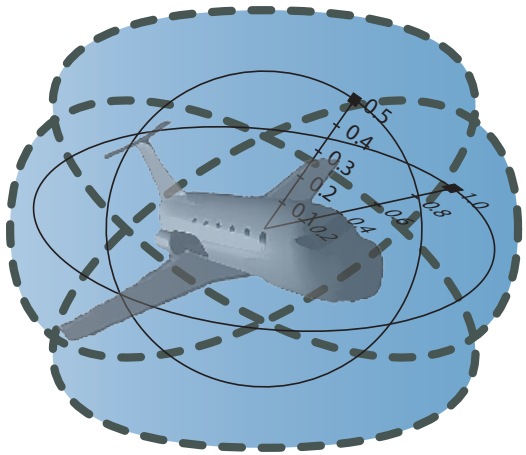
06: Truth & Proof

15-424: Foundations of Cyber-Physical Systems

André Platzer

aplatzer@cs.cmu.edu
Computer Science Department
Carnegie Mellon University, Pittsburgh, PA



Outline

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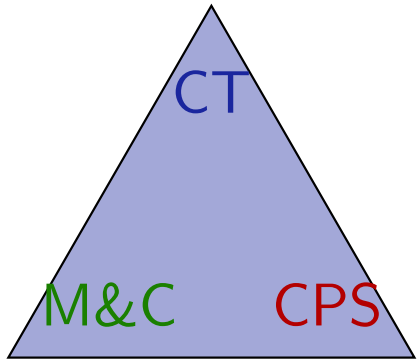
Learning Objectives

Truth & Proof

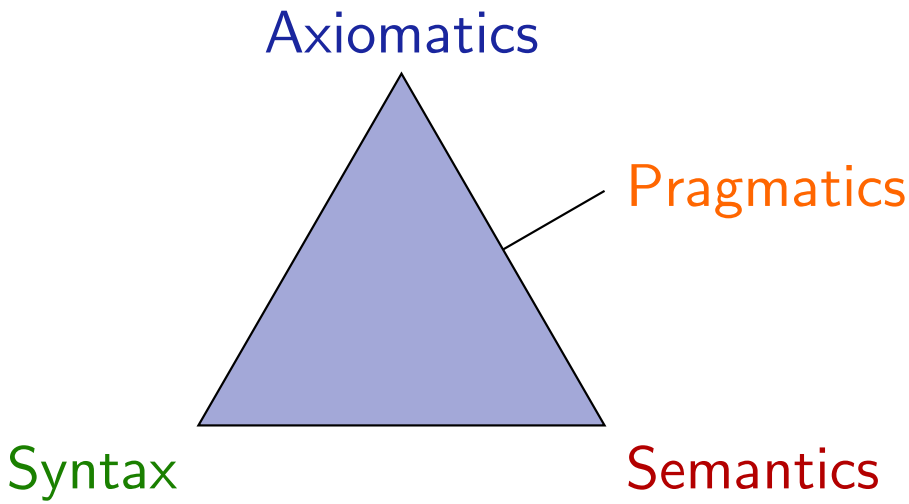
systematic reasoning for CPS
verifying CPS models at scale
pragmatics: how to use axiomatics to justify truth
structure of proofs and their arithmetic



discrete+continuous relation
with evolution domains

analytic skills for CPS

Logical Trinity with Extra Leg



Syntax defines the notation
What problems are we allowed to write down?

Semantics what carries meaning.
What real or mathematical objects does the syntax stand for?

Axiomatics internalizes semantic relations into universal syntactic transformations.

Pragmatics how to use axiomatics to justify syntactic rendition of semantical concepts. How to do a proof?

Outline

Sequent Calculus

Definition (Sequent)

$$\Gamma \vdash \Delta$$

has the same meaning as $\bigwedge_{P \in \Gamma} P \rightarrow \bigvee_{Q \in \Delta} Q$.
The *antecedent* Γ and *succedent* Δ are finite sets of d$\mathcal{L}$ formulas.

