05: Dynamical Systems & Dynamic Axioms

15-424: Foundations of Cyber-Physical Systems

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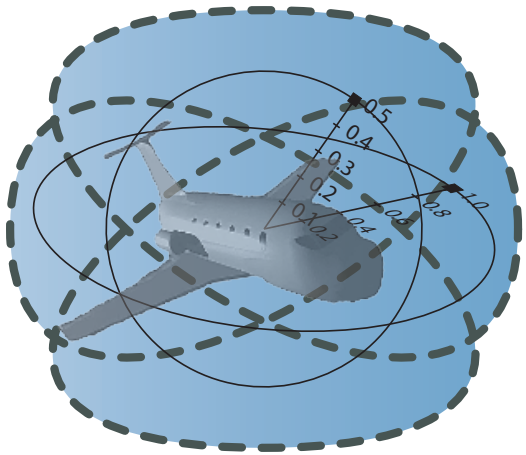


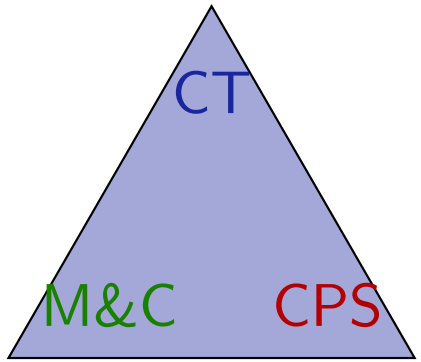
Outline

Outline

Learning Objectives

Dynamical Systems & Dynamic Axioms

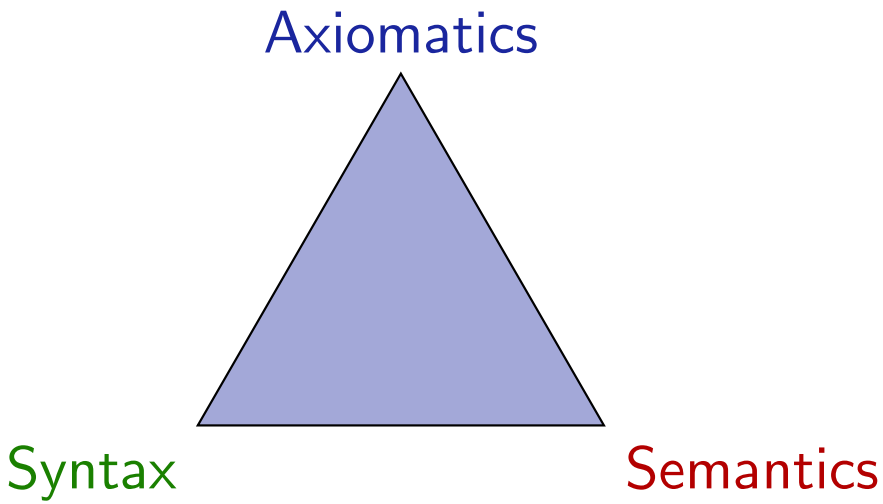
rigorous reasoning about CPS
d$\mathcal{L}$ as verification language



cyber+physics interaction
relate discrete+continuous

align semantics+reasoning
operational CPS effects

Logical Trinity



Syntax defines the notation
What problems are we allowed to write down?

Semantics what carries meaning.
What real or mathematical objects does the syntax stand for?

Axiomatics internalizes semantic relations into universal syntactic transformations.
How does the semantics of A relate to semantics of $A \wedge B$, syntactically? If A is true, is $A \wedge B$ true, too? Conversely?

Outline

Approach

Logical guiding principle: Compositionality

1. Every CPS is modeled by a hybrid program (or game . . .)
2. All hybrid programs are combinations of simpler hybrid programs (by a program operator such as $\cup$ and ; and *)
3. All CPS can be analyzed if only we identify one suitable analysis technique for each operator.

Outline

Differential Dynamic Logic d$\mathcal{L}$: Semantics

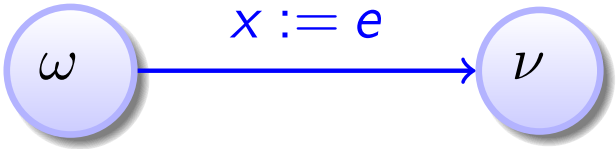
Definition (Hybrid program semantics) $([\![\cdot]\!] : \text{HP} \to \wp(\mathcal{S} \times \mathcal{S}))$

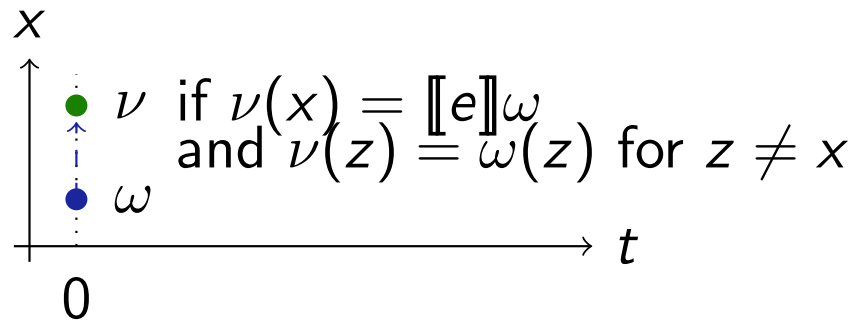
$$
\begin{aligned}
[\![x := e]\!] &= \{(\omega, \nu) \ : \ \nu = \omega \text{ except } [\![x]\!]\nu = [\![e]\!]\omega\} \\
[\![?Q]\!] &= \{(\omega, \omega) \ : \ \omega \in [\![Q]\!]\} \\
[\![x' = f(x)]\!] &= \{(\varphi(0), \varphi(r)) \ : \ \varphi \models x' = f(x) \text{ for some duration } r\} \\
[\![\alpha \cup \beta]\!] &= [\![\alpha]\!] \cup [\![\beta]\!] \\
[\![\alpha; \beta]\!] &= [\![\alpha]\!] \circ [\![\beta]\!] \\
[\![\alpha^*]\!] &= \bigcup_{n \in \mathbb{N}} [\![\alpha^n]\!]
\end{aligned}
$$

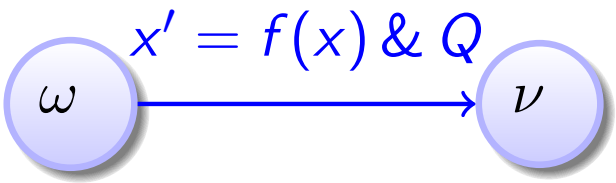
Definition (d$\mathcal{L}$ semantics) $([\![\cdot]\!] : \text{Fml} \to \wp(\mathcal{S}))$

