15-424: Foundations of Cyber-Physical Systems

Lecture Notes on Safety & Contracts

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Lecture 4

1. Introduction

In the previous lectures, we have studied models of cyber-physical systems. Hybrid programs provide a programming language for cyber-physical systems [Pla12c, Pla08, Pla10] with the most prominent features being differential equations and nondeterminism alongside the usual classical control structures and discrete assignments. This gives powerful and flexible ways of modeling even very challenging systems and very complex control principles. This lecture will start studying ways of making sure that the resulting behavior, however flexible and powerful it may be, also meets the required safety and correctness standards.

In the 15-122 Principles of Imperative Computation course, you have experienced how contracts can be used to make properties of programs explicit. You have seen how contracts can be checked dynamically at runtime, which, if they fail, will alert you right away to flaws in the design of the programs. You have experienced first hand that it is much easier to find and fix problems in programs starting from the first contract that failed in the middle of the program, rather than from the mere observation about the symptoms that ultimately surface when the final output is not as expected (which you may not notice either unless the output is checked dynamically).

Another aspect of contracts that you have had the opportunity to observe in Principles of Imperative Computation is that they can be used in proofs that show that every program run will satisfy the contracts. Unlike in dynamic checking, the scope of correctness arguments with proofs extends far beyond the test cases that have been tried, however clever the tests may have been chosen. Both uses of contracts, dynamic checking and rigorous proofs, are very helpful to check whether a system does what we intend it to, as has been argued on numerous occasions in various contexts in the literature, e.g., [Flo67, Hoa69, Pra76, Mey92, XJC09, PCL11, Log11].

The principles of contracts help cyber-physical systems [Pla08, Pla10, Pla13, DLTT13] as well. Yet, their use in proving may, arguably, be more important than their use in dynamic checking. The reason has to do with the physical impact of CPS and the (relative) non-negotiability of the laws of physics. The reader is advised to imagine a situation where a self-driving car is propelling him or her down the street. Suppose the car's control software is covered with contracts all over, but all of them are exclusively for dynamic checking, none have been proved. If that self-driving car speeds up to 100mph on a 55mph highway and drives up very close to a car in front of it, then dynamically checking the contract "distance to car in front should be more than 1 meter" does not help. If that contract fails, the car's software would know that it made a mistake, but it has become too late to do anything about it, because the brakes of the car will never work out in time. So the car would be "trapped in its own physics", in the sense that it has run out of all safe control options. There are still effective ways of making use of dynamic contract checking in CPS [MP16], but the design of those contracts then requires proof to ensure that safety is always maintained.

For those reasons, this course will focus on the role of proofs as correctness arguments much more than on dynamical checking of contracts. Because of the physical consequences of malfunctions, correctness requirements on CPS are also more stringent. And their proofs involve significantly more challenging arguments than in Principles of Imperative Computation. For those reasons, we will approach CPS proofs with much more rigor than what you have seen in Principles of Imperative Computation. But that is a story for a later lecture. The focus of today's lecture will be to understand CPS contracts and the first basics of reasoning about CPS. Subsequent lectures will ultimately identify a much cleaner, more elegant, and more general style of reasoning about CPS of which the reasoning approach developed in today's lecture are a special case. But today's lecture is a useful stepping stone for reaching that generality.

This material is based on correctness specifications and proofs for CPS [Pla12c, Pla07, Pla08, Pla10]. We will come back to more details in later lectures, where we will also use the KeYmaera X prover for verifying CPS [PQ08]. More information about safety and contracts can be found in [Pla10, Chapter 2.2,2.3].

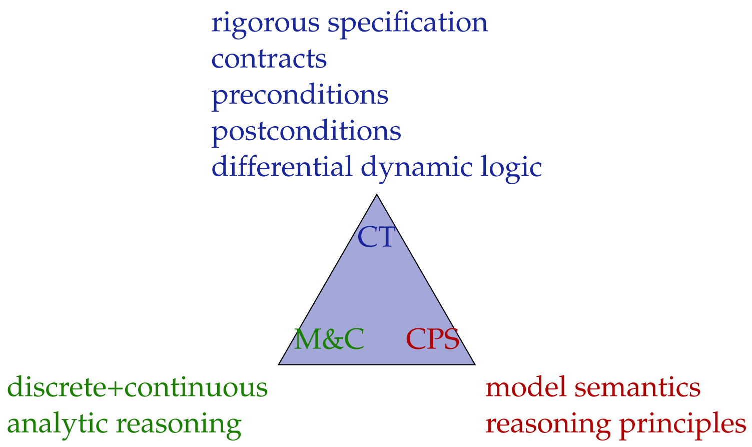
The focus of today's lecture is on developing and studying a model of a bouncing ball and on identifying all requirements for it to be safe. Along the way, however, this lecture develops an intuitive understanding for the role of requirements and contracts in CPS as well as important ways of formalizing CPS properties and their analyzes. The most important learning goals of this lecture are:

Modeling and Control: We deepen our understanding of the core principles behind CPS by relating discrete and continuous aspects of CPS to analytic reasoning principles.

Computational Thinking: We go through a simple but very instructive example to learn how to identify specifications and critical properties of CPS. Even if the example we look at, the bouncing ball, is a rather impoverished CPS, it still formidably conveys the subtleties involved with hybrid systems models, which are crucial for understanding CPS. This lecture is further devoted to contracts in the form of pre-

and post-conditions for CPS models. We will begin to reason rigorously about CPS models, which is critical to getting CPS right. CPS designs can be flawed for very subtle reasons. Without sufficient rigor in their analysis it can be impossible to spot the flaws, and even more challenging to say for sure whether and why a design is no longer faulty. This lecture introduces *differential dynamic logic* d$\mathcal{L}$ [Pla12c, Pla08, Pla10] as the specification and verification language for CPS that we will be using throughout this course.

CPS Skills: We will begin to deepen our understanding of the semantics of CPS models by relating it to their reasoning principles. A full study of this alignment will only be covered in the next lecture, though.



2. The Adventures of Quantum the Bouncing Ball

Lecture 3 considered hybrid programs that model a choice of increasing acceleration or braking.

$$\begin{aligned}&\Big(\big((?x-o>5; a:=a+1)\cup a:=-b\big);\\ &x'=v, v'=a\Big)^*\end{aligned} \tag{1}$$

That model did perform interesting control choices and we could continue to study it in this lecture.

In order to sharpen our intuition about CPS, we will, however, study a very simple but also very intuitive system instead. Once upon a time, there was a little bouncing ball called *Quantum*. Day in, day out, Quantum had nothing else to do but bounce up and down the street until it was tired of doing that, which, in fact, rarely happened, because bouncing was such a joy (Fig. 1). The bouncing ball, Quantum, was not much of a CPS, because Quantum does not actually have any interesting decisions to make. But it nevertheless formed a perfectly reasonable hybrid system, because, after a closer look, it turns out to involve discrete and continuous dynamics. The continuous dynamics is

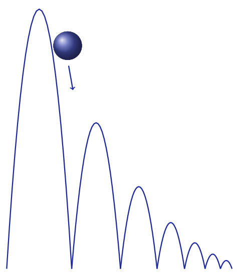


Figure 1: Sample trajectory of a bouncing ball (plotted as height over time)

caused by gravity, which is pulling the ball down and makes it fall from the sky in the first place. The discrete dynamics comes from the singular discrete event of what happens when the ball hits the ground and bounces back up. There are a number of ways of modeling the ball and its impact on the ground with physics. They include a whole range of different more or less realistic physical effects including gravity, aerodynamic resistance, the elastic deformation on the ground, and so on and so on. But the little bouncing ball, Quantum, didn't study enough physics to know anything about those effects. And so Quantum had to go about understanding the world in easier terms. Quantum was a clever bouncing ball, though, so it had experienced the phenomenon of sudden change and was trying to use that to its advantage.

If we are looking for a very simple model of what the bouncing ball does, it is easier to describe as a hybrid system. The ball at height x is falling subject to gravity:

$$x'' = -g$$

When it hits the ground, which is assumed at height $x = 0$, the ball bounces back and jumps back up in the air. Yet, as every child knows, the ball tends to come back up a little less high than before. Given enough time to bounce around, it will ultimately lie flat on the ground forever. Until it is picked up again and thrown high up in the air.

Let us model the impact on the ground as a discrete phenomenon and describe what happens so that the ball jumps back up then. One attempt of understanding this could be to make the ball jump back up rather suddenly by increasing its height by, say, 10 when it hit the ground $x = 0$:

$$\begin{aligned} &x'' = -g; \\ &\texttt{if}(x = 0)\, x := x + 10 \end{aligned} \tag{2}$$

Such a model may be useful for other systems, but would be rather at odds with our physical experience with bouncing balls, because the ball is indeed slowly climbing back up rather than suddenly being way up in the air again.

Quantum ponders about what happens when it hits the ground. Quantum does not suddenly get teleported to a new position above ground like (2) would suggest. Instead, the ball suddenly changes its direction. A moment ago, Quantum used to fall down with a negative velocity (i.e. one that is pointing down into the ground) and suddenly climbs back up with a positive velocity (pointing up into the sky). In order to be able to write such a model, the velocity v will be made explicit in the bouncing ball's differential equation:

$$\begin{aligned}&x' = v, v' = -g;\\&\texttt{if}(x = 0)\, v := -v\end{aligned} \tag{3}$$

Of course, something happens after the bouncing ball reversed its direction because it hit the ground. Physics continues until it hits the ground again.

$$\begin{aligned}&x' = v, v' = -g;\\&\texttt{if}(x = 0)\, v := -v\\&x' = v, v' = -g;\\&\texttt{if}(x = 0)\, v := -v\end{aligned} \tag{4}$$

Then, of course, physics moves on again, so the model actually involves a repetition:

$$\begin{aligned}&\big(x' = v, v' = -g;\\&\;\texttt{if}(x = 0)\, v := -v\big)^*\end{aligned} \tag{5}$$

Yet, Quantum is now rather surprised. For if it follows that HP (5), it seems as if it should always be able to come back up to its initial height again. Excited about that possibility, Quantum tries and tries again but never succeeds to bounce back up as high as it was before. So there must be something wrong with the model in (5), Quantum concludes and sets out to fix (5).