Definition (Soundness of sequent calculus proof rules)

$$\frac{\Gamma_1 \vdash \Delta_1 \quad \ldots \quad \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta}$$

is *sound* iff validity of all premises implies validity of conclusion:

$$\text{If } \models (\Gamma_1 \vdash \Delta_1) \text{ and } \ldots \text{ and } \models (\Gamma_n \vdash \Delta_n) \text{ then } \models (\Gamma \vdash \Delta)$$

Sequent Calculus

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Definition (Soundness of sequent calculus proof rules)

construct proofs 

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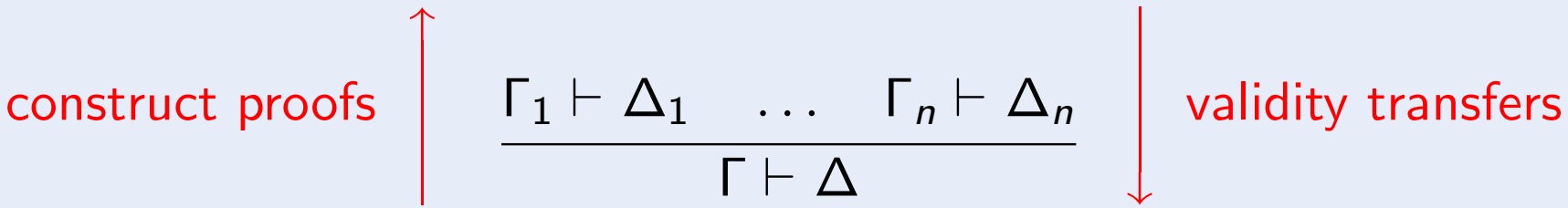

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Developed on the board:

1. Proof rules for propositional logic
2. Proofs with dynamics
3. Contextual equivalence rewriting / congruence
4. Quantifier proof rules
5. Real arithmetic
6. Structural proof rules

See lecture notes for details [1].

Simple Propositional Example Proof in Sequent Calculus

$$\rightarrow R \frac{}{\vdash v^2 \leq 10 \land b > 0 \rightarrow b > 0 \land (\neg(v \geq 0) \lor v^2 \leq 10)}$$

Simple Propositional Example Proof in Sequent Calculus

$$\dfrac{\dfrac{}{v^2 \leq 10 \land b > 0 \vdash b > 0 \land (\neg(v \geq 0) \lor v^2 \leq 10)}\,\land R}{\vdash v^2 \leq 10 \land b > 0 \rightarrow b > 0 \land (\neg(v \geq 0) \lor v^2 \leq 10)}\,\rightarrow R$$

Simple Propositional Example Proof in Sequent Calculus

$$\dfrac{\dfrac{\dfrac{}{v^2 \leq 10 \land b > 0 \vdash b > 0}\land L \qquad \dfrac{}{v^2 \leq 10 \land b > 0 \vdash \neg(v \geq 0) \lor v^2 \leq 10}\lor R}{v^2 \leq 10 \land b > 0 \vdash b > 0 \land (\neg(v \geq 0) \lor v^2 \leq 10)}\land R}{\vdash v^2 \leq 10 \land b > 0 \to b > 0 \land (\neg(v \geq 0) \lor v^2 \leq 10)}\to R$$

Simple Propositional Example Proof in Sequent Calculus

$$\dfrac{\dfrac{\dfrac{??\ \dfrac{}{v^2 \leq 10, b > 0 \vdash b > 0}}{\wedge L\ \dfrac{}{v^2 \leq 10 \wedge b > 0 \vdash b > 0}} \qquad \vee R\ \dfrac{}{v^2 \leq 10 \wedge b > 0 \vdash \neg(v \geq 0) \vee v^2 \leq 10}}{\wedge R\ \dfrac{}{v^2 \leq 10 \wedge b > 0 \vdash b > 0 \wedge (\neg(v \geq 0) \vee v^2 \leq 10)}}}{\rightarrow R\ \dfrac{}{\vdash v^2 \leq 10 \wedge b > 0 \rightarrow b > 0 \wedge (\neg(v \geq 0) \vee v^2 \leq 10)}}$$