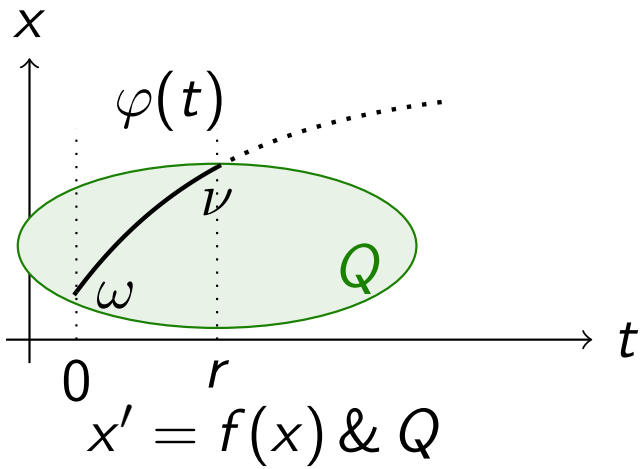
$$
\begin{aligned}
[\![\theta \geq \eta]\!] &= \{\omega \ : \ [\![\theta]\!]\omega \geq [\![\eta]\!]\omega\} \\
[\![\neg\phi]\!] &= ([\![\phi]\!])^{\complement} \\
[\![\phi \wedge \psi]\!] &= [\![\phi]\!] \cap [\![\psi]\!] \\
[\![\langle\alpha\rangle\phi]\!] &= [\![\alpha]\!] \circ [\![\phi]\!] = \{\omega \ : \ \nu \in [\![\phi]\!] \text{ for some } \nu : \ (\omega, \nu) \in [\![\alpha]\!]\} \\
[\![[\alpha]\phi]\!] &= [\![\neg\langle\alpha\rangle\neg\phi]\!] = \{\omega \ : \ \nu \in [\![\phi]\!] \text{ for all } \nu : \ (\omega, \nu) \in [\![\alpha]\!]\} \\
[\![\exists x \, \phi]\!] &= \{\omega \ : \ \omega_x^r \in [\![\phi]\!] \text{ for some } r \in \mathbb{R}\}
\end{aligned}
$$

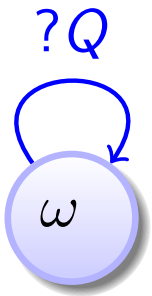
Differential Dynamic Logic d$\mathcal{L}$: Transition Semantics



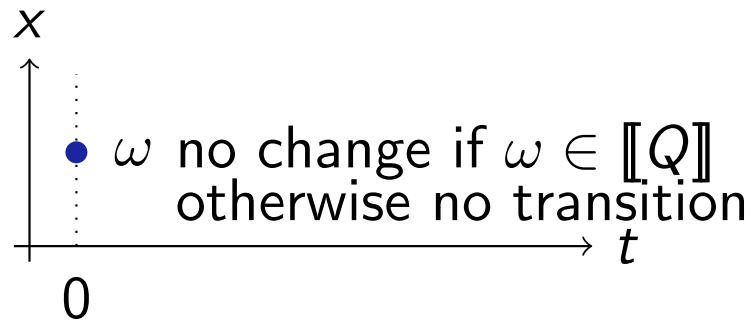




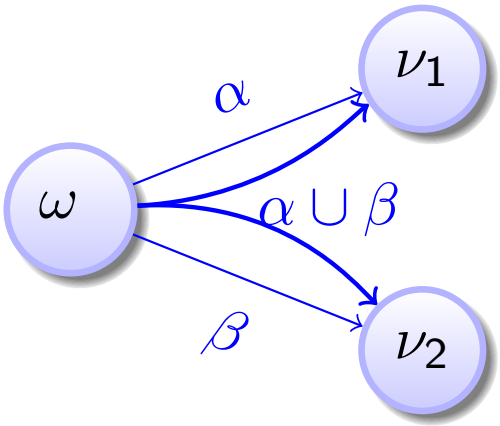


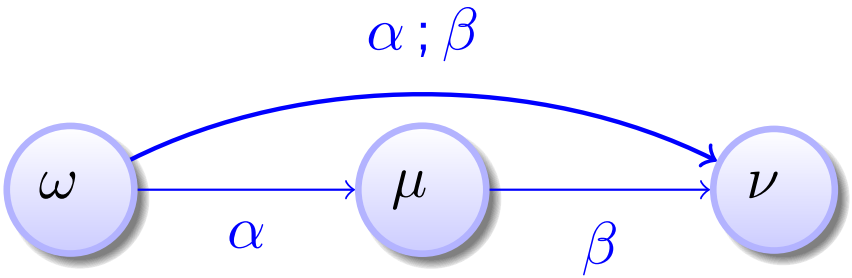


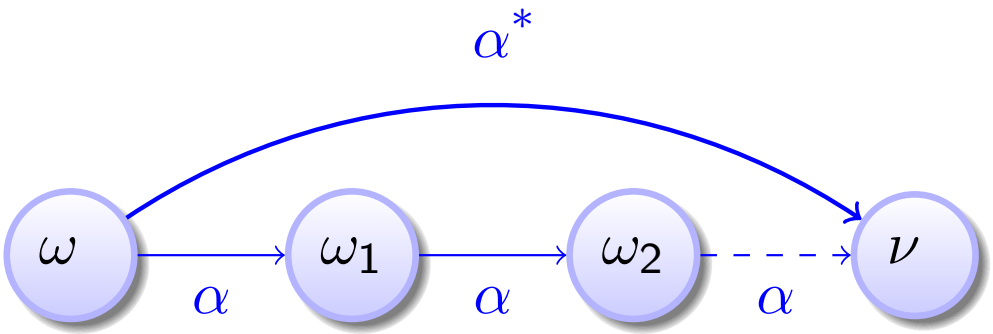
if $\omega \in [\![Q]\!]$

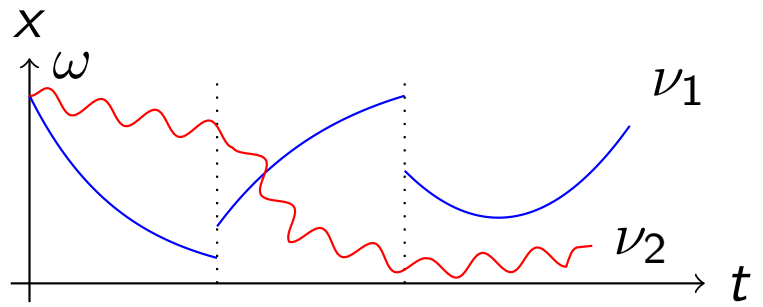


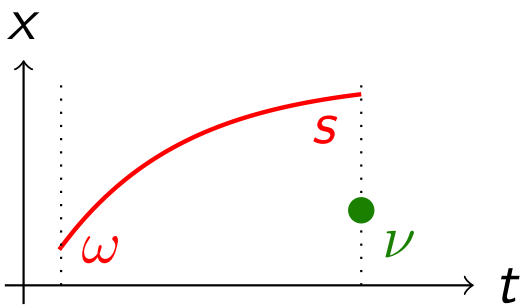
Differential Dynamic Logic d$\mathcal{L}$: Transition Semantics