Having observed itself rather carefully when bouncing around, Quantum concludes that it feels slower when bouncing back up than it used to be when falling on down. Indeed, Quantum feels less energetic on its way up. So its velocity must not only flip direction from down to up, at a bounce, but also seems to shrink in magnitude. Quantum swiftly calls the corresponding damping factor c and quickly comes up with a better model of itself:

$$\begin{aligned}&\big(x' = v, v' = -g;\\&\;\texttt{if}(x = 0)\, v := -cv\big)^*\end{aligned} \tag{6}$$

Yet, running that model in clever ways, Quantum observes that model (6) could make it fall through the cracks in the ground. Terrified at that thought, Quantum quickly tries to set the physics right, lest it falls through the cracks in space before it had a chance to fix its physics. The issue with (6) is that its differential equation isn't told when to stop. Yet, Quantum luckily remembers that this is quite exactly what evolution domains were meant for. Above ground is where it wants to remain, and so $x \geq 0$ is what Quantum asks dear physics to obey, since the floor underneath Quantum is of rather sturdy built:

$$\begin{aligned}&\big(x' = v, v' = -g \,\&\, x \geq 0;\\&\;\texttt{if}(x = 0)\, v := -cv\big)^*\end{aligned} \tag{7}$$

Now, indeed, physics will have to stop evolving before gravity has made our little bouncing ball Quantum fall through the ground. Yet, physics could still choose to stop evolving while the ball is still high up in the sky. In that case, the ball will not yet be on the ground and line 2 of (7) would have no effect because $x \neq 0$ still. This is not a catastrophe, however, because the loop in (7) could simply repeat, which would allow physics to continue to evolve the differential equation further.

Quite happy with model (7) for itself, the bouncing ball Quantum goes on to explore whether the model does what the ball expects it to do. Of course, had Quantum taken this course already, it would have marched right into a rigorous analysis of the model. Since Quantum is still a rookie, it takes a detour along visualization road, and first shoots a couple of pictures of what happens when simulating model (7). Thanks to a really good simulator, these simulations all came out looking characteristically similar to Fig. 1.

3. How Quantum Discovered a Crack in the Fabric of Time

After a little while of idly simulating its very own model, Quantum decides to take out its temporal magnifying glasses and zoom in real close to see what happens when its model (7) bounces on the ground ($x = 0$). At that point in time, its differential equation is forced to stop due to the evolution domain $x \geq 0$, so the continuous evolution stops and a discrete action happens that inspects the height and, if $x = 0$, discretely changes the velocity around to $-cv$ instantly in no time.

At the continuous point in time of the first bounce—Quantum recorded the time t_1—the ball observes a succession of different states. First at continuous time t_1, Quantum has position $x = 0$ and velocity $v = -5$. But then, after the discrete assignment of (7) ran, still at real time t_1, it has position $x = 0$ and velocity $v = 4$. This chaos cannot possibly go on like that, thought Quantum, and decided to give an extra natural number index $j \in \mathbb{N}$ to distinguish the two occurrences of continuous time t_1. So, for the sake of illustration, it called $(t_1, 0)$ the first point in time where it was in state $x = 0, v = -5$ and then called $(t_1, 1)$ the second point in time where it was in state $x = 0, v = 4$.

In fact, Quantum's temporal magnifying glasses worked so well that it suddenly discovered it had accidentally invented an extra dimension for time: the discrete time step $i \in \mathbb{N}$ in addition to the continuous time coordinate $t \in \mathbb{R}$. Quantum plots the continuous $\mathbb{R}$-valued time coordinate in the t axis of Fig. 2 while separating the $\mathbb{N}$-valued discrete step count of its hybrid time into the j axis and leaving the x axis for position. Quantum now observed the first simulation of model (7) with its temporal magnifiers activated to fully appreciate its hybrid nature in its full blossom. And, indeed, if Quantum looks at the hybrid time simulation from Fig. 2 and turns its temporal magnifiers off again, the extra dimension of discrete steps j vanishes again, leaving behind only the shadow of the execution in the x over t face, which agrees with the layman's simulation shown in Fig. 1. And even the projection of the hybrid time simulation from Fig. 2 to the j over t face leads to a curious illustration, shown in Fig. 3, of what the temporal magnifying glasses revealed about how hybrid time has evolved in this particular simulation.

Armed with the additional intuition about the operations of (7) that these sample

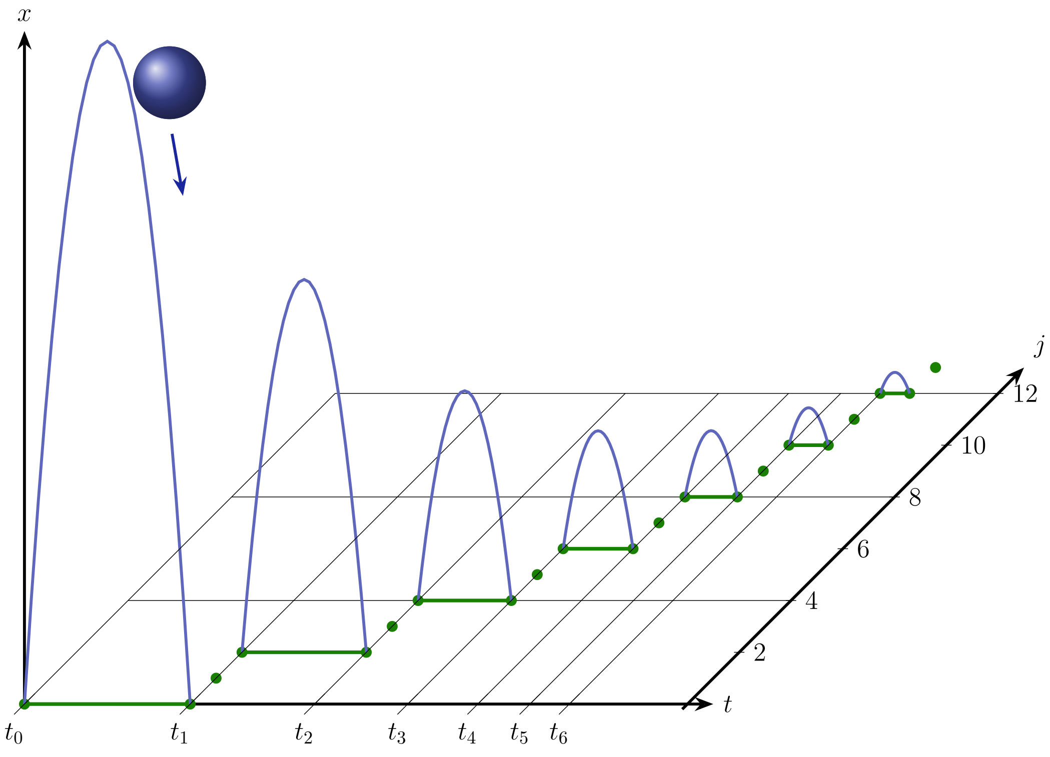


Figure 2: Sample trajectory of a bouncing ball plotted as position x over its hybrid time domain with discrete time step j and continuous time t

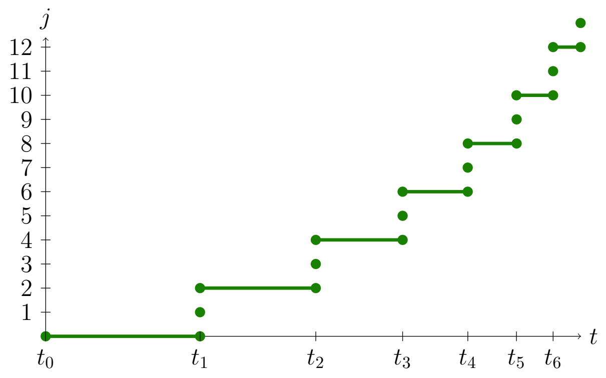


Figure 3: Hybrid time domain for the sample trajectory of a bouncing ball with discrete time step j and continuous time t

executions in Fig. 1 and its hybrid version Fig. 2 provide, Quantum now feels prepared to ask the deeper questions. Will its model (7) always do the right thing? Or was it just lucky in the case shown in Fig. 1? What even is the right behavior for a proper bouncing ball? What are its important properties? And for what purpose? How could they be specified unambiguously? And, ultimately, how could Quantum convince itself about these properties being true?

4. Postcondition Contracts for CPS

Hybrid programs are interesting models for CPS. They describe the behavior of a CPS, ultimately captured by their semantics $[\![\alpha]\!]$, which is a reachability relation on states (Lecture 3 on Choice & Control). Yet, reliable development of CPS also needs a way of ensuring that the behavior will be as expected. So, for example, we may want the behavior of a CPS to always satisfy certain crucial safety properties. A robot, for example, should never do something unsafe like running over a human being.[1]

Quantum, the little bouncing ball, may consider itself less safety-critical, except that it may be interested in its own safety. Quantum still wants to make sure that it couldn't ever fall through the cracks in the ground. And even though it would love to jump all the way up to the moon, Quantum is also rather terrified of big heights and would never want to jump any higher than it was in the very beginning. So, when H denotes the initial height, Quantum would love to know whether its height will always stay within $0 \leq x \leq H$ when following HP (7).