Simple Propositional Example Proof in Sequent Calculus

$$\dfrac{\dfrac{\dfrac{\dfrac{}{v^2 \leq 10, b>0 \vdash b>0}\,*}{v^2 \leq 10 \land b>0 \vdash b>0}\,??}{v^2 \leq 10 \land b>0 \vdash b>0 \land (\neg(v \geq 0) \lor v^2 \leq 10)}\,\land R \qquad \dfrac{}{v^2 \leq 10 \land b>0 \vdash \neg(v \geq 0) \lor v^2 \leq 10}\,\lor R}{\vdash v^2 \leq 10 \land b>0 \to b>0 \land (\neg(v \geq 0) \lor v^2 \leq 10)}\,\land L \quad \to R$$

Simple Propositional Example Proof in Sequent Calculus

$$\dfrac{\dfrac{\dfrac{}{v^2 \leq 10, b > 0 \vdash b > 0}\,{}^{*}}{v^2 \leq 10 \land b > 0 \vdash b > 0}\,{}^{??}\land L \qquad \dfrac{\dfrac{}{v^2 \leq 10 \land b > 0 \vdash \neg(v \geq 0), v^2 \leq 10}\,{}^{\land L}}{v^2 \leq 10 \land b > 0 \vdash \neg(v \geq 0) \lor v^2 \leq 10}\,{}^{\lor R}}{v^2 \leq 10 \land b > 0 \vdash b > 0 \land (\neg(v \geq 0) \lor v^2 \leq 10)}\,{}^{\land R}$$
$$\vdash v^2 \leq 10 \land b > 0 \to b > 0 \land (\neg(v \geq 0) \lor v^2 \leq 10)\,{}^{\to R}$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{}{v^2\leq 10, b>0 \vdash b>0}\,{}^{??}\,*}{v^2\leq 10 \land b>0 \vdash b>0}\,\land L \qquad \dfrac{\dfrac{}{v^2\leq 10, b>0 \vdash \neg(v\geq 0), v^2\leq 10}\,{}^{??}}{v^2\leq 10 \land b>0 \vdash \neg(v\geq 0), v^2\leq 10}\,\land L}{v^2\leq 10 \land b>0 \vdash \neg(v\geq 0) \lor v^2\leq 10}\,\lor R}{v^2\leq 10 \land b>0 \vdash b>0 \land (\neg(v\geq 0) \lor v^2\leq 10)}\,\land R$$

$$\dfrac{}{\vdash v^2\leq 10 \land b>0 \to b>0 \land (\neg(v\geq 0) \lor v^2\leq 10)}\,\to R$$

Simple Propositional Example Proof in Sequent Calculus

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{??}\;\dfrac{}{v^2\leq 10, b>0 \vdash b>0}\,*}{\wedge L\;v^2\leq 10 \wedge b>0 \vdash b>0} \qquad \dfrac{\dfrac{\dfrac{}{??}\;\dfrac{}{v^2\leq 10, b>0 \vdash \neg(v\geq 0), v^2\leq 10}\,*}{\wedge L\;v^2\leq 10 \wedge b>0 \vdash \neg(v\geq 0), v^2\leq 10}}{\vee R\;v^2\leq 10 \wedge b>0 \vdash \neg(v\geq 0) \vee v^2\leq 10}}{\wedge R\;v^2\leq 10 \wedge b>0 \vdash b>0 \wedge (\neg(v\geq 0) \vee v^2\leq 10)}}{\to R\;\vdash v^2\leq 10 \wedge b>0 \to b>0 \wedge (\neg(v\geq 0) \vee v^2\leq 10)}
$$

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Simple Dynamics Example Proof in Sequent Calculus

$$[;]\ \frac{}{\vdash [a := -b; c := 10]\big(v^2 \leq 10 \land -a > 0 \rightarrow b > 0 \land (\neg(v \geq 0) \lor v^2 \leq c)\big)}$$