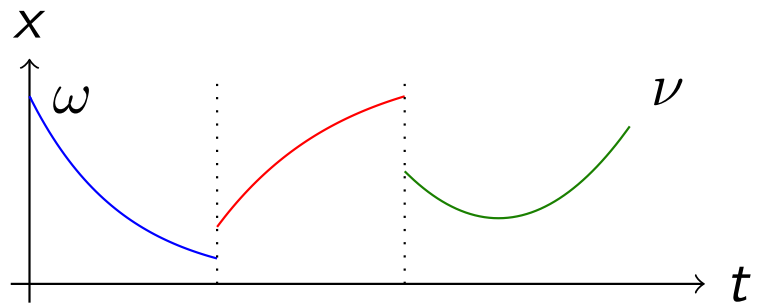




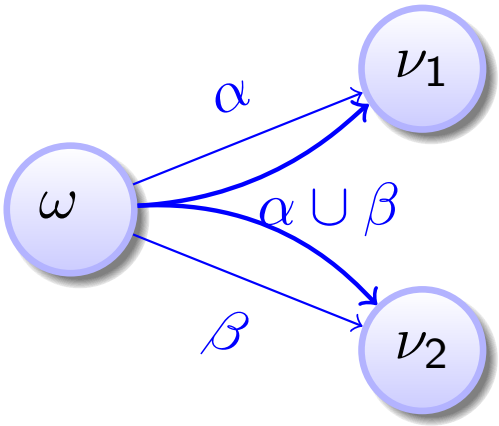


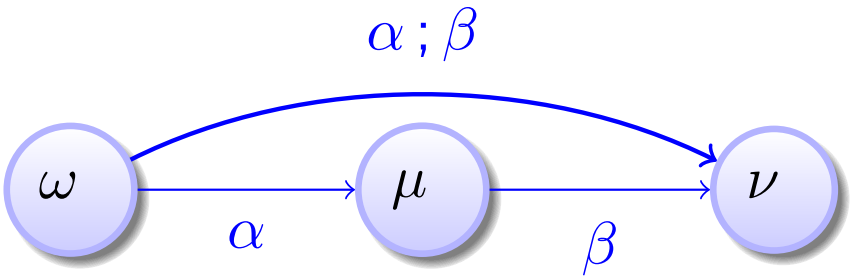


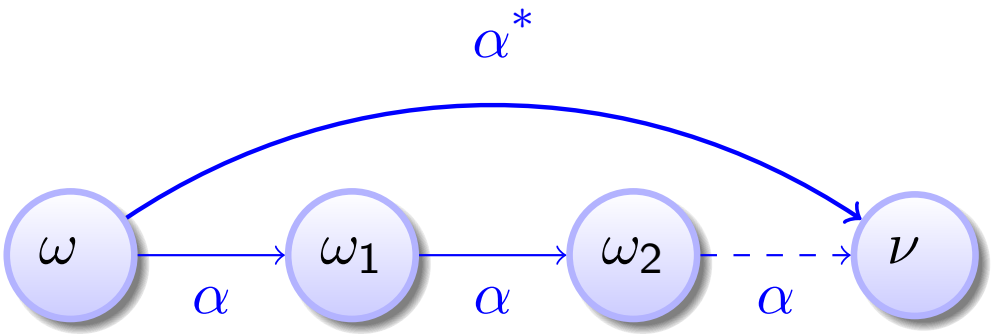


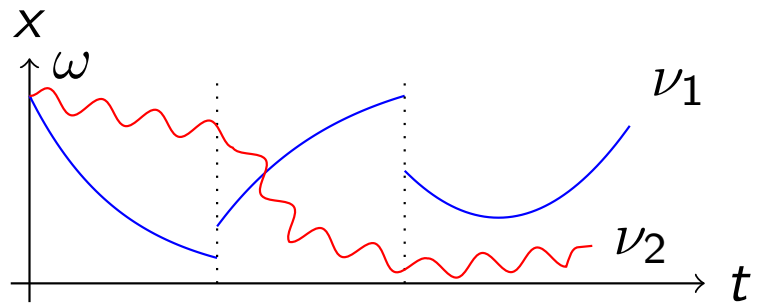


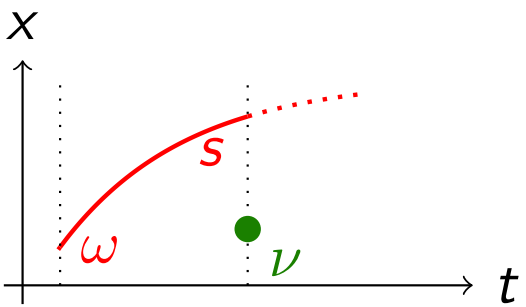
Differential Dynamic Logic d$\mathcal{L}$: Transition Semantics