Scared of what otherwise might happen to it if $0 \leq x \leq H$ should ever be violated, Quantum decides to make its goals for the HP (7) explicit. Fortunately, Quantum excelled in the course 15-122 Principles of Imperative Computation and recalls that contracts such as `@requires` and `@ensures` have been used in that course to make behavioral expectations for C and C0 programs[2] explicit. Even though Quantum clearly no longer deals with plain C program, but rather a hybrid program, it still puts an `@ensures`(F) contract in front of HP (7) to express that all runs of that HP are expected to lead only to states in which logical formula F is true. Quantum even uses `@ensures`

[1]Safety of robots has, of course, been aptly defined by Asimov [Asi42] with his Three Laws of Robotics:

1. A robot may not injure a human being or, through inaction, allow a human being to come to harm.
2. A robot must obey the orders given to it by human beings, except where such orders would conflict with the First Law.
3. A robot must protect its own existence as long as such protection does not conflict with the First or Second Law.

Sadly, their exact rendition in logic or anything else that would be precise still remains a bit of a challenge due to language ambiguities and similar minor nuisances that kept scientists busy for the good deal of a century since. The Three Laws of Robotics are not the answer. They are the inspiration!

[2]C0 is a small safe subset of the C programming language, which is featured in the 15-122 Principles of Imperative Computation course.

twice, once for each of its expectations.

$$\begin{aligned}
&\texttt{@ensures}(0 \leq x)\\
&\texttt{@ensures}(x \leq H)\\
&\big(x' = v, v' = -g \,\&\, x \geq 0;\\
&\ \ \texttt{if}(x = 0)\, v := -cv\big)^*
\end{aligned} \tag{8}$$

5. Precondition Contracts for CPS

Having learned a lot from the Principles of Imperative Computation experience, Quantum immediately starts thinking about whether the `@ensures` contracts in (8) would, in fact, always be true after running that HP. After all, Quantum would really love to know that it can rely on that contract never failing. In fact, it would prefer to see that logical contract met before it ever dares trying another thoughtless bounce again.

Wondering about whether the `@ensures` contract in (8) would always succeed, Quantum notices that this would have to depend on what values the bouncing ball starts with. It called H its initial height, but the HP (8) cannot know that. For one thing, the contracts in (8) would be hard to fulfill if $H = -5$, because $0 \leq x$ and $x \leq H$ can impossibly both be true then.

So, Quantum figures it should demand a `@requires` contract with the precondition $x = H$ to say that the height, x, of the bouncing ball is initially H. Because that still does not (obviously) ensure that $0 \leq x$ has a chance of holding, Quantum requires $0 \leq H$ to hold initially:

$$\begin{aligned}
&\texttt{@requires}(x = H)\\
&\texttt{@requires}(0 \leq H)\\
&\texttt{@ensures}(0 \leq x)\\
&\texttt{@ensures}(x \leq H)\\
&\big(x' = v, v' = -g \,\&\, x \geq 0;\\
&\ \ \texttt{if}(x = 0)\, v := -cv\big)^*
\end{aligned} \tag{9}$$

6. Invariant Contracts for CPS

Quantum remembers the prominent role that invariants have played in the course Principles of Imperative Computation. So, Quantum ventures including an invariant with its HP. In C0 programs, invariants were associated with loops, e.g.

```
i = 0;
while (i < 10)
  //@loop_invariant 0 <= i && i <= 10;
  {
    i++;
}
```

Quantum, thus, figures that invariants for loops in HPs should also be associated with a loop, which is written α^* for nondeterministic repetition. After a moment's thought, Quantum decides that falling through the cracks in the ground is still it's biggest worry, so the invariant it'd like to maintain is $x \geq 0$:

$$\begin{aligned}
&\texttt{@requires}(x = H)\\
&\texttt{@requires}(0 \leq H)\\
&\texttt{@ensures}(0 \leq x)\\
&\texttt{@ensures}(x \leq H)\\
&\big(x' = v, v' = -g \,\&\, x \geq 0;\\
&\ \texttt{if}(x = 0)\, v := -cv\big)^*\texttt{@invariant}(x \geq 0)
\end{aligned} \tag{10}$$

On second thought, Quantum is less sure what exactly the @invariant(F) contract would mean for a CPS. So it decides to first give more thought to the proper way of phrasing CPS contracts and what they mean to begin with.

We will get back to the @invariant(F) construct in a later lecture, when we fully understand the role and meaning of the other contracts.

7. Logical Formulas for Hybrid Programs

CPS contracts play a very useful role in the development of CPS models and CPS programs. Using them as part of their design right from the very beginning is a good idea, probably even more crucial than it was in 15-122 Principles of Imperative Computation for the development of C0 programs, because CPS have more stringent requirements on safety.

Yet, we do not only want to program CPS, we also want to and have to understand thoroughly what they mean, what their contracts mean, and how we convince ourselves that the CPS contracts are respected by the CPS program. It turns out that this is where mere contracts are at a disadvantage compared to full logic.

Note 1 (Logic is for specification and reasoning)**.** *Logic allows not only the specification of a whole CPS program, but also an analytic inspection of its parts as well as argumentative relations between contracts and program parts.*

Differential dynamic logic (d$\mathcal{L}$) [Pla12c, Pla08, Pla12a, Pla07, Pla10] is the logic of hybrid systems that this courses uses for specification and verification of cyber-physical systems. There are more aspects of logic for cyber-physical systems [Pla12c, Pla12b], which will be studied (to some extent) in later parts of this course.

The most unique feature of differential dynamic logic for our purposes is that it allows us to refer to hybrid systems. Lecture 2 on Differential Equations & Domains introduced first-order logic of real arithmetic.

Note 2 (Limits of first-order logic for CPS)**.** *First-order logic of real arithmetic is a crucial basis for describing what is true and false about CPS, because it allows us to refer to real-valued quantities like positions and velocities and their arithmetic relations. Yet, that is not enough, because first-order logic describes what is true in a single state of a system. It has no way of referring to what will be true in future states of a CPS, nor of describing the relationship of the initial state of the CPS to the final state of the CPS.*

Recall that this relationship, $[\![\alpha]\!]$, is what ultimately constitutes the semantics of HP α.

Note 3 (Differential dynamic logic principle)**.** *Differential dynamic logic (d$\mathcal{L}$) extends first-order logic of real arithmetic with operators that refer to the future states of a CPS in the sense of referring to the states that are reachable by running a given HP. The logic d$\mathcal{L}$ provides a modal operator $[\alpha]$, parametrized by α, that refers to all states reachable by HP α according to the reachability relation $[\![\alpha]\!]$ of its semantics. This modal operator can be placed in front of any d$\mathcal{L}$ formula ϕ. The d$\mathcal{L}$ formula*

$$[\alpha]\phi$$

expresses that all *states reachable by HP α satisfy formula ϕ.*

The logic d$\mathcal{L}$ also provides a modal operator $\langle\alpha\rangle$, parametrized by α, that can be placed in front of any d$\mathcal{L}$ formula ϕ. The d$\mathcal{L}$ formula

$$\langle\alpha\rangle\phi$$

expresses that there is at least one *state reachable by HP α for which ϕ holds. The modalities $[\alpha]$ and $\langle\alpha\rangle$can be used to express necessary or possible properties of the transition behavior of α, because they refer to all or some runs of α*

An `@ensures`(E) postcondition for a HP α can be expressed directly as a logical formula in d$\mathcal{L}$:

$$[\alpha]E$$

So, the first CPS postcondition `@ensures`$(0 \leq x)$ for the bouncing ball HP in (8) can be stated as a d$\mathcal{L}$ formula:

$$[\big(x' = v, v' = -g \,\&\, x \geq 0;\ \texttt{if}(x = 0)\, v := -cv\big)^*]\, 0 \leq x \tag{11}$$

The second CPS postcondition `@ensures`$(x \leq H)$ for the bouncing ball HP in (8) can be stated as a d$\mathcal{L}$ formula as well:

$$[\big(x' = v, v' = -g \,\&\, x \geq 0;\ \texttt{if}(x = 0)\, v := -cv\big)^*]\, x \leq H \tag{12}$$

The logic d$\mathcal{L}$ allows all other logical operators from first-order logic, including conjunction ($\wedge$). So, the two d$\mathcal{L}$ formulas (11) and (12) can be stated together as a single d$\mathcal{L}$ formula:

$$\begin{aligned} &[\big(x' = v, v' = -g \,\&\, x \geq 0;\ \texttt{if}(x = 0)\, v := -cv\big)^*]\, 0 \leq x \\ \wedge\, &[\big(x' = v, v' = -g \,\&\, x \geq 0;\ \texttt{if}(x = 0)\, v := -cv\big)^*]\, x \leq H \end{aligned} \tag{13}$$

Stepping back, we could also have combined the two postconditions `@ensures`$(0 \leq x)$ and `@ensures`$(x \leq H)$ into a single postcondition `@ensures`$(0 \leq x \land x \leq H)$. The translation of that into d$\mathcal{L}$ would have gotten us an alternative way of combining both statements about the lower and upper bound on the height of the bouncing ball into a single d$\mathcal{L}$ formula:

$$[\big(x' = v, v' = -g \,\&\, x \geq 0;\ \texttt{if}(x = 0)\, v := -cv\big)^*]\,(0 \leq x \land x \leq H) \tag{14}$$

Which way of representing what we expect bouncing balls to do is better? Like (13) or like (14)? Are they equivalent? Or do they express different things?