Simple Dynamics Example Proof in Sequent Calculus

$$[:=]\ \frac{\vdash [a := -b][c := 10]\big(v^2 \leq 10 \land -a > 0 \rightarrow b > 0 \land (\neg(v \geq 0) \lor v^2 \leq c)\big)}{[;]\ \vdash [a := -b; c := 10]\big(v^2 \leq 10 \land -a > 0 \rightarrow b > 0 \land (\neg(v \geq 0) \lor v^2 \leq c)\big)}$$

Simple Dynamics Example Proof in Sequent Calculus

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\dfrac{}{\vdash [c := 10]\big(v^2 \leq 10 \land -(-b) > 0 \rightarrow b > 0 \land (\neg(v \geq 0) \lor v^2 \leq c)\big)}[:=]
}{\vdash [a := -b][c := 10]\big(v^2 \leq 10 \land -a > 0 \rightarrow b > 0 \land (\neg(v \geq 0) \lor v^2 \leq c)\big)}[:=]
}{\vdash [a := -b; c := 10]\big(v^2 \leq 10 \land -a > 0 \rightarrow b > 0 \land (\neg(v \geq 0) \lor v^2 \leq c)\big)}[;]
$$

Simple Dynamics Example Proof in Sequent Calculus

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash v^2 \leq 10 \land -(-b) > 0 \rightarrow b > 0 \land (\neg(v \geq 0) \lor v^2 \leq 10)}}{\vdash [c := 10](v^2 \leq 10 \land -(-b) > 0 \rightarrow b > 0 \land (\neg(v \geq 0) \lor v^2 \leq c))}\,[:=]}{\vdash [a := -b][c := 10](v^2 \leq 10 \land -a > 0 \rightarrow b > 0 \land (\neg(v \geq 0) \lor v^2 \leq c))}\,[:=]}{\vdash [a := -b; c := 10](v^2 \leq 10 \land -a > 0 \rightarrow b > 0 \land (\neg(v \geq 0) \lor v^2 \leq c))}\,[;]$$

Simple Dynamics Example Proof in Sequent Calculus

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\dfrac{*}{v^2\leq 10, b>0 \vdash b>0}\,??
}{v^2\leq 10 \land b>0 \vdash b>0}\,\land L
\qquad
\dfrac{*}{v^2\leq 10, b>0 \vdash \neg(v\geq 0), v^2\leq 10}\,??
}{v^2\leq 10 \land b>0 \vdash \neg(v\geq 0), v^2\leq 10}\,\land L
}{v^2\leq 10 \land b>0 \vdash \neg(v\geq 0) \lor v^2\leq 10}\,\lor R
}{v^2\leq 10 \land b>0 \vdash b>0 \land (\neg(v\geq 0) \lor v^2\leq 10)}\,\land R
}{\vdash v^2\leq 10 \land b>0 \to b>0 \land (\neg(v\geq 0) \lor v^2\leq 10)}\,\to R
$$

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\vdash v^2\leq 10 \land -(-b)>0 \to b>0 \land (\neg(v\geq 0) \lor v^2\leq 10)
}{\vdash [c:=10]\big(v^2\leq 10 \land -(-b)>0 \to b>0 \land (\neg(v\geq 0) \lor v^2\leq c)\big)}\,[:=]
}{\vdash [a:=-b][c:=10]\big(v^2\leq 10 \land -a>0 \to b>0 \land (\neg(v\geq 0) \lor v^2\leq c)\big)}\,[:=]
}{\vdash [a:=-b; c:=10]\big(v^2\leq 10 \land -a>0 \to b>0 \land (\neg(v\geq 0) \lor v^2\leq c)\big)}\,[;]
$$

Simple Dynamics Example Proof in Sequent Calculus

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\dfrac{*}{v^2\leq 10, b>0 \vdash b>0}\,??
}{v^2\leq 10 \land b>0 \vdash b>0}\,\land L
\qquad
\dfrac{*}{v^2\leq 10, b>0 \vdash \neg(v\geq 0), v^2\leq 10}\,??
}{v^2\leq 10 \land b>0 \vdash \neg(v\geq 0), v^2\leq 10}\,\land L
}{v^2\leq 10 \land b>0 \vdash b>0 \land (\neg(v\geq 0) \lor v^2\leq 10)}\,\land R
}{\vdash v^2\leq 10 \land b>0 \to b>0 \land (\neg(v\geq 0) \lor v^2\leq 10)}\,\to R
$$

