Proving Quantum the Acrophobic Bouncing Ball

$0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \vdash j(x,v)$
$j(x,v) \vdash [\{x'=v, v'=-g \,\&\, x \geq 0\}](j(x,v))$
$j(x,v), x=0 \vdash j(x,(-cv))$
$j(x,v), x \neq 0 \vdash j(x,v)$
$j(x,v) \vdash 0 \leq x \wedge x \leq H$

2. $j(x,v) \equiv 0 \leq x \wedge x \leq H$ weak: fails ODE if $v \gg 0$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$
$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$
$$x''=.. \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

# Proving Quantum the Acrophobic Bouncing Ball

$0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \vdash j(x,v)$
$j(x,v) \vdash [\{x'=v, v'=-g \,\&\, x \geq 0\}](j(x,v))$
$j(x,v), x=0 \vdash j(x,(-cv))$
$j(x,v), x \neq 0 \vdash j(x,v)$
$j(x,v) \vdash 0 \leq x \wedge x \leq H$

1. $j(x,v) \equiv x \geq 0$
2. $j(x,v) \equiv 0 \leq x \wedge x \leq H$ weak: fails ODE if $v \gg 0$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$
$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$
$$x''=.. \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

# Proving Quantum the Acrophobic Bouncing Ball

$0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \vdash j(x,v)$
$j(x,v) \vdash [\{x'=v, v'=-g \,\&\, x \geq 0\}](j(x,v))$
$j(x,v), x=0 \vdash j(x,(-cv))$
$j(x,v), x \neq 0 \vdash j(x,v)$
$j(x,v) \vdash 0 \leq x \wedge x \leq H$

1. $j(x,v) \equiv x \geq 0$ — weaker: fails postcondition if $x > H$
2. $j(x,v) \equiv 0 \leq x \wedge x \leq H$ — weak: fails ODE if $v \gg 0$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$
$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$
$$x''=.. \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

# Proving Quantum the Acrophobic Bouncing Ball

$0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \vdash j(x,v)$
$j(x,v) \vdash [\{x'=v, v'=-g \,\&\, x \geq 0\}](j(x,v))$
$j(x,v), x=0 \vdash j(x,(-cv))$
$j(x,v), x \neq 0 \vdash j(x,v)$
$j(x,v) \vdash 0 \leq x \wedge x \leq H$

1. $j(x,v) \equiv x \geq 0$
2. $j(x,v) \equiv 0 \leq x \wedge x \leq H$
3. $j(x,v) \equiv x = 0 \wedge v = 0$

weaker: fails postcondition if $x > H$

weak: fails ODE if $v \gg 0$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$
$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$
$$x''=.. \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

# Proving Quantum the Acrophobic Bouncing Ball

$0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \vdash j(x,v)$
$j(x,v) \vdash [\{x'=v, v'=-g \,\&\, x \geq 0\}](j(x,v))$
$j(x,v), x=0 \vdash j(x,(-cv))$
$j(x,v), x \neq 0 \vdash j(x,v)$
$j(x,v) \vdash 0 \leq x \wedge x \leq H$

1. $j(x,v) \equiv x \geq 0$ — weaker: fails postcondition if $x > H$
2. $j(x,v) \equiv 0 \leq x \wedge x \leq H$ — weak: fails ODE if $v \gg 0$
3. $j(x,v) \equiv x = 0 \wedge v = 0$ — strong: fails initial condition if $x > 0$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$
$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$
$$x''=.. \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

# Proving Quantum the Acrophobic Bouncing Ball

$0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \vdash j(x,v)$
$j(x,v) \vdash [\{x'=v, v'=-g \,\&\, x \geq 0\}](j(x,v))$
$j(x,v), x=0 \vdash j(x,(-cv))$
$j(x,v), x \neq 0 \vdash j(x,v)$
$j(x,v) \vdash 0 \leq x \wedge x \leq H$