It turns out that there is a very simple argument within the logic $\mathsf{d}\mathcal{L}$ that shows that (13) and (14) are equivalent. And not just that those two particular logical formulas are equivalent but that the same equivalence holds for any $\mathsf{d}\mathcal{L}$ formulas of this form. This will be investigated formally in a later lecture, but it is useful to observe now already to sharpen our intuition.

Having said that, do we believe $\mathsf{d}\mathcal{L}$ formula (13) should be valid? Should (14) be valid? Before we study this question in any further detail, the first question should be what it means for a modal formula $[\alpha]\phi$ to be true. What is its semantics? Better yet, what exactly is its syntax in the first place?

8. Syntax of Differential Dynamic Logic

The formulas of differential dynamic logic are defined like the formulas of first-order logic of real arithmetic with the additional capability of using modal operators for any hybrid program α.

Definition 1 ($\mathsf{d}\mathcal{L}$ formula)**.** The *formulas of differential dynamic logic* ($\mathsf{d}\mathcal{L}$) are defined by the grammar (where ϕ, ψ are $\mathsf{d}\mathcal{L}$ formulas, θ_1, θ_2 (polynomial) terms, x a variable, α a HP):

$$\phi, \psi \ ::= \ \theta_1 = \theta_2 \mid \theta_1 \geq \theta_2 \mid \neg\phi \mid \phi \land \psi \mid \phi \lor \psi \mid \phi \rightarrow \psi \mid \forall x\, \phi \mid \exists x\, \phi \mid [\alpha]\phi \mid \langle\alpha\rangle\phi$$

Operators $>, \leq, <, \leftrightarrow$ *can be defined as usual, e.g.,* $\phi \leftrightarrow \psi \equiv (\phi \rightarrow \psi) \land (\psi \rightarrow \phi)$.

We use the notational convention that unary operators (including $\neg$ and quantifiers $\forall x, \exists x$ and modalities $[\alpha], \langle\alpha\rangle$)[3] bind stronger than binary operators. In particular, quantifiers and modal operators bind strong, i.e. their scope only extends to the formula immediately after. Thus, $[\alpha]\phi \land \psi \ \equiv \ ([\alpha]\phi) \land \psi$ and $\forall x\, \phi \land \psi \ \equiv \ (\forall x\, \phi) \land \psi$. In our notation, we also let $\land$ bind stronger than $\lor$, which binds stronger than $\rightarrow, \leftrightarrow$. We also associate $\rightarrow$ to the right so that $\phi \rightarrow \psi \rightarrow \varphi \equiv \phi \rightarrow (\psi \rightarrow \varphi)$. To avoid confusion, we do not adopt precedence conventions between $\rightarrow, \leftrightarrow$ but expect explicit parentheses. So $\phi \rightarrow \psi \leftrightarrow \varphi$ would be considered illegal and explicit parentheses are required to distinguish $\phi \rightarrow (\psi \leftrightarrow \varphi)$ from $(\phi \rightarrow \psi) \leftrightarrow \varphi$. Likewise $\phi \leftrightarrow \psi \rightarrow \varphi$ would be considered illegal and explicit parentheses are required to distinguish $\phi \leftrightarrow (\psi \rightarrow \varphi)$ from $(\phi \leftrightarrow \psi) \rightarrow \varphi$.

[3] Quantifiers are only quite arguably understood as unary operators. Yet, $\forall x$ is a unary operator on formulas while $\forall$ would be an operator with arguments of mixed syntactic categories. In a higher-order context, it can also be understood more formally by understanding $\forall x\, \phi$ as an operator on functions: $\forall(\lambda x.\phi)$. Similar cautionary remarks apply to the understanding of modalities as unary operators. The primary reason for adopting this understanding is that it simplifies the precedence rules.

9. Semantics of Differential Dynamic Logic

For $\mathsf{d}\mathcal{L}$ formulas that are also formulas of first-order real arithmetic (i.e. formulas without modalities), the semantics of $\mathsf{d}\mathcal{L}$ formulas is the same as that of first-order real arithmetic. The semantics of modalities $[\alpha]$ and $\langle\alpha\rangle$ quantifies over all ($[\alpha]$) or some ($\langle\alpha\rangle$) of the (final) states reachable by following HP α, respectively.

Definition 2 (d$\mathcal{L}$ semantics)**.** The *satisfaction relation* $\omega \in [\![\phi]\!]$ for a d$\mathcal{L}$ formula ϕ in state ω is defined inductively:

- $\omega \in [\![(\theta_1 = \theta_2)]\!]$ iff $[\![\theta_1]\!]\omega = [\![\theta_2]\!]\omega$.
  That is, an equation is true in a state ω iff the terms on both sides evaluate to the same number.
- $\omega \in [\![(\theta_1 \geq \theta_2)]\!]$ iff $[\![\theta_1]\!]\omega \geq [\![\theta_2]\!]\omega$.
  That is, a greater-or-equals inequality is true in a state ω iff the term on the left evaluate to a number that is greater or equal to the value of the right term.
- $\omega \in [\![\neg\phi]\!]$ iff $\omega \notin [\![\phi]\!]$, i.e. if it is not the case that $\omega \in [\![\phi]\!]$.
  That is, a negated formula $\neg\phi$ is true in state ω iff the formula ϕ itself is not true in ω.
- $\omega \in [\![\phi \wedge \psi]\!]$ iff $\omega \in [\![\phi]\!]$ and $\omega \in [\![\psi]\!]$.
  That is, a conjunction is true in a state iff both conjuncts are true in said state.
- $\omega \in [\![\phi \vee \psi]\!]$ iff $\omega \in [\![\phi]\!]$ or $\omega \in [\![\psi]\!]$.
  That is, a disjunction is true in a state iff either of its disjuncts is true in said state.
- $\omega \in [\![\phi \rightarrow \psi]\!]$ iff $\omega \notin [\![\phi]\!]$ or $\omega \in [\![\psi]\!]$.
  That is, an implication is true in a state iff either its left-hand side is false or its right-hand side true in said state.
- $\omega \in [\![\phi \leftrightarrow \psi]\!]$ iff ($\omega \in [\![\phi]\!]$ and $\omega \in [\![\psi]\!]$) or ($\omega \notin [\![\phi]\!]$ and $\omega \notin [\![\psi]\!]$).
  That is, a biimplication is true in a state iff both sides are true or both sides are false in said state.
- $\omega \in [\![\forall x\, \phi]\!]$ iff $\omega_x^d \in [\![\phi]\!]$ for all $d \in \mathbb{R}$.
  That is, a universally quantified formula $\forall x\, \phi$ is true in a state iff its kernel ϕ is true in all variations of the state, no matter what real number d the quantified variable x evaluates to in the variation ω_x^d.
- $\omega \in [\![\exists x\, \phi]\!]$ iff $\omega_x^d \in [\![\phi]\!]$ for some $d \in \mathbb{R}$.
  That is, an existentially quantified formula $\exists x\, \phi$ is true in a state iff its kernel ϕ is true in some variation of the state, for a suitable real number d that the quantified variable x evaluates to in the variation ω_x^d.
- $\omega \in [\![[\alpha]\phi]\!]$ iff $\nu \in [\![\phi]\!]$ for all ν with $(\omega, \nu) \in [\![\alpha]\!]$.
  That is, a box modal formula $[\alpha]\phi$ is true in state ω iff postcondition ϕ is true in all states ν that are reachable by running α from ω.
- $\omega \in [\![\langle\alpha\rangle\phi]\!]$ iff $\nu \in [\![\phi]\!]$ for some ν with $(\omega, \nu) \in [\![\alpha]\!]$.
  That is, a diamond modal formula $\langle\alpha\rangle\phi$ is true in state ω iff postcondition ϕ is true in at least one state ν that is reachable by running α from ω.

If $\omega \in [\![\phi]\!]$, then we say that ϕ is true at ω or that ω is a model of ϕ. A formula ϕ is *valid*, written $\vDash \phi$, iff $\omega \in [\![\phi]\!]$ for all states ω. A formula ϕ is a *consequence* of a set of formulas Γ, written $\Gamma \vDash \phi$, iff, for each ω: ($\omega \in [\![\psi]\!]$ for all $\psi \in \Gamma$) implies that $\omega \in [\![\phi]\!]$.

A formula ϕ is called *satisfiable* iff there is a ω such that $\omega \in [\![\phi]\!]$. The formula ϕ is called *unsatisfiable* iff there is no such ω.

The formula $x > 0 \land x < 1$ is satisfiable, because all it takes for it to be true is a state ω in which, indeed, the value of x is a real number between zero and one. The formula $x > 0 \land x < 0$ is unsatisfiable, because it is kind of hard (read: impossible) to find a state which satisfies both conjuncts. The formula $x > 0 \lor x < 1$ is valid, because there is no state in which it would not be true, because, surely, x will either be positive or smaller than one.