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\vdash v^2\leq 10 \land -(-b)>0 \to b>0 \land (\neg(v\geq 0) \lor v^2\leq 10)
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}{\vdash [a:=-b][c:=10]\big(v^2\leq 10 \land -a>0 \to b>0 \land (\neg(v\geq 0) \lor v^2\leq c)\big)}\,[:=]
}{\vdash [a:=-b; c:=10]\big(v^2\leq 10 \land -a>0 \to b>0 \land (\neg(v\geq 0) \lor v^2\leq c)\big)}\,[;]
$$

Need some real arithmetic

Here: to glue previous propositional proof with this dynamic proof

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Taming Arithmetic: Weakening

$$\text{WR}\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash P, \Delta} \qquad \text{WL}\ \frac{\Gamma \vdash \Delta}{\Gamma, P \vdash \Delta}$$

$$\text{WL}\frac{r \geq 0 \vdash 0 \leq r \leq r}{A, r \geq 0 \vdash 0 \leq r \leq r}$$

Throw arithmetic distraction A away by weakening since proof is independent of A.

Occam's assumption razor

Think how hard it would be to prove a theorem with all the facts in all books of mathematics as assumptions.
Compared to a proof from just the two facts that matter.

Taming Arithmetic: Extreme Instantiation

$$\forall R\ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta}(y \notin \Gamma, \Delta) \qquad \exists R\ \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta}$$

$$\forall L\ \frac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta} \qquad \exists L\ \frac{\Gamma, p(y) \vdash \Delta}{\Gamma, \exists x\, p(x) \vdash \Delta}(y \notin \Gamma, \Delta)$$

$$[']\ \frac{}{\Gamma \vdash [x' = f(x)\,\&\, Q]P}$$

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$$\forall R\ \frac{}{\Gamma \vdash \forall t{\geq}0\ \big((\forall 0{\leq}s{\leq}t\ [x := y(s)]Q){\rightarrow}[x := y(t)]P\big)}$$
$$[']\ \frac{}{\Gamma \vdash [x' = f(x)\,\&\,Q]P}$$

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$$\cfrac{\cfrac{\cfrac{}{\Gamma \vdash t{\geq}0 \rightarrow \big((\forall 0{\leq}s{\leq}t\, [x := y(s)]Q) \rightarrow [x := y(t)]P\big)}\rightarrow R}{\Gamma \vdash \forall t{\geq}0\, \big((\forall 0{\leq}s{\leq}t\, [x := y(s)]Q) \rightarrow [x := y(t)]P\big)}\forall R}{\Gamma \vdash [x' = f(x)\,\&\, Q]P}[']$$

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$$\rightarrow R\ \frac{\Gamma, t{\geq}0, \forall 0{\leq}s{\leq}t\, [x := y(s)]Q \vdash [x := y(t)]P}{\Gamma, t{\geq}0 \vdash (\forall 0{\leq}s{\leq}t\, [x := y(s)]Q) \rightarrow [x := y(t)]P}$$

$$\rightarrow R\ \frac{\Gamma, t{\geq}0 \vdash (\forall 0{\leq}s{\leq}t\, [x := y(s)]Q) \rightarrow [x := y(t)]P}{\Gamma \vdash t{\geq}0 \rightarrow ((\forall 0{\leq}s{\leq}t\, [x := y(s)]Q) \rightarrow [x := y(t)]P)}$$