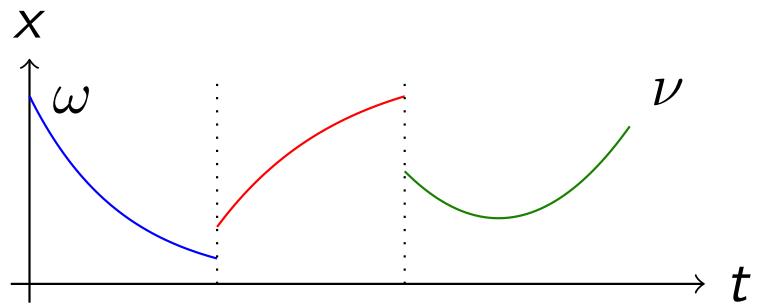




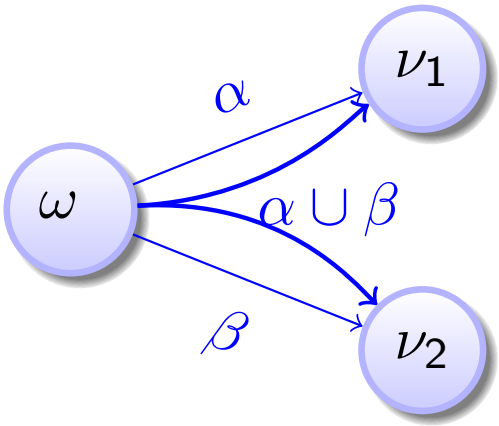


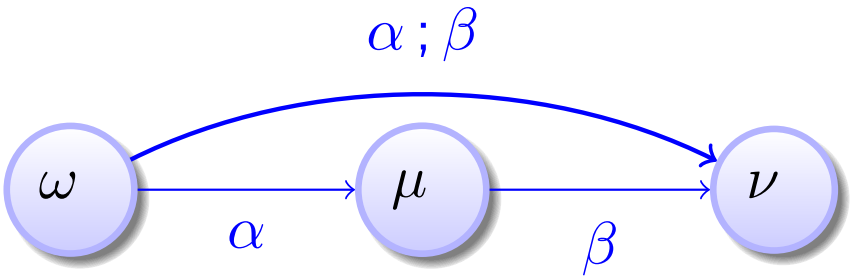


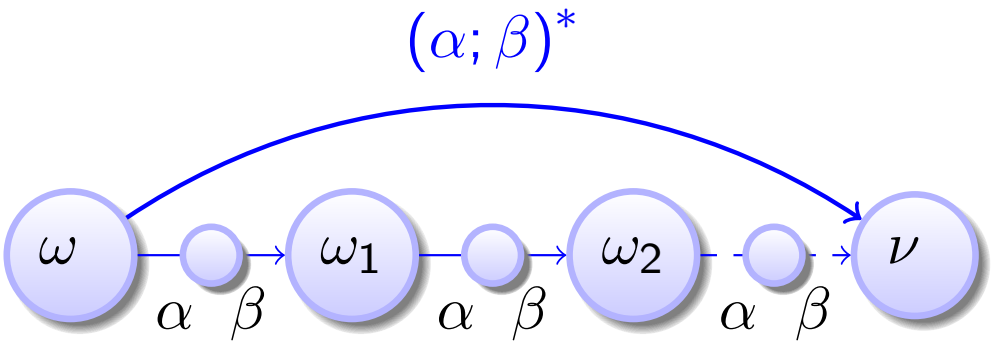


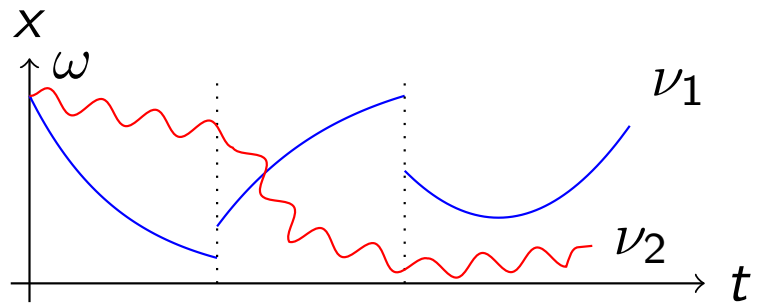


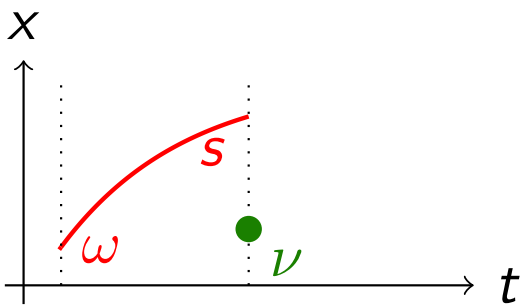
Differential Dynamic Logic d$\mathcal{L}$: Transition Semantics

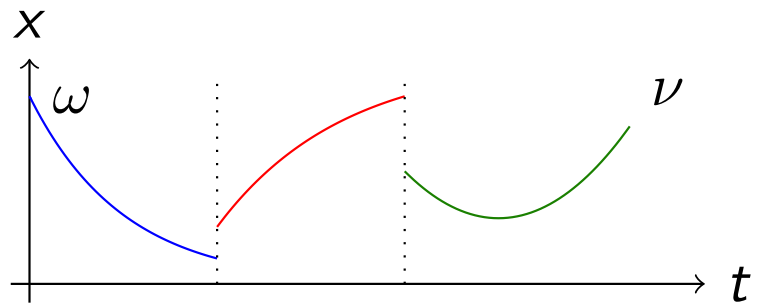






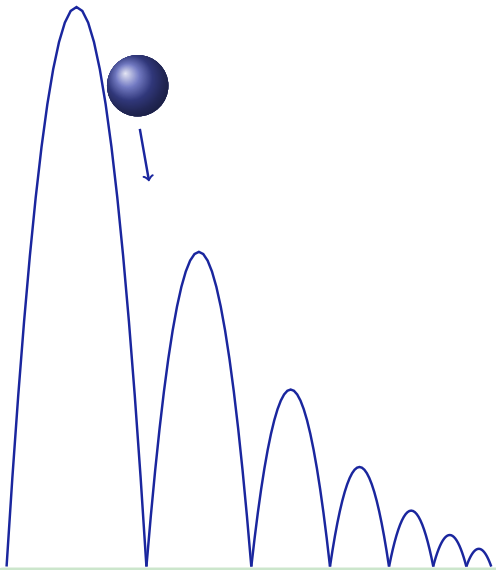






Outline

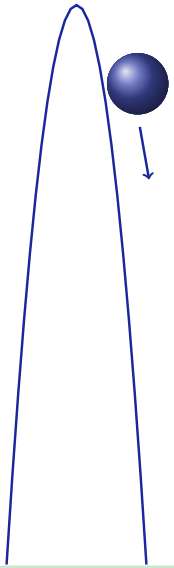
Conjecture: Quantum the Acrophobic Bouncing Ball



Example (Quantum the Bouncing Ball)

$$0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \rightarrow$$
$$\left[\left(x' = v, v' = -g \,\&\, x \geq 0; (?x = 0; v := -cv \cup ?x \neq 0)\right)^*\right](0 \leq x \wedge x \leq H)$$

Conjecture: Quantum the Acrophobic Bouncing Ball

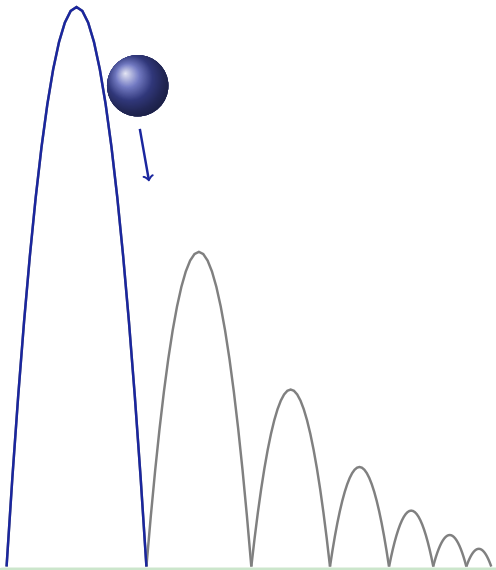


Example (Quantum the Bouncing Ball) (Single-hop)

$$0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \rightarrow$$
$$\left[x' = v, v' = -g \,\&\, x \geq 0; (?x = 0; v := -cv \cup ?x \neq 0) \right] (0 \leq x \wedge x \leq H)$$

Removing the repetition grotesquely changes the behavior to a single hop

Conjecture: Quantum the Acrophobic Bouncing Ball



Example (Quantum the Bouncing Ball) (Single-hop)

$$0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \rightarrow$$
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Removing the repetition grotesquely changes the behavior to a single hop

Developed on the board:

1. Intermediate condition proof rule $G[;]$ for sequential compositions
2. Dynamic axioms for dynamical systems
3. Example-driven sketch of single-hop bouncing ball proof

See lecture notes for details [1].