10. CPS Contracts in Logic

Now that we know what truth and validity are, let's go back to the previous question. Is dℒ formula (13) valid? Is (14) valid? Actually, let's first ask if they are equivalent, i.e. the dℒ formula

$$(13) \leftrightarrow (14)$$

is valid. Expanding the abbreviations this is the question whether the following dℒ formula is valid:

$$\begin{aligned} &\Big([\big(x' = v, v' = -g \,\&\, x \geq 0;\ \texttt{if}(x = 0)\, v := -cv\big)^*]\, 0 \leq x \\ &\land [\big(x' = v, v' = -g \,\&\, x \geq 0;\ \texttt{if}(x = 0)\, v := -cv\big)^*]\, x \leq H\Big) \\ \leftrightarrow\ & [\big(x' = v, v' = -g \,\&\, x \geq 0;\ \texttt{if}(x = 0)\, v := -cv\big)^*]\, (0 \leq x \land x \leq H) \end{aligned} \tag{15}$$

Exercise 1 gives you an opportunity to convince yourself that the equivalence (13) $\leftrightarrow$ (14) is indeed valid.[4] So if (13) is valid, then so should (14) be (Exercise 2). But is (13) even valid?

[4]This equivalence also foreshadows the fact that CPS provide ample opportunity for questions how multiple system models relate. The dℒ formula (15) relates three different properties of three occurrences of one and the same hybrid program, for example. Over the course of the semester, the need to relate different properties of different CPS will arise more and more even if it may lie dormant for the moment. You are advised to already take notice that this is possible, because dℒ can form any arbitrary combination and nesting of all its logical operators.

Certainly, (13) is not true in a state ω where $\omega(x) < 0$, because from that initial state, no repetitions of the loop (which is allowed by nondeterministic repetition, Exercise 4), will lead to a state $\nu \overset{\text{def}}{=} \omega$ in which $\nu \notin [\![0 \leq x]\!]$. Thus, (13) only has a chance of being valid in initial states that satisfy further assumptions, including $0 \leq x$ and $x \leq H$. In fact, that is what the preconditions were meant for in Sect. 5. How can we express a precondition contract in a d$\mathcal{L}$ formula?

Preconditions serve a very different role than postconditions do. Postconditions of HP α are what we want to hold true after every run of α. The meaning of a postcondition is what is rather difficult to express in first-order logic (to say the least). That is what d$\mathcal{L}$ has modalities for. Do we also need any extra logical operator to express preconditions?

The meaning of a precondition `@requires`(A) of a HP α is that it is assumed to hold before the HP starts. *If* A holds when the HP starts, then its postcondition `@ensures`(B) holds after all runs of HP α. What if A does not hold when the HP starts?

If precondition A does not hold initially, then all bets are off, because the person who started the HP did not obey its requirements, which says that it should only be run if its preconditions are met. The CPS contract `@requires`(A) `@ensures`(B) for a HP α promises that B will always hold after running α if A was true initially when α started. Thus, the meaning of a precondition can be expressed easily using an implication

$$A \rightarrow [\alpha]B \tag{16}$$

because an implication is valid if, in every state in which the left-hand side is true, the right-hand side is also true. The implication (16) is valid ($\vDash A \rightarrow [\alpha]B$), if, indeed, for every state ω in which precondition A holds ($\omega \in [\![A]\!]$), it is the case that all runs of HP α lead to states ν (with $(\omega, \nu) \in [\![\alpha]\!]$) in which postcondition B holds ($\nu \in [\![B]\!]$). The d$\mathcal{L}$ formula (16) does not say what happens in states ω in which the precondition A does not hold ($\omega \notin [\![A]\!]$).

How does formula (16) talk about the runs of a HP and postcondition B again? Recall that the d$\mathcal{L}$ formula $[\alpha]B$ is true in exactly those states in which all runs of HP α lead only to states in which postcondition B is true. The implication in (16), thus, ensures that this holds in all (initial) states that satisfy precondition A.

Note 6 (Contracts to d$\mathcal{L}$ Formulas)**.** *Consider a HP α with a CPS contract using a single* `@requires`(A) *precondition and a single* `@ensures`(B) *postcondition:*

$$\begin{array}{l}\texttt{@requires}(A)\\ \texttt{@ensures}(B)\\ \alpha\end{array}$$

This CPS contract can be expressed directly as a logical formula in d$\mathcal{L}$*:*

$$A \rightarrow [\alpha]B$$

CPS contracts with multiple preconditions and multiple postconditions can directly be expressed as a d$\mathcal{L}$ formula as well (Exercise 7).

Recall HP (10), which is shown here in a slightly simplified form:

$$\begin{aligned}&\texttt{@requires}(0 \leq x \land x = H)\\ &\texttt{@ensures}(0 \leq x \land x \leq H)\\ &\big(x' = v, v' = -g \,\&\, x \geq 0;\\ &\ \texttt{if}(x = 0)\, v := -cv\big)^*\end{aligned} \tag{17}$$

The d$\mathcal{L}$ formula expressing that the CPS contract for HP (17) holds is:

$$0 \leq x \land x = H \to \big[\big(x' = v, v' = -g \,\&\, x \geq 0;\ \texttt{if}(x = 0)\, v := -cv\big)^*\big]\,(0 \leq x \land x \leq H) \tag{18}$$

So to find out whether (17) satisfies its CPS contract, we ask whether the d$\mathcal{L}$ formula (18) is valid.

In order to find out whether such a formula is valid, i.e. true in all states, we need some operational way that allows us to tell whether it is valid, because mere inspection of the semantics alone is not a particularly scalable way of approaching validity question.

11. Identifying Requirements of a CPS

Before trying to prove any formulas to be valid, it is a pretty good idea to check whether all required assumptions have been found that are necessary for the formula to hold. Otherwise, the proof will fail and we need to start over after having identified the missing requirements from the failed proof attempt. So let us scrutinize d$\mathcal{L}$ formula (18) and ponder whether there are any circumstances under which it is not true. Even though the bouncing ball is a rather impoverished CPS (it suffers from a disparate lack of control), its immediate physical intuition still makes the ball a particularly insightful example for illustrating how critical it is to identify the right requirements. Besides, unlike for heavy duty CPS, we trust you have had ample opportunities to make yourself familiar with the behavior of bouncing balls.

Maybe the first thing to notice is that the HP mentions g, which is meant to represent the standard gravity constant, but the formula (18) never says that. Certainly, if gravity were negative ($g < 0$), bouncing balls would function rather differently in a quite astonishing way. They would suddenly be floating balls disappearing into the sky and would lose all the joy of bouncing around; see Fig. 4.

So let's modify (18) to assume $g = 9.81$:

$$0 \leq x \land x = H \land \boldsymbol{g = 9.81} \to \big[\big(x' = v, v' = -g \,\&\, x \geq 0;\ \texttt{if}(x = 0)\, v := -cv\big)^*\big]\,(0 \leq x \land x \leq H) \tag{19}$$

Let's undo unnecessarily strict requirements right away, though. What would the bouncing ball do if it were set loose on the moon instead of on Earth? Would it still fall? Things are much lighter on the moon. Yet they still fall down ultimately, which

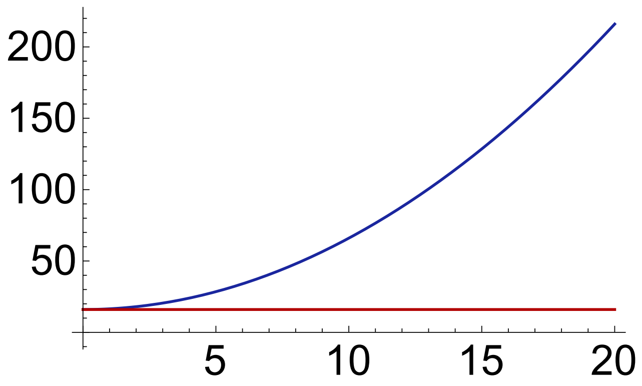


Figure 4: Sample trajectory of a bouncing ball in an anti-gravity field with $g < 0$

is again the phenomenon known as gravity, just with a different constant (1.6 on the moon and 25.9 on Jupiter). Besides, none of those constants was particularly precise. Earth's gravity is more like 9.8067. The behavior of the bouncing ball depends on the value of that parameter g. But its qualitative behavior and whether it obeys (18) does not.

Note 7 (Parameters)**.** *A common feature of CPS is that their behavior is subject to parameters, which can have quite a non-negligible impact. Yet, it is very hard to determine precise values for all parameters by measurements. When a particular concrete value for a parameter has been assumed to prove a property of a CPS, it is not clear whether that property holds for the true system, which may in reality have a slightly different parameter value.*

Instead of a numerical value for a parameter, our analysis can proceed just fine by treating the parameter as a symbolic parameter, *i.e. a variable such as g, which is not assumed to hold a specific numerical value like 9.81. Instead, we would only assume certain constraints about the parameter, say $g > 1$ without choosing a specific value. If we then analyze the CPS with this symbolic parameter g, all analysis results will continue to hold for any concrete choice of g respecting its constraints (here $g > 1$). That results in a stronger statement about the system, which is less fragile as it does not break down just because the true g is ≈ 9.8067 rather than the previously assumed $g = 9.81$. Often times, those more general statements with symbolic parameters can even be easier to prove than statements about systems with specific magic numbers chosen for their parameters.*

In light of these thoughts, we could assume $9 < g < 10$ to be the gravity constant for Earth. Yet, we can also just consider all bouncing balls on all planets in the solar system or elsewhere at once by assuming only $g > 0$ instead of $g = 9.81$ as in (19), since this is the only aspect of gravity that the usual behavior of a bouncing ball depends on:

$$0 \leq x \wedge x = H \wedge \boldsymbol{g > 0} \rightarrow \left[\left(x' = v, v' = -g \,\&\, x \geq 0;\ \mathtt{if}(x = 0)\, v := -cv\right)^*\right] (0 \leq x \wedge x \leq H) \tag{20}$$

Do we expect dℒ formula (20) to be valid, i.e. true in all states? What could go wrong? The insight from modifying (18) to (19) and finally to (20) started with the observation that (18) did not include any assumptions about g. It is worth noting that (20) also does not assume anything about c. Bouncing balls clearly would not work as expected if $c > 1$, because such anti-damping would cause the bouncing ball to jump back up higher and higher and higher and ultimately as high up as the moon, clearly falsifying (20); see Fig. 5.