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$$\forall R\ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta}(y \notin \Gamma, \Delta) \qquad \exists R\ \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta}$$

$$\forall L\ \frac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta} \qquad \exists L\ \frac{\Gamma, p(y) \vdash \Delta}{\Gamma, \exists x\, p(x) \vdash \Delta}(y \notin \Gamma, \Delta)$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{t{\geq}0 \vdash 0{\leq}t{\leq}t, [x := y(t)]P}\,*\qquad \overline{\Gamma, t{\geq}0, [x := y(t)]Q \vdash [x := y(t)]P}}{\Gamma, t{\geq}0, 0{\leq}t{\leq}t{\to}[x := y(t)]Q \vdash [x := y(t)]P}\,{\to}L}{\Gamma, t{\geq}0, \forall 0{\leq}s{\leq}t\, [x := y(s)]Q \vdash [x := y(t)]P}\,\forall L}{\Gamma, t{\geq}0 \vdash (\forall 0{\leq}s{\leq}t\, [x := y(s)]Q){\to}[x := y(t)]P}\,{\to}R}{\Gamma \vdash t{\geq}0{\to}\big((\forall 0{\leq}s{\leq}t\, [x := y(s)]Q){\to}[x := y(t)]P\big)}\,{\to}R}{\Gamma \vdash \forall t{\geq}0\, \big((\forall 0{\leq}s{\leq}t\, [x := y(s)]Q){\to}[x := y(t)]P\big)}\,\forall R}{\Gamma \vdash [x' = f(x)\,\&\, Q]P}\,[']$$

Taming Arithmetic: Extreme Instantiation

$$\forall R\ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta}(y \notin \Gamma, \Delta) \qquad \exists R\ \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta}$$

$$\forall L\ \frac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta} \qquad \exists L\ \frac{\Gamma, p(y) \vdash \Delta}{\Gamma, \exists x\, p(x) \vdash \Delta}(y \notin \Gamma, \Delta)$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{*}{t{\geq}0 \vdash 0{\leq}t{\leq}t, [x := y(t)]P} \quad \cfrac{\cdots}{\Gamma, t{\geq}0, [x := y(t)]Q \vdash [x := y(t)]P}}{\Gamma, t{\geq}0, 0{\leq}t{\leq}t{\to}[x := y(t)]Q \vdash [x := y(t)]P}\,{\to}L}{\Gamma, t{\geq}0, \forall 0{\leq}s{\leq}t\, [x := y(s)]Q \vdash [x := y(t)]P}\,\forall L}{\Gamma, t{\geq}0 \vdash (\forall 0{\leq}s{\leq}t\, [x := y(s)]Q){\to}[x := y(t)]P}\,{\to}R}{\Gamma \vdash t{\geq}0{\to}\big((\forall 0{\leq}s{\leq}t\, [x := y(s)]Q){\to}[x := y(t)]P\big)}\,{\to}R}{\Gamma \vdash \forall t{\geq}0\, \big((\forall 0{\leq}s{\leq}t\, [x := y(s)]Q){\to}[x := y(t)]P\big)}\,\forall R}{\Gamma \vdash [x' = f(x)\,\&\,Q]P}\,[']$$

Taming Arithmetic: Creative Cuts

$$=\mathrm{R}\ \frac{\Gamma, x = e \vdash p(e), \Delta}{\Gamma, x = e \vdash p(x), \Delta}$$

$$=\mathrm{L}\ \frac{\Gamma, x = e, p(e) \vdash \Delta}{\Gamma, x = e, p(x) \vdash \Delta}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(x-y)^2 \leq 0 \vdash x = y}\,\mathbb{R}^{*}}{(x-y)^2 \leq 0 \vdash x = y, p(x)}\,\mathrm{WR}}{(x-y)^2 \leq 0, p(y) \vdash x = y, p(x)}\,\mathrm{WL} \qquad \dfrac{\dfrac{\dfrac{}{p(y), x = y \vdash p(y)}\,??^{*}}{p(y), x = y \vdash p(x)}\,=\mathrm{R}}{(x-y)^2 \leq 0, p(y), x = y \vdash p(x)}\,\mathrm{WL}}{(x-y)^2 \leq 0, p(y) \vdash p(x)}\,\mathrm{cut}}{(x-y)^2 \leq 0 \wedge p(y) \vdash p(x)}\,\wedge\mathrm{L}}{\vdash (x-y)^2 \leq 0 \wedge p(y) \to p(x)}\,\to\mathrm{R}$$


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