1. $j(x,v) \equiv x \geq 0$ — weaker: fails postcondition if $x > H$
2. $j(x,v) \equiv 0 \leq x \wedge x \leq H$ — weak: fails ODE if $v \gg 0$
3. $j(x,v) \equiv x = 0 \wedge v = 0$ — strong: fails initial condition if $x > 0$
4. $j(x,v) \equiv x = 0 \vee x = H \wedge v = 0$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$
$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$
$$x''=.. \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

# Proving Quantum the Acrophobic Bouncing Ball

$0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0 \vdash j(x,v)$
$j(x,v) \vdash [\{x'=v, v'=-g \,\&\, x \geq 0\}](j(x,v))$
$j(x,v), x=0 \vdash j(x,(-cv))$
$j(x,v), x \neq 0 \vdash j(x,v)$
$j(x,v) \vdash 0 \leq x \land x \leq H$

1. $j(x,v) \equiv x \geq 0$ — weaker: fails postcondition if $x > H$
2. $j(x,v) \equiv 0 \leq x \land x \leq H$ — weak: fails ODE if $v \gg 0$
3. $j(x,v) \equiv x = 0 \land v = 0$ — strong: fails initial condition if $x > 0$
4. $j(x,v) \equiv x = 0 \lor x = H \land v = 0$ — no space for intermediate states

$$A \equiv 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$
$$B(x,v) \equiv 0 \leq x \land x \leq H$$
$$x''=.. \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

# Proving Quantum the Acrophobic Bouncing Ball

$0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0 \vdash j(x,v)$
$j(x,v) \vdash [\{x'=v, v'=-g \,\&\, x \geq 0\}](j(x,v))$
$j(x,v), x=0 \vdash j(x,(-cv))$
$j(x,v), x \neq 0 \vdash j(x,v)$
$j(x,v) \vdash 0 \leq x \land x \leq H$

1. $j(x,v) \equiv x \geq 0$ — weaker: fails postcondition if $x > H$
2. $j(x,v) \equiv 0 \leq x \land x \leq H$ — weak: fails ODE if $v \gg 0$
3. $j(x,v) \equiv x = 0 \land v = 0$ — strong: fails initial condition if $x > 0$
4. $j(x,v) \equiv x = 0 \lor x = H \land v = 0$ — no space for intermediate states
5. $j(x,v) \equiv 2gx = 2gH - v^2 \land x \geq 0$

$$A \equiv 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$
$$B(x,v) \equiv 0 \leq x \land x \leq H$$
$$x''=.. \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

# Proving Quantum the Acrophobic Bouncing Ball

$0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0 \vdash j(x,v)$
$j(x,v) \vdash [\{x'=v, v'=-g \,\&\, x \geq 0\}](j(x,v))$
$j(x,v), x=0 \vdash j(x,(-cv))$
$j(x,v), x \neq 0 \vdash j(x,v)$
$j(x,v) \vdash 0 \leq x \land x \leq H$

1. $j(x,v) \equiv x \geq 0$ — weaker: fails postcondition if $x > H$
2. $j(x,v) \equiv 0 \leq x \land x \leq H$ — weak: fails ODE if $v \gg 0$
3. $j(x,v) \equiv x = 0 \land v = 0$ — strong: fails initial condition if $x > 0$
4. $j(x,v) \equiv x = 0 \lor x = H \land v = 0$ — no space for intermediate states
5. $j(x,v) \equiv 2gx = 2gH - v^2 \land x \geq 0$ — works: implicitly links $v$ and $x$

$$A \equiv 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$
$$B(x,v) \equiv 0 \leq x \land x \leq H$$
$$x''=.. \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

# Proving Quantum the Acrophobic Bouncing Ball

$0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \vdash 2gx{=}2gH{-}v^2 \wedge x{\geq}0$
$2gx{=}2gH{-}v^2 \wedge x{\geq}0 \vdash [\{x'{=}v, v'{=}{-}g \,\&\, x{\geq}0\}](2gx{=}2gH{-}v^2 \wedge x{\geq}0)$
$2gx{=}2gH{-}v^2 \wedge x{\geq}0, x{=}0 \vdash 2gx{=}2gH{-}(-cv)^2 \wedge x{\geq}0$
$2gx{=}2gH{-}v^2 \wedge x{\geq}0, x{\neq}0 \vdash 2gx{=}2gH{-}v^2 \wedge x{\geq}0$
$2gx{=}2gH{-}v^2 \wedge x{\geq}0 \vdash 0 \leq x \wedge x \leq H$