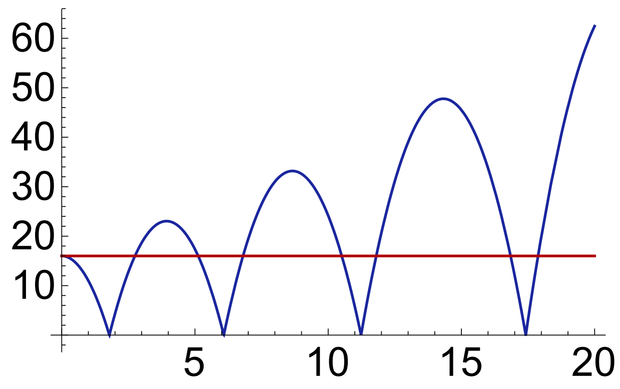


Figure 5: Sample trajectory of a bouncing ball with anti-damping $c > 1$

Consequently, (20) only has a chance of being true when assuming that c is not too big:

$$0 \leq x \wedge x = H \wedge g > 0 \wedge \mathbf{1} > \boldsymbol{c} \geq \mathbf{0} \rightarrow \\ \left[\left(x' = v, v' = -g \,\&\, x \geq 0;\ \mathtt{if}(x = 0)\, v := -cv\right)^*\right] (0 \leq x \wedge x \leq H) \quad (21)$$

Is (21) valid now? Or does its truth depend on more assumptions that have not been identified yet? Now, all parameters (H, g, c) have some assumptions in (21). Is there some requirement we forgot about? Or did we find them all?

Before you read on, see if you can find the answer for yourself.

What about variable v? Why is there no assumption about it yet? Should there be one? Velocity v changes over time. What is its initial value allowed to be? What could go wrong?

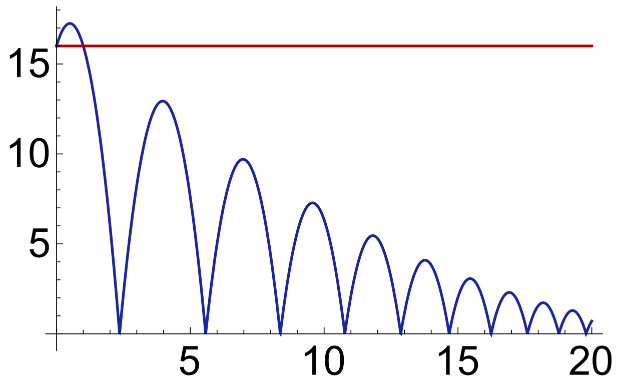


Figure 6: Sample trajectory of a bouncing ball climbing with upwards initial velocity $v > 0$

Indeed, the initial velocity v of the bouncing ball could be positive ($v > 0$), which would make the bouncing ball climb initially, clearly exceeding its initial height H; see Fig. 6. This would correspond to the bouncing ball being thrown high up in the air in the beginning, so that its initial velocity v is upwards from its initial height $x = H$. Consequently, (21) has to be modified to assume $v \leq 0$ holds initially:

$$0 \leq x \wedge x = H \wedge \boldsymbol{v \leq 0} \wedge g > 0 \wedge 1 > c \geq 0 \rightarrow \\ \big[\big(x' = v, v' = -g \,\&\, x \geq 0;\ \texttt{if}(x = 0)\, v := -cv\big)^*\big]\,(0 \leq x \wedge x \leq H) \tag{22}$$

Now there's finally assumptions about all parameters and variables of (22). That does not mean that we found the right assumptions, yet, but is still a good sanity check. Before wasting cycles on trying to prove or otherwise justify (22), let's try once more whether we can find an initial state ω that satisfies all assumptions $v \leq 0 \wedge 0 \leq x \wedge x = H \wedge g > 0 \wedge 1 > c \geq 0$ in the antecedent (i.e. left-hand side of the implication) of (22) so that ω does not satisfy the succedent (i.e. right-hand side of implication) of (22). Such an initial state ω falsifies (22) and would, thus, represent a *counterexample*.

Is there still a counterexample to (22)? Or have we successfully identified all assumptions so that it is now valid?

Before you read on, see if you can find the answer for yourself.

Formula (22) still has a problem. Even if the initial state satisfies all requirements in the antecedent of (22), the bouncing ball might still jump higher than it ought to, i.e. higher than its initial height H. That happens if the bouncing ball has a very big downwards velocity, so if v is a lot smaller than 0 (sometimes written $v \ll 0$). If v is a little smaller than 0, then the damping c will eat up enough the ball's kinetic energy so that it cannot jump back up higher than it was initially (H). But if v is a lot smaller than 0, then it starts falling down with so much kinetic energy that the damping on the ground does not slow it down enough, so the ball will come bouncing back higher than it was originally like when dribbling a basket ball; see Fig. 7. Under which circumstance this happens depends on the relationship of the initial velocity and height to the damping coefficient.

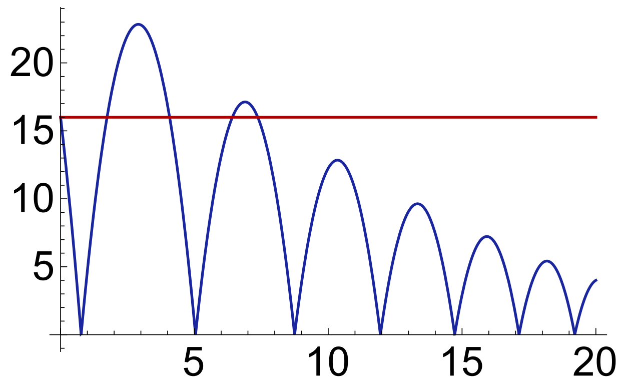


Figure 7: Sample trajectory of a bouncing ball dribbling with fast initial velocity $v < 0$

We could explore this relationship in more detail. But it is actually easier to infer this relationship by conducting a proof. So we modify (22) to simply assume $v = 0$ initially:

$$0 \le x \land x = H \land \boldsymbol{v = 0} \land g > 0 \land 1 > c \ge 0 \rightarrow \\ \left[\left(x' = v, v' = -g \,\&\, x \ge 0;\ \mathtt{if}(x = 0)\, v := -cv\right)^*\right] (0 \le x \land x \le H) \qquad (23)$$

Is d$\mathcal{L}$ formula (23) valid now? Or does it still have a counterexample?
Before you read on, see if you can find the answer for yourself.

It seems like all required assumptions have been identified to make the d$\mathcal{L}$ formula (23) valid so that the bouncing ball described in (23) satisfies the postcondition $0 \le x \le H$. But after so many failed starts and missing assumptions and requirements for the bouncing ball, it is a good idea to prove (23) once and for all beyond any doubt.

In order to be able to prove d$\mathcal{L}$ formula (23), however, we need to investigate how proving works. How can d$\mathcal{L}$ formulas be proved? And, since first-order formulas are d$\mathcal{L}$ formulas as well, one part of the question will be: how can first-order formulas be proved? How can real arithmetic be proved? How can requirements for the safety of CPS be identified systematically? All these questions will be answered in this course, but not all of them in this lecture.

In order to make sure we only need to worry about a minimal set of operators of d$\mathcal{L}$ for proving purposes, let's simplify (23) by getting rid of `if-then-else` (Exercise 13):

$$0 \le x \land x = H \land v = 0 \land g > 0 \land 1 > c \ge 0 \rightarrow \\ \big[\big(x' = v, v' = -g \,\&\, x \ge 0;\ (?x = 0; v := -cv \cup ?x \ne 0)\big)^*\big]\,(0 \le x \land x \le H) \quad (24)$$

Observing the non-negligible difference between the original conjecture (19) and the revised and improved conjecture (24), leads us to often adopt the following principle.

Note 8 (Principle of Cartesian Doubt). *In 1641, René Descartes suggested an attitude of systematic doubt where he would be skeptical about the truth of all beliefs until he found reason that they were justified. This principle is now known as* Cartesian Doubt *or skepticism.*

We will have perfect justifications: proofs. But until we have found proof, it is often helpful to adopt the principle of Cartesian Doubt in a very weak and pragmatic form. Before setting out on the journey to prove a conjecture, we first scrutinize it to see if we can find a counterexample that would make it false. For such a counterexample will not only save us a lot of misguided effort in trying to prove a false conjecture, but also helps us identify missing assumptions in conjectures and justifies the assumptions to be necessary. Surely, if, without assumption A, a counterexample to a conjecture exists, then A must be rather necessary.

12. Summary

This lecture introduced differential dynamic logic (d$\mathcal{L}$), whose operators and their informal meaning is summarized in Note 9.

The appendix of this lecture also features first reasoning aspects for CPS. But reasoning for CPS will be investigated systematically in subsequent lectures, one operator at a time, which establishes separate reasoning principles for each operator, rather than proceeding in the more ad-hoc style of this lecture along an example. For future lectures, we should keep the bouncing ball example and its surprising subtleties in mind, though.