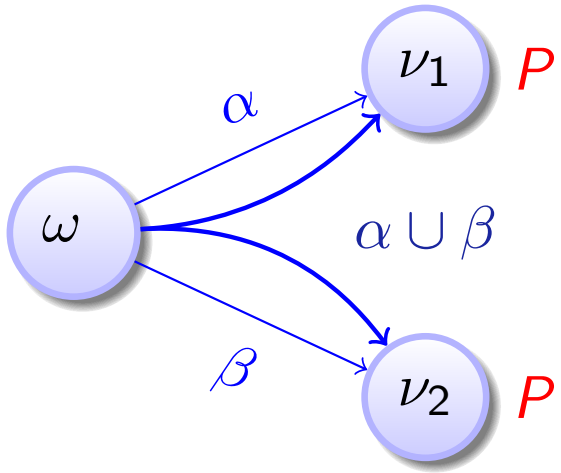
Outline

Dynamic Axioms for Dynamical Systems

compositional semantics $\Rightarrow$ compositional rules!

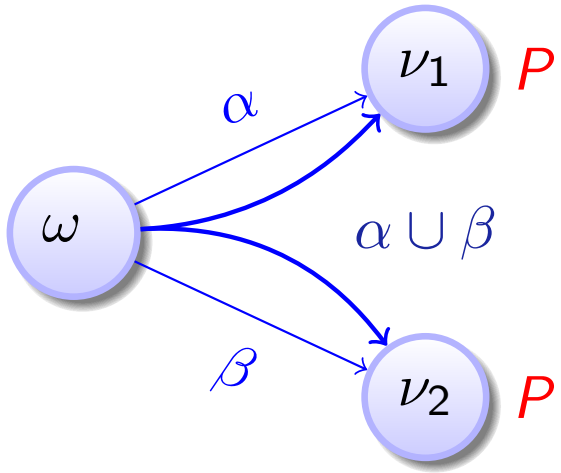
Dynamic Axioms for Dynamical Systems

[∪] $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$

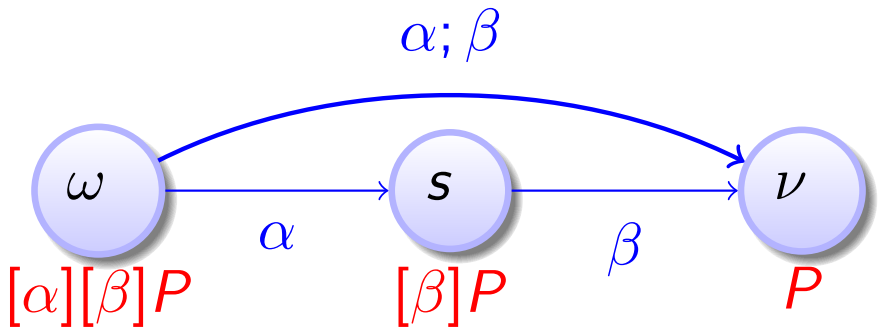


Dynamic Axioms for Dynamical Systems

[∪] $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$

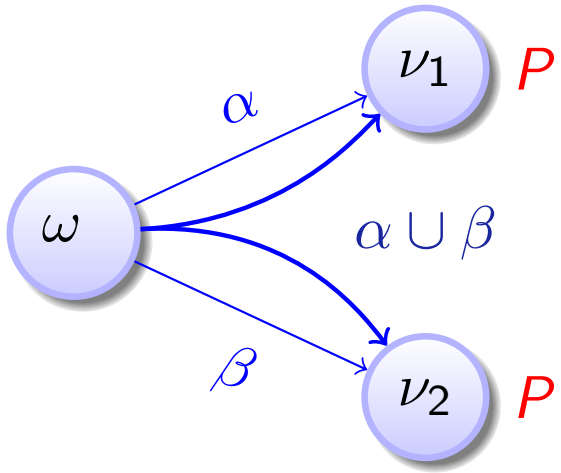


[;] $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$

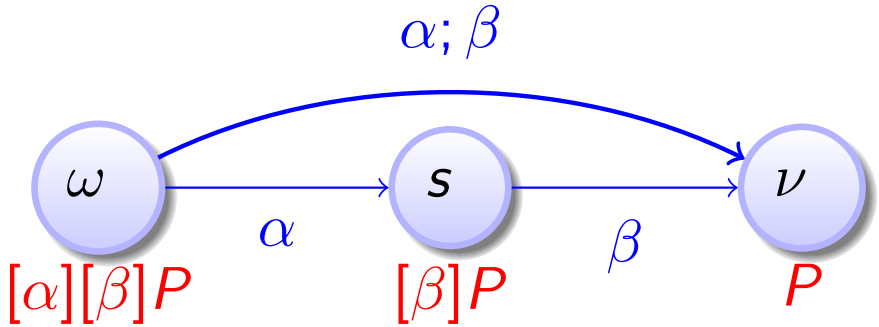


Dynamic Axioms for Dynamical Systems

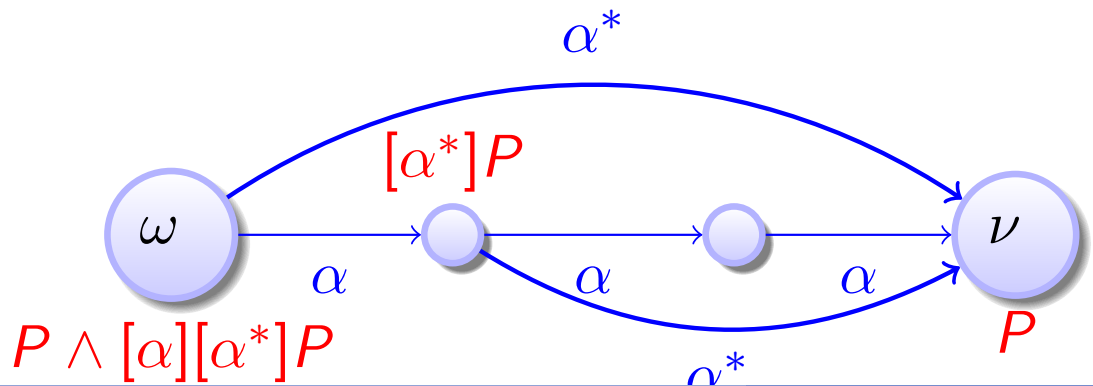
[∪] $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$