1. $j(x,v) \equiv x \geq 0$ — weaker: fails postcondition if $x > H$
2. $j(x,v) \equiv 0 \leq x \wedge x \leq H$ — weak: fails ODE if $v \gg 0$
3. $j(x,v) \equiv x = 0 \wedge v = 0$ — strong: fails initial condition if $x > 0$
4. $j(x,v) \equiv x = 0 \vee x = H \wedge v = 0$ — no space for intermediate states
5. $j(x,v) \equiv 2gx{=}2gH{-}v^2 \wedge x{\geq}0$ — works: implicitly links $v$ and $x$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$
$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$
$$x''{=}.. \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

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$2gx{=}2gH{-}v^2 \wedge x{\geq}0, x{=}0 \vdash 2gx{=}2gH{-}(-cv)^2 \wedge x{\geq}0$ if $c = 1 \ldots$

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$$[']\ \frac{}{j(x,v) \vdash [x'=v, v'=-g \,\&\, x\geq 0]j(x,v)}$$

# Proving Quantum the Acrophobic Bouncing Ball

$$[;]\ \frac{}{j(x,v) \vdash \forall t{\geq}0\, [x{:=}H{-}\frac{g}{2}t^2;\, v{:=}{-}gt](x{\geq}0 \rightarrow j(x,v))}$$

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\dfrac{\dfrac{\dfrac{}{j(x,v) \vdash \forall t{\geq}0\, [x{:=}H{-}\frac{g}{2}t^2][v{:=}{-}gt](x{\geq}0 \rightarrow j(x,v))}[:=]}{j(x,v) \vdash \forall t{\geq}0\, [x{:=}H{-}\frac{g}{2}t^2; v{:=}{-}gt](x{\geq}0 \rightarrow j(x,v))}[;]}{j(x,v) \vdash [x'{=}v, v'{=}{-}g \,\&\, x{\geq}0]j(x,v)}[']
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\cfrac{}{j(x,v) \vdash t\geq 0 \rightarrow H-\frac{g}{2}t^2\geq 0 \rightarrow j(H-\frac{g}{2}t^2,-gt)}\rightarrow R
}{j(x,v) \vdash \forall t\geq 0\,(H-\frac{g}{2}t^2\geq 0 \rightarrow j(H-\frac{g}{2}t^2,-gt))}\forall R
}{j(x,v) \vdash \forall t\geq 0\,[x:=H-\frac{g}{2}t^2](x\geq 0 \rightarrow j(x,-gt))}[:=]
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$$

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$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{j(x,v),\ t\geq 0,\ H-\frac{g}{2}t^2\geq 0 \vdash j(H-\frac{g}{2}t^2,-gt)}{j(x,v) \vdash t\geq 0 \rightarrow H-\frac{g}{2}t^2\geq 0 \rightarrow j(H-\frac{g}{2}t^2,-gt)}\,{\rightarrow}R}{j(x,v) \vdash \forall t\geq 0\,\big(H-\frac{g}{2}t^2\geq 0 \rightarrow j(H-\frac{g}{2}t^2,-gt)\big)}\,\forall R}{j(x,v) \vdash \forall t\geq 0\,[x:=H-\frac{g}{2}t^2]\big(x\geq 0 \rightarrow j(x,-gt)\big)}\,[:=]}{j(x,v) \vdash \forall t\geq 0\,[x:=H-\frac{g}{2}t^2][v:=-gt]\big(x\geq 0 \rightarrow j(x,v)\big)}\,[:=]}{j(x,v) \vdash \forall t\geq 0\,[x:=H-\frac{g}{2}t^2;\ v:=-gt]\big(x\geq 0 \rightarrow j(x,v)\big)}\,[;]}{j(x,v) \vdash [x'=v,\ v'=-g\ \&\ x\geq 0]j(x,v)}\,[']
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# Proving Quantum the Acrophobic Bouncing Ball