Note 9 (Operators and (informal) meaning in differential dynamic logic (d$\mathcal{L}$)).

d$\mathcal{L}$	*Operator*	*Meaning*
$\theta = \eta$	*equals*	*true iff values of θ and η are equal*
$\theta \geq \eta$	*equals*	*true iff value of θ greater-or-equal to η*
$\neg\phi$	*negation / not*	*true if ϕ is false*
$\phi \land \psi$	*conjunction / and*	*true if both ϕ and ψ are true*
$\phi \lor \psi$	*disjunction / or*	*true if ϕ is true or if ψ is true*
$\phi \rightarrow \psi$	*implication / implies*	*true if ϕ is false or ψ is true*
$\phi \leftrightarrow \psi$	*bi-implication / equivalent*	*true if ϕ and ψ are both true or both false*
$\forall x\, \phi$	*universal quantifier / for all*	*true if ϕ is true for all values of variable x*
$\exists x\, \phi$	*existential quantifier / exists*	*true if ϕ is true for some values of variable x*
$[\alpha]\phi$	*$[\cdot]$ modality / box*	*true if ϕ is true after all runs of HP α*
$\langle\alpha\rangle\phi$	*$\langle\cdot\rangle$ modality / diamond*	*true if ϕ is true after at least one run of HP α*

A. Intermediate Conditions for CPS

These appendices begin a semiformal study of the bouncing ball, which is a useful preparation for the next lecture.

Before proceeding any further with ways of proving d$\mathcal{L}$ formulas, let's simplify (24) grotesquely by removing the loop:

$$0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 > c \geq 0 \rightarrow \\ \big[x' = v, v' = -g \,\&\, x \geq 0; (?x = 0; v := -cv \cup ?x \neq 0)\big]\,(0 \leq x \land x \leq H) \quad (25)$$

Removing the loop clearly changes the behavior of the bouncing ball. It no longer bounces particularly well. All it can do now is fall and, if it reaches the floor, have its velocity reverted without actually climbing back up. So if we manage to prove (25), we certainly have not shown the actual d$\mathcal{L}$ formula (24). But it's a start, because the behavior modeled in (25) is a part of the behavior of (24). So it is useful (and easier) to understand (25) first.

The d$\mathcal{L}$ formula (25) has a number of assumptions $0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 > c \geq 0$ that can be used during the proof. It claims that the postcondition $0 \leq x \land x \leq H$ holds after all runs of the HP in the $[\cdot]$ modality. The top-level operator in the modality of (25) is a sequential composition (;), for which we need to find a proof argument.[5]

The HP in (25) follows a differential equation first and then, after the sequential composition (;), proceeds to run a discrete program $(?x = 0; v := -cv \cup ?x \neq 0)$. Depending on how long the HP follows its differential equation, the intermediate state after the differential equation and before the discrete program will be rather different.

[5] The way we proceed here to prove (25) is actually not the recommended way. Later on, we will see a much easier way. But it is instructive to understand the more verbose approach we take first. This also prepares us for the challenges that lie ahead when proving properties of loops.

Note 10 (Intermediate states of sequential compositions)**.** *This phenomenon happens in general for sequential compositions $\alpha;\beta$. The first HP α may reach a whole range of states, which represent intermediate states for the sequential composition $\alpha;\beta$, i.e. states that are final states for α and initial states for β. The intermediate states of $\alpha;\beta$ are the states μ in the semantics $[\![\alpha;\beta]\!]$ from Lecture 3:*

$$[\![\alpha;\beta]\!] = [\![\beta]\!] \circ [\![\alpha]\!] = \{(\omega,\nu) : (\omega,\mu) \in [\![\alpha]\!], (\mu,\nu) \in [\![\beta]\!]\}$$

Can we find a way of summarizing what all intermediate states between the differential equation and the discrete program of (25) have in common? They differ by how long the CPS has followed the differential equation.

If the system has followed the differential equation of (25) for time t, then the resulting velocity $v(t)$ at time t and height $x(t)$ at time t will be

$$v(t) = -gt, x(t) = H - \frac{g}{2}t^2 \tag{26}$$

This answer can be found by integrating or solving the differential equations. This knowledge (26) is useful but it is not (directly) clear how to use it to describe what all intermediate states have in common, because the time t in (26) is not available as a variable in the HP (25).[6] Can the intermediate states be described by a relation of the variables that (unlike t) are actually in the system? That is, an (arithmetic) formula relating x, v, g, H?

Before you read on, see if you can find the answer for yourself.

[6] Following these thoughts a bit further reveals how (26) can actually be used perfectly well to describe intermediate states when changing the HP (25) a little bit. But working with solutions is still not the way that gets us to the goal the quickest, usually.

One way of producing a relation from (26) is to get the units aligned and get rid of time t. Time drops out of the "equation" when squaring the identity for velocity:

$$v(t)^2 = g^2t^2, \quad x(t) = H - \frac{g}{2}t^2$$

and multiplying the identity for position by $2g$:

$$v(t)^2 = g^2t^2, \quad 2gx(t) = 2gH - 2\frac{g^2}{2}t^2$$

Then substituting the first equation into the second yields

$$2gx(t) = 2gH - v(t)^2$$

This equation does not depend on time t, so we expect it to hold after all runs of the differential equation irrespective of t:

$$2gx = 2gH - v^2 \tag{27}$$

We conjecture the intermediate condition (27) to hold in the intermediate state of the sequential composition in (25). In order to prove (25) we can decompose our reasoning into two parts. The first part will prove that the intermediate condition (27) holds after all runs of the first differential equation. The second part will assume (27) to hold and prove that all runs of the discrete program in (25) from any state satisfying (27) satisfy the postcondition $0 \le x \land x \le H$.

Note 11 (Intermediate conditions as contracts for sequential composition)**.** *For a HP that is a sequential composition $\alpha;\beta$ an* intermediate condition *is a formula that characterizes the intermediate states in between HP α and β. That is, for a* d$\mathcal{L}$ *formula*

$$A \to [\alpha;\beta]B$$

an intermediate condition is a formula E such that the following d$\mathcal{L}$ *formulas are valid:*

$$A \to [\alpha]E \qquad \textit{and} \qquad E \to [\beta]B$$

The first d$\mathcal{L}$ *formula expresses that intermediate condition E characterizes the intermediate states accurately, i.e. E actually holds after all runs of HP α from states satisfying A. The second* d$\mathcal{L}$ *formula says that the intermediate condition E characterizes intermediate states well enough, i.e. E is all we need to know about a state to conclude that all runs of β end up in B. That is, from all states satisfying E (in particular from those that result by running α from a state satisfying A), B holds after all runs of β.*

For proving (25), we conjecture that (27) is an intermediate condition, which requires us to prove the following two d$\mathcal{L}$ formulas:

$$\begin{aligned} 0 \le x \land x = H \land v = 0 \land g > 0 \land 1 > c \ge 0 \to [x' = v, v' = -g \,\&\, x \ge 0]2gx = 2gH - v^2 \\ 2gx = 2gH - v^2 \to [?x = 0; v := -cv \cup ?x \ne 0]\,(0 \le x \land x \le H) \end{aligned} \tag{28}$$

Let's focus on the latter formula. Do we expect to be able to prove it? Do we expect it to be valid?

Before you read on, see if you can find the answer for yourself.

The second formula of (28) claims that $0 \leq x$ holds after all runs of $?x = 0; v := -cv \cup ?x \neq 0$ from all states that satisfy $2gx = 2gH - v^2$. That is a bit much to hope for, however, because $0 \leq hx$ is not even ensured in the precondition of this second formula. So the second formula of (28) is not valid. How can this problem be resolved? By adding $0 \leq x$ into the intermediate condition, thus, requiring us to prove:

$$\begin{aligned} 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 > c \geq 0 \to [x' = v, v' = -g \,\&\, x \geq 0](2gx = 2gH - v^2 \land x \geq 0) \\ 2gx = 2gH - v^2 \land x \geq 0 \to [?x = 0; v := -cv \cup ?x \neq 0]\,(0 \leq x \land x \leq H) \end{aligned} \tag{29}$$

Proving the first formula in (29) requires us to handle differential equations, which we will get to later. The second formula in (29) is the one whose proof is discussed first.

B. A Proof of Choice

The second formula in (29) has a nondeterministic choice ($\cup$) as the top-level operator in its $[\cdot]$ modality. How can we prove a formula of the form

$$A \to [\alpha \cup \beta]B \tag{30}$$

Recalling its semantics from Lecture 3,

$$[\![\alpha \cup \beta]\!] = [\![\alpha]\!] \cup [\![\beta]\!]$$

HP $\alpha \cup \beta$ has two possible behaviors. It could run as HP α does or as β does. And it is chosen nondeterministically which of the two behaviors happens. Since the behavior of $\alpha \cup \beta$ could be either α or β, proving (30) requires proving B to hold after α and after β. More precisely, (30) assumes A to hold initially, otherwise (30) is vacuously true. Thus, proving (30) allows us to assume A and requires us to prove that B holds after all runs of α (which is permitted behavior for $\alpha \cup \beta$) and to prove that, assuming A holds initially, that B holds after all runs of β (which is also permitted behavior of $\alpha \cup \beta$).