[;] $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$



[*] $[\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$



Outline

A Proof of a Short Single-hop Bouncing Ball

$$[;]\ \frac{}{A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)}$$

$$A \overset{\text{def}}{\equiv} 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \overset{\text{def}}{\equiv} 0 \leq x \wedge x \leq H$$

$$(x'' = -g) \overset{\text{def}}{\equiv} (x' = v, v' = -g)$$

A Proof of a Short Single-hop Bouncing Ball

$$[\cup]\ \frac{A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v)}{A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)}\ [;]$$

$$A \overset{\text{def}}{\equiv} 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

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$$(x'' = -g) \overset{\text{def}}{\equiv} (x' = v, v' = -g)$$

A Proof of a Short Single-hop Bouncing Ball

$$[;]\ \frac{A \vdash [x'' = -g]\big([?x = 0; v := -cv]B(x,v) \wedge [?x \geq 0]B(x,v)\big)}{A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v)}$$

$$[\cup]\ \frac{A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v)}{A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)}$$

$$[;]\ \frac{A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)}{}$$

$$A \overset{\text{def}}{\equiv} 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

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$$(x'' = -g) \overset{\text{def}}{\equiv} (x' = v, v' = -g)$$

A Proof of a Short Single-hop Bouncing Ball

$$\text{[?],[?]}\ \frac{A \vdash [x'' = -g]\big([?x = 0][v := -cv]B(x,v) \wedge [?x \geq 0]B(x,v)\big)}{A \vdash [x'' = -g]\big([?x = 0; v := -cv]B(x,v) \wedge [?x \geq 0]B(x,v)\big)}\ [;]$$

$$\frac{A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v)}{A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)}\ [;]\ \ \frac{A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v)}{A \vdash [x'' = -g]\big([?x = 0][v := -cv]B(x,v) \wedge [?x \geq 0]B(x,v)\big)}\ [\cup]$$

$$A \overset{\text{def}}{\equiv} 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \overset{\text{def}}{\equiv} 0 \leq x \wedge x \leq H$$

$$(x'' = -g) \overset{\text{def}}{\equiv} (x' = v, v' = -g)$$

A Proof of a Short Single-hop Bouncing Ball

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash [x'' = -g]\big((x = 0 \to [v := -cv]B(x,v)) \wedge (x \geq 0 \to B(x,v))\big)}[:=]}{A \vdash [x'' = -g]\big([?x = 0][v := -cv]B(x,v) \wedge [?x \geq 0]B(x,v)\big)}[?],[?]}{A \vdash [x'' = -g]\big([?x = 0; v := -cv]B(x,v) \wedge [?x \geq 0]B(x,v)\big)}[;]}{A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v)}[\cup]}{A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)}[;]$$

$$A \overset{\text{def}}{\equiv} 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \overset{\text{def}}{\equiv} 0 \leq x \wedge x \leq H$$

$$(x'' = -g) \overset{\text{def}}{\equiv} (x' = v, v' = -g)$$

A Proof of a Short Single-hop Bouncing Ball

$$
\begin{array}{ll}
[']\ & \dfrac{}{A \vdash [x'' = -g]\big((x = 0 \to B(x,-cv)) \wedge (x \geq 0 \to B(x,v))\big)} \\[2ex]
[:=]\ & \dfrac{A \vdash [x'' = -g]\big((x = 0 \to [v := -cv]B(x,v)) \wedge (x \geq 0 \to B(x,v))\big)}{A \vdash [x'' = -g]\big((x = 0 \to [v := -cv]B(x,v)) \wedge (x \geq 0 \to B(x,v))\big)} \\[2ex]
[?],[?]\ & \dfrac{A \vdash [x'' = -g]\big([?x = 0][v := -cv]B(x,v) \wedge [?x \geq 0]B(x,v)\big)}{A \vdash [x'' = -g]\big((x = 0 \to [v := -cv]B(x,v)) \wedge (x \geq 0 \to B(x,v))\big)} \\[2ex]
[;]\ & \dfrac{A \vdash [x'' = -g]\big([?x = 0; v := -cv]B(x,v) \wedge [?x \geq 0]B(x,v)\big)}{A \vdash [x'' = -g]\big([?x = 0][v := -cv]B(x,v) \wedge [?x \geq 0]B(x,v)\big)} \\[2ex]
[\cup]\ & \dfrac{A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v)}{A \vdash [x'' = -g]\big([?x = 0; v := -cv]B(x,v) \wedge [?x \geq 0]B(x,v)\big)} \\[2ex]
[;]\ & \dfrac{A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)}{A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v)}
\end{array}
$$

$$A \overset{\text{def}}{\equiv} 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \overset{\text{def}}{\equiv} 0 \leq x \wedge x \leq H$$

$$(x'' = -g) \overset{\text{def}}{\equiv} (x' = v, v' = -g)$$

A Proof of a Short Single-hop Bouncing Ball

$$[;]\ \frac{A \vdash \forall t{\geq}0\,[x := H - \frac{g}{2}t^2; v := -gt]\big((x{=}0 \to B(x,-cv)) \wedge (x{\geq}0 \to B(x,v))\big)}{A \vdash [x'' = -g]\big((x = 0 \to B(x,-cv)) \wedge (x \geq 0 \to B(x,v))\big)}[']$$

$$[:=]\ \frac{A \vdash [x'' = -g]\big((x = 0 \to [v := -cv]B(x,v)) \wedge (x \geq 0 \to B(x,v))\big)}{A \vdash [x'' = -g]\big([?x = 0][v := -cv]B(x,v) \wedge [?x \geq 0]B(x,v)\big)}[?],[?]$$

$$[;]\ \frac{A \vdash [x'' = -g]\big([?x = 0; v := -cv]B(x,v) \wedge [?x \geq 0]B(x,v)\big)}{A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v)}[\cup]$$

$$[;]\ \frac{A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v)}{A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)}$$

$$A \overset{\text{def}}{\equiv} 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \overset{\text{def}}{\equiv} 0 \leq x \wedge x \leq H$$