$$\frac{}{2gx=2gH-v^2\land x\geq 0, H-\frac{g}{2}t^2\geq 0 \vdash 2g(H-\frac{g}{2}t^2)=2gH-(gt)^2\land(H-\frac{g}{2}t^2)\geq 0}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{j(x,v),\ t\geq 0,\ H-\frac{g}{2}t^2\geq 0 \vdash j(H-\frac{g}{2}t^2,-gt)}{j(x,v) \vdash t\geq 0 \to H-\frac{g}{2}t^2\geq 0 \to j(H-\frac{g}{2}t^2,-gt)}\,{\to}R}{j(x,v) \vdash \forall t\geq 0\,(H-\frac{g}{2}t^2\geq 0 \to j(H-\frac{g}{2}t^2,-gt))}\,\forall R}{j(x,v) \vdash \forall t\geq 0\,[x:=H-\frac{g}{2}t^2](x\geq 0 \to j(x,-gt))}\,[:=]}{j(x,v) \vdash \forall t\geq 0\,[x:=H-\frac{g}{2}t^2][v:=-gt](x\geq 0 \to j(x,v))}\,[:=]}{j(x,v) \vdash \forall t\geq 0\,[x:=H-\frac{g}{2}t^2;\ v:=-gt](x\geq 0 \to j(x,v))}\,[;]}{j(x,v) \vdash [x'=v,\ v'=-g\ \&\ x\geq 0]j(x,v)}\,[']$$

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$$\wedge R\,\frac{2gx{=}2gH{-}v^2 \vdash 2g(H{-}\frac{g}{2}t^2){=}2gH{-}(gt)^2 \qquad H{-}\frac{g}{2}t^2{\geq}0 \vdash H{-}\frac{g}{2}t^2{\geq}0}{2gx{=}2gH{-}v^2 \wedge x{\geq}0, H{-}\frac{g}{2}t^2{\geq}0 \vdash 2g(H{-}\frac{g}{2}t^2){=}2gH{-}(gt)^2 \wedge (H{-}\frac{g}{2}t^2){\geq}0}$$

$$\rightarrow R\,\frac{j(x,v), t{\geq}0, H{-}\frac{g}{2}t^2{\geq}0 \vdash j(H{-}\frac{g}{2}t^2,{-}gt)}{j(x,v) \vdash t{\geq}0 \rightarrow H{-}\frac{g}{2}t^2{\geq}0 \rightarrow j(H{-}\frac{g}{2}t^2,{-}gt)}$$

$$\forall R\,\frac{j(x,v) \vdash t{\geq}0 \rightarrow H{-}\frac{g}{2}t^2{\geq}0 \rightarrow j(H{-}\frac{g}{2}t^2,{-}gt)}{j(x,v) \vdash \forall t{\geq}0\,(H{-}\frac{g}{2}t^2{\geq}0 \rightarrow j(H{-}\frac{g}{2}t^2,{-}gt))}$$

$$[:=]\,\frac{j(x,v) \vdash \forall t{\geq}0\,(H{-}\frac{g}{2}t^2{\geq}0 \rightarrow j(H{-}\frac{g}{2}t^2,{-}gt))}{j(x,v) \vdash \forall t{\geq}0\,[x{:=}H{-}\frac{g}{2}t^2](x{\geq}0 \rightarrow j(x,{-}gt))}$$

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- Is Quantum done with his safety proof?