Note 12 (Proving choices). *For a HP that is a nondeterministic choice $\alpha \cup \beta$, we can prove*

$$A \to [\alpha \cup \beta]B$$

by proving the following d$\mathcal{L}$ *formulas:*

$$A \to [\alpha]B \qquad \text{and} \qquad A \to [\beta]B$$

Using these thoughts on the second formula of (29), we could prove that formula if we would manage to prove both of the following d$\mathcal{L}$ formulas:

$$\begin{aligned} 2gx = 2gH - v^2 \land x \geq 0 \to [?x = 0; v := -cv]\,(0 \leq x \land x \leq H) \\ 2gx = 2gH - v^2 \land x \geq 0 \to [?x \neq 0]\,(0 \leq x \land x \leq H) \end{aligned} \tag{31}$$

C. Proofs of Tests

Consider the second formula of (31). Proving it requires us to understand how to handle a test $?Q$ in a modality $[?Q]$. The semantics of a test $?Q$ from Lecture 3 on Choice & Control

$$[\![?Q]\!] = \{(\omega, \omega) \; : \; \omega \in [\![Q]\!]\} \tag{32}$$

says that a test $?Q$ completes successfully without changing the state in any state ω in which Q holds (i.e. $\omega \in [\![Q]\!]$) and fails to run in all other states (i.e. where $\omega \notin [\![Q]\!]$). How can we prove a formula with a test:

$$A \rightarrow [?Q]B \tag{33}$$

This formula expresses that from all initial states satisfying A all runs of $?Q$ reach states satisfying B. When is there a run of $?Q$ at all? There is a run from state ω if and only if Q holds in ω. So the only cases to worry about those initial states that satisfy Q as, otherwise, the HP in (33) cannot execute at all by fails miserably so that the run is discarded. Hence, we get to assume Q holds, as the HP $?Q$ does not otherwise execute. In all states that the HP $?Q$ reaches from states satisfying A, (33) conjectures that B holds. Now, by (32), the final states that $?Q$ reaches are the same as the initial state (as long as they satisfy Q so that HP $?Q$ can be executed at all). That is, postcondition B needs to hold in all states from which $?Q$ runs (i.e. that satisfy Q) and that satisfy the precondition A. So (33) can be proved by proving

$$A \land Q \rightarrow B$$

Note 13 (Proving tests)**.** *For a HP that is a test $?Q$, we can prove*

$$A \rightarrow [?Q]B$$

by proving the following d$\mathcal{L}$ *formula:*

$$A \land Q \rightarrow B$$

Using this for the second formula of (31), Note 13 reduces proving the second formula of (31)

$$2gx = 2gH - v^2 \land x \geq 0 \rightarrow [?x \neq 0]\,(0 \leq x \land x \leq H)$$

to proving

$$2gx = 2gH - v^2 \land x \geq 0 \land x \neq 0 \rightarrow 0 \leq x \land x \leq H \tag{34}$$

Now we are left with arithmetic that we need to prove. Proofs for arithmetic and propositional logical operators such as $\land$ and $\rightarrow$ will be considered in a later lecture. For now, we notice that the formula $0 \leq x$ in the right-hand side of $\rightarrow$ is justified by assumption $x \geq 0$ if we flip the inequality around. And that $x \leq H$ does not exactly have a justification in (34), because we lost the assumptions about H somewhere.

How could that happen? We used to know $x \leq H$ in (25). We also still knew about it in the first formula of (29). But we let it disappear from the second formula of (29), because we chose an intermediate condition that was too weak when constructing (29).

This is a common problem in trying to prove properties of CPS or of any other mathematical statements. One of our intermediate steps might have been too weak, so that our attempt of proving it fails and we need to revisit how we got there. For sequential compositions, this is actually a nonissue as soon as we move on (in the next lecture) to a proof technique that is more useful than the intermediate conditions from Note 11. But similar difficulties can arise in other parts of proof attempts.

In this case, the fact that we lost $x \leq H$ can be fixed by including it in the intermediate conditions, because it can be shown to hold after the differential equation. Other crucial assumptions have also suddenly disappeared in our reasoning. An extra assumption $1 > c \geq 0$, for example, is crucially needed to justify the first formula of (31). It is easier to see why that particular assumption can be added to the intermediate contract without changing the argument much. The reason is that c never ever changes during the system run.

Note 14. *It is very difficult to come up with bug-free code. Just thinking about your assumptions really hard does not ensure correctness, but we can gain confidence that our system does what we want it to by proving that certain properties are satisfied.*

Changing the assumptions and arguments in a hybrid program around during the search for a proof of safety is something that happens frequently. It is easy to make subtle mistakes in informal arguments such as "I need to know C here and I would know C if I had included it here or there, so now I hope the argument holds". This is one of many reasons why we are better off if our CPS proofs are rigorous, because we would rather not end up in trouble because of a subtle flaw in a correctness argument. A rigorous, formal proof calculus for differential dynamic logic (d$\mathcal{L}$) will help us avoid the pitfalls of informal arguments. The theorem prover KeYmaera X that you will use in this course implements a proof calculus for d$\mathcal{L}$.

A related observation from our informal arguments in this lecture is that we desperately need a way to keep an argument consistent as a single argument justifying one conjecture. Quite the contrary to the informal loose threads of argumentation we have pursued in this lecture for the sake of developing an intuition. Consequently, we will investigate what holds all arguments together and what constitutes an actual proof in subsequent lectures. A proof in which the relationship of premises to conclusions via proof steps is rigorous.

Moreover, there's two loose ends in our arguments. For one, the differential equation in (29) is still waiting for an argument that could help us prove it. Also, the assignment in (31) still needs to be handled and its sequential composition needs an intermediate contract (Exercise 15). Both will be pursued in the next lecture, where we move to a systematic and rigorous reasoning style for CPS.

Exercises

Exercise 1. Show that (15) is valid. It is okay to focus only on this case, even though the argument is more general, because the following d$\mathcal{L}$ formula is valid for any hybrid program α:

$$[\alpha]F \land [\alpha]G \leftrightarrow [\alpha](F \land G)$$

Exercise 2. Let A, B be d$\mathcal{L}$ formulas. Suppose $A \leftrightarrow B$ is valid and A is valid. Is B valid? Prove or disprove.

Exercise 3. Let A, B be d$\mathcal{L}$ formulas. Suppose $A \leftrightarrow B$ is true in state ω and A is true in state ω. That is, $\omega \in [\![A \leftrightarrow B]\!]$ and $\omega \in [\![A]\!]$. Is B true in state ω? Prove or disprove. Is B valid? Prove or disprove.

Exercise 4. Let α be an HP. Let ω be a state with $\omega \notin [\![\phi]\!]$. Does $\omega \notin [\![[\alpha^*]\phi]\!]$ hold? Prove or disprove.

Exercise 5. Let α be an HP. Let ω be a state with $\omega \in [\![\phi]\!]$. Does $\omega \in [\![[\alpha^*]\phi]\!]$ hold? Prove or disprove.

Exercise 6. Let α be an HP. Let ω be a state with $\omega \in [\![\phi]\!]$. Does $\omega \in [\![\langle\alpha^*\rangle\phi]\!]$ hold? Prove or disprove.

Exercise 7. Suppose you have a HP α with a CPS contract using multiple preconditions $A_1, \dots, A_n$ and multiple postconditions $B_1, \dots, B_m$:

$$\begin{array}{l}
\texttt{@requires}(A_1) \\
\texttt{@requires}(A_2) \\
\vdots \\
\texttt{@requires}(A_n) \\
\texttt{@ensures}(B_1) \\
\texttt{@ensures}(B_2) \\
\vdots \\
\texttt{@ensures}(B_m) \\
\alpha
\end{array}$$

How can this CPS contract be expressed in a d$\mathcal{L}$ formula? If there are multiple alternatives on how to express it, discuss the advantages and disadvantages of each option.

Exercise 8. For each of the following d$\mathcal{L}$ formulas, determine if they are valid, satisfiable, *and/or* unsatisfiable:

1. $[?x \geq 0]x \geq 0$.
2. $[?x \geq 0]x \leq 0$.
3. $[?x \geq 0]x < 0$.

4. $[?true]true$.
5. $[?true]false$.
6. $[?false]true$.
7. $[?false]false$.
8. $[x' = 1 \,\&\, true]true$.
9. $[x' = 1 \,\&\, true]false$.
10. $[x' = 1 \,\&\, false]true$.
11. $[x' = 1 \,\&\, false]false$.
12. $[(x' = 1 \,\&\, true)^*]true$.
13. $[(x' = 1 \,\&\, true)^*]false$.
14. $[(x' = 1 \,\&\, false)^*]true$.
15. $[(x' = 1 \,\&\, false)^*]false$.

Exercise 9. For each of the following d$\mathcal{L}$ formulas, determine if they are valid, satisfiable, *and/or* unsatisfiable:

1. $x > 0 \rightarrow [x' = 1]x > 0$
2. $x > 0 \land \langle x' = 1\rangle x < 0$
3. $x > 0 \rightarrow [x' = -1]x < 0$
4. $x > 0 \rightarrow [x' = -1]x \geq 0$
5. $x > 0 \rightarrow [(x := x + 1)^*]x > 0$
6. $x > 0 \rightarrow [(x := x + 1)^*]x > 1$
7. $[x := x^2 + 1; x' = 1]x > 0$.
8. $[(x := x^2 + 1; x' = 1)^*]x > 0$.
9. $[(x := x + 1; x' = -1)^*; ?x > 0; x' = 2]x > 0$

Exercise 10. For each $j, k \in \{\text{satisfiable}, \text{unsatisfiable}, \text{valid}\}$ answer whether there is a formula that is j but not k. Also answer for each such j, k whether there is a formula that is j but its negation is not k. Briefly justify each answer.

Exercise 11. There are at least two styles if giving a meaning to a logical formula. One way is, as in Def. 2, to inductively define a satisfaction