$$(x'' = -g) \overset{\text{def}}{\equiv} (x' = v, v' = -g)$$

A Proof of a Short Single-hop Bouncing Ball

$$
\begin{array}{ll}
[:=] & \dfrac{A \vdash \forall t{\geq}0\,[x := H - \frac{g}{2}t^2][v := -gt]\big((x{=}0 \to B_{(x,-cv)}) \wedge (x{\geq}0 \to B_{(x,v)})\big)}{} \\
[;] & \dfrac{A \vdash \forall t{\geq}0\,[x := H - \frac{g}{2}t^2; v := -gt]\big((x{=}0 \to B_{(x,-cv)}) \wedge (x{\geq}0 \to B_{(x,v)})\big)}{} \\
['] & \dfrac{A \vdash [x'' = -g]\big((x = 0 \to B_{(x,-cv)}) \wedge (x \geq 0 \to B_{(x,v)})\big)}{} \\
[:=] & \dfrac{A \vdash [x'' = -g]\big((x = 0 \to [v := -cv]B_{(x,v)}) \wedge (x \geq 0 \to B_{(x,v)})\big)}{} \\
[?],[?] & \dfrac{A \vdash [x'' = -g]\big([?x = 0][v := -cv]B_{(x,v)} \wedge [?x \geq 0]B_{(x,v)}\big)}{} \\
[;] & \dfrac{A \vdash [x'' = -g]\big([?x = 0; v := -cv]B_{(x,v)} \wedge [?x \geq 0]B_{(x,v)}\big)}{} \\
[\cup] & \dfrac{A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B_{(x,v)}}{} \\
[;] & \dfrac{A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B_{(x,v)}}{}
\end{array}
$$

$$A \overset{\text{def}}{\equiv} 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B_{(x,v)} \overset{\text{def}}{\equiv} 0 \leq x \wedge x \leq H$$

$$(x'' = -g) \overset{\text{def}}{\equiv} (x' = v, v' = -g)$$

A Proof of a Short Single-hop Bouncing Ball

$$[:=]\ \frac{A \vdash \forall t{\geq}0\,[x := H - \frac{g}{2}t^2]\big((x{=}0 \to B_{(x,-c(-gt))}) \wedge (x{\geq}0 \to B_{(x,-gt)})\big)}{A \vdash \forall t{\geq}0\,[x := H - \frac{g}{2}t^2][v := -gt]\big((x{=}0 \to B_{(x,-cv)}) \wedge (x{\geq}0 \to B_{(x,v)})\big)}$$

$$[;]\ \frac{A \vdash \forall t{\geq}0\,[x := H - \frac{g}{2}t^2; v := -gt]\big((x{=}0 \to B_{(x,-cv)}) \wedge (x{\geq}0 \to B_{(x,v)})\big)}{A \vdash \forall t{\geq}0\,[x := H - \frac{g}{2}t^2][v := -gt]\big((x{=}0 \to B_{(x,-cv)}) \wedge (x{\geq}0 \to B_{(x,v)})\big)}$$

$$[']\ \frac{A \vdash [x'' = -g]\big((x = 0 \to B_{(x,-cv)}) \wedge (x \geq 0 \to B_{(x,v)})\big)}{A \vdash \forall t{\geq}0\,[x := H - \frac{g}{2}t^2; v := -gt]\big((x{=}0 \to B_{(x,-cv)}) \wedge (x{\geq}0 \to B_{(x,v)})\big)}$$

$$[:=]\ \frac{A \vdash [x'' = -g]\big((x = 0 \to [v := -cv]B_{(x,v)}) \wedge (x \geq 0 \to B_{(x,v)})\big)}{A \vdash [x'' = -g]\big((x = 0 \to B_{(x,-cv)}) \wedge (x \geq 0 \to B_{(x,v)})\big)}$$

$$[?],[?]\ \frac{A \vdash [x'' = -g]\big([?x = 0][v := -cv]B_{(x,v)} \wedge [?x \geq 0]B_{(x,v)}\big)}{A \vdash [x'' = -g]\big((x = 0 \to [v := -cv]B_{(x,v)}) \wedge (x \geq 0 \to B_{(x,v)})\big)}$$

$$[;]\ \frac{A \vdash [x'' = -g]\big([?x = 0; v := -cv]B_{(x,v)} \wedge [?x \geq 0]B_{(x,v)}\big)}{A \vdash [x'' = -g]\big([?x = 0][v := -cv]B_{(x,v)} \wedge [?x \geq 0]B_{(x,v)}\big)}$$

$$[\cup]\ \frac{A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B_{(x,v)}}{A \vdash [x'' = -g]\big([?x = 0; v := -cv]B_{(x,v)} \wedge [?x \geq 0]B_{(x,v)}\big)}$$

$$[;]\ \frac{A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B_{(x,v)}}{A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B_{(x,v)}}$$

$$A \overset{\text{def}}{\equiv} 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B_{(x,v)} \overset{\text{def}}{\equiv} 0 \leq x \wedge x \leq H$$

$$(x'' = -g) \overset{\text{def}}{\equiv} (x' = v, v' = -g)$$

A Proof of a Short Single-hop Bouncing Ball

$$\dfrac{A \vdash \forall t{\geq}0 \left((H-\frac{g}{2}t^2{=}0 \rightarrow B(H-\frac{g}{2}t^2,-c(-gt))) \wedge (H-\frac{g}{2}t^2{\geq}0 \rightarrow B(H-\frac{g}{2}t^2,-gt)\right)}{[:=]\ A \vdash \forall t{\geq}0\, [x := H - \frac{g}{2}t^2]\big((x{=}0 \rightarrow B(x,-c(-gt))) \wedge (x{\geq}0 \rightarrow B(x,-gt))\big)}$$

$$\dfrac{A \vdash \forall t{\geq}0\, [x := H - \frac{g}{2}t^2][v := -gt]\big((x{=}0 \rightarrow B(x,-cv)) \wedge (x{\geq}0 \rightarrow B(x,v))}{[:=]\ A \vdash \forall t{\geq}0\, [x := H - \frac{g}{2}t^2][v := -gt]\big((x{=}0 \rightarrow B(x,-cv)) \wedge (x{\geq}0 \rightarrow B(x,v))}$$

$$\dfrac{A \vdash \forall t{\geq}0\, [x := H - \frac{g}{2}t^2; v := -gt]\big((x{=}0 \rightarrow B(x,-cv)) \wedge (x{\geq}0 \rightarrow B(x,v))}{[;]\ A \vdash \forall t{\geq}0\, [x := H - \frac{g}{2}t^2; v := -gt]\big((x{=}0 \rightarrow B(x,-cv)) \wedge (x{\geq}0 \rightarrow B(x,v))}$$