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$$\wedge R\,\dfrac{\mathbb{R}\,\dfrac{*}{2gx=2gH-v^2 \vdash 2g(H-\frac{g}{2}t^2)=2gH-(gt)^2} \quad \mathrm{id}\,\dfrac{*}{H-\frac{g}{2}t^2\geq 0 \vdash H-\frac{g}{2}t^2\geq 0}}{2gx=2gH-v^2\wedge x\geq 0, H-\frac{g}{2}t^2\geq 0 \vdash 2g(H-\frac{g}{2}t^2)=2gH-(gt)^2\wedge(H-\frac{g}{2}t^2)\geq 0}$$

$$\begin{array}{ll}
 & j(x,v),\ t\geq 0,\ H-\frac{g}{2}t^2\geq 0 \vdash j(H-\frac{g}{2}t^2,-gt) \\
\to R & \hline
 & j(x,v) \vdash t\geq 0 \to H-\frac{g}{2}t^2\geq 0 \to j(H-\frac{g}{2}t^2,-gt) \\
\forall R & \hline
 & j(x,v) \vdash \forall t\geq 0\,(H-\frac{g}{2}t^2\geq 0 \to j(H-\frac{g}{2}t^2,-gt)) \\
[:=] & \hline
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[:=] & \hline
 & j(x,v) \vdash \forall t\geq 0\,[x:=H-\frac{g}{2}t^2][v:=-gt](x\geq 0 \to j(x,v)) \\
[;] & \hline
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['] & \hline
 & j(x,v) \vdash [x'=v,\ v'=-g\ \&\ x\geq 0]j(x,v)
\end{array}$$

- Is Quantum done with his safety proof?
- Oh no! The solutions we sneaked into $[']$ only solve the ODE/IVP if $x = 0, v = 0$ which $j(x,v)$ can't guarantee!

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{j(x,v),\, t\geq 0,\, H-\frac{g}{2}t^2\geq 0 \vdash j(H-\frac{g}{2}t^2,-gt)}{j(x,v) \vdash t\geq 0 \to H-\frac{g}{2}t^2\geq 0 \to j(H-\frac{g}{2}t^2,-gt)}\,\to R}{j(x,v) \vdash \forall t\geq 0\,(H-\frac{g}{2}t^2\geq 0 \to j(H-\frac{g}{2}t^2,-gt))}\,\forall R}{j(x,v) \vdash \forall t\geq 0\,[x:=H-\frac{g}{2}t^2](x\geq 0 \to j(x,-gt))}\,[:=]}{j(x,v) \vdash \forall t\geq 0\,[x:=H-\frac{g}{2}t^2][v:=-gt](x\geq 0 \to j(x,v))}\,[:=]}{j(x,v) \vdash \forall t\geq 0\,[x:=H-\frac{g}{2}t^2; v:=-gt](x\geq 0 \to j(x,v))}\,[;]}{j(x,v) \vdash [x'=v, v'=-g \,\&\, x\geq 0]j(x,v)}\,[']$$

- Is Quantum done with his safety proof?
- Oh no! The solutions we sneaked into $[']$ only solve the ODE/IVP if $x=0, v=0$ which $j(x,v)$ can't guarantee!
- Never use solutions without proof! $\rightsquigarrow$ redo proof with true solution

# Quantum the Provably Safe Bouncing Ball

**Proposition (Quantum can bounce around safely)**

$$0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 = c \rightarrow$$
$$[(x' = v, v' = -g \,\&\, x \geq 0; (?x = 0; v{:=}-cv \cup ?x \neq 0))^*](0 \leq x \wedge x \leq H)$$

$$\texttt{@requires}(0 \leq x \wedge x = H \wedge v = 0)$$
$$\texttt{@requires}(g > 0 \wedge c = 1)$$
$$\texttt{@ensures}(0 \leq x \wedge x \leq H)$$
$$\{\{x' = v, v' = -g \,\&\, x \geq 0\};$$
$$(?x = 0; v{:=}-cv \cup ?x \neq 0))\}^* \texttt{@invariant}(2gx = 2gH - v^2 \wedge x \geq 0)$$

**Invariant Contracts**

Invariants play a crucial role in CPS design. Capture them if you can. Use `@invariant` contracts in your hybrid programs.

Note: constants $c = 1 \wedge g > 0$ that never change are often elided

# Outline

# Invariants

The lion's share of understanding comes from understanding what does change (variants/progress measures) and what doesn't change (invariants).

**Invariants are a fundamental force of CS**

**Variants are another fundamental force of CS**


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