$$\dfrac{A \vdash [x'' = -g]\big((x = 0 \rightarrow B(x,-cv)) \wedge (x \geq 0 \rightarrow B(x,v))\big)}{[']\ A \vdash [x'' = -g]\big((x = 0 \rightarrow B(x,-cv)) \wedge (x \geq 0 \rightarrow B(x,v))\big)}$$

$$\dfrac{A \vdash [x'' = -g]\big((x = 0 \rightarrow [v := -cv]B(x,v)) \wedge (x \geq 0 \rightarrow B(x,v))\big)}{[:=]\ A \vdash [x'' = -g]\big((x = 0 \rightarrow [v := -cv]B(x,v)) \wedge (x \geq 0 \rightarrow B(x,v))\big)}$$

$$\dfrac{A \vdash [x'' = -g]\big([?x = 0][v := -cv]B(x,v) \wedge [?x \geq 0]B(x,v)\big)}{[?],[?]\ A \vdash [x'' = -g]\big([?x = 0][v := -cv]B(x,v) \wedge [?x \geq 0]B(x,v)\big)}$$

$$\dfrac{A \vdash [x'' = -g]\big([?x = 0; v := -cv]B(x,v) \wedge [?x \geq 0]B(x,v)\big)}{[;]\ A \vdash [x'' = -g]\big([?x = 0; v := -cv]B(x,v) \wedge [?x \geq 0]B(x,v)\big)}$$

$$\dfrac{A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v)}{[\cup]\ A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v)}$$

$$\dfrac{A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)}{[;]\ A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)}$$

$$A \overset{\text{def}}{\equiv} 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \overset{\text{def}}{\equiv} 0 \leq x \wedge x \leq H$$

$$(x'' = -g) \overset{\text{def}}{\equiv} (x' = v, v' = -g)$$

A Proof of a Short Single-hop Bouncing Ball

$$A \vdash \forall t{\geq}0 \left((H-\tfrac{g}{2}t^2{=}0 \rightarrow B(H-\tfrac{g}{2}t^2,-c(-gt))) \land (H-\tfrac{g}{2}t^2{\geq}0 \rightarrow B(H-\tfrac{g}{2}t^2,-gt))\right)$$

[:=] $A \vdash \forall t{\geq}0\, [x := H - \frac{g}{2}t^2]\big((x{=}0 \rightarrow B(x,-c(-gt))) \land (x{\geq}0 \rightarrow B(x,-gt))\big)$

[:=] $A \vdash \forall t{\geq}0\, [x := H - \frac{g}{2}t^2][v := -gt]\big((x{=}0 \rightarrow B(x,-cv)) \land (x{\geq}0 \rightarrow B(x,v))\big)$

[;] $A \vdash \forall t{\geq}0\, [x := H - \frac{g}{2}t^2; v := -gt]\big((x{=}0 \rightarrow B(x,-cv)) \land (x{\geq}0 \rightarrow B(x,v))\big)$

['] $A \vdash [x'' = -g]\big((x = 0 \rightarrow B(x,-cv)) \land (x \geq 0 \rightarrow B(x,v))\big)$

[:=] $A \vdash [x'' = -g]\big((x = 0 \rightarrow [v := -cv]B(x,v)) \land (x \geq 0 \rightarrow B(x,v))\big)$

[?],[?] $A \vdash [x'' = -g]\big([?x = 0][v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\big)$

[;] $A \vdash [x'' = -g]\big([?x = 0; v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\big)$

[∪] $A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v)$

[;] $A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)$

$$A \overset{\text{def}}{\equiv} 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$

$$B(x,v) \overset{\text{def}}{\equiv} 0 \leq x \land x \leq H$$

$$(x'' = -g) \overset{\text{def}}{\equiv} (x' = v, v' = -g)$$

A Proof of a Short Single-hop Bouncing Ball

Resolving abbreviations at the premise yields:

$$0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \rightarrow$$
$$\forall t \geq 0 \left((H - \frac{g}{2}t^2 = 0 \rightarrow 0 \leq H - \frac{g}{2}t^2 \wedge H - \frac{g}{2}t^2 \leq H) \right.$$
$$\left. \wedge (H - \frac{g}{2}t^2 \geq 0 \rightarrow 0 \leq H - \frac{g}{2}t^2 \wedge H - \frac{g}{2}t^2 \leq H) \right)$$

which is provable by arithmetic (since $g > 0$ and $t^2 \geq 0$).

A Proof of a Short Single-hop Bouncing Ball

Resolving abbreviations at the premise yields:

$$0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \rightarrow$$
$$\forall t \geq 0 \left((H - \frac{g}{2} t^2 = 0 \rightarrow 0 \leq H - \frac{g}{2} t^2 \wedge H - \frac{g}{2} t^2 \leq H) \right.$$
$$\left. \wedge (H - \frac{g}{2} t^2 \geq 0 \rightarrow 0 \leq H - \frac{g}{2} t^2 \wedge H - \frac{g}{2} t^2 \leq H) \right)$$

which is provable by arithmetic (since $g > 0$ and $t^2 \geq 0$).

A Proof of a Short Single-hop Bouncing Ball

Resolving abbreviations at the premise yields:

$$0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \rightarrow$$
$$\forall t \geq 0 \left(\left(H - \frac{g}{2} t^2 = 0 \rightarrow 0 \leq H - \frac{g}{2} t^2 \wedge H - \frac{g}{2} t^2 \leq H \right) \right.$$
$$\left. \wedge \left(H - \frac{g}{2} t^2 \geq 0 \rightarrow 0 \leq H - \frac{g}{2} t^2 \wedge H - \frac{g}{2} t^2 \leq H \right) \right)$$

which is provable by arithmetic (since $g > 0$ and $t^2 \geq 0$).

Exciting!

We have just formally verified our very first CPS!

A Proof of a Short Single-hop Bouncing Ball

Resolving abbreviations at the premise yields:

$$0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \rightarrow$$
$$\forall t \geq 0 \left((H - \frac{g}{2}t^2 = 0 \rightarrow 0 \leq H - \frac{g}{2}t^2 \wedge H - \frac{g}{2}t^2 \leq H) \right.$$
$$\left. \wedge (H - \frac{g}{2}t^2 \geq 0 \rightarrow 0 \leq H - \frac{g}{2}t^2 \wedge H - \frac{g}{2}t^2 \leq H) \right)$$

which is provable by arithmetic (since $g > 0$ and $t^2 \geq 0$).

Exciting!

We have just formally verified our very first CPS!

Okay, alright, it was a grotesquely simplified single-hop bouncing ball. But the axioms of our proof technique were completely general and not specific to bouncing balls, so they should carry us forward to true CPS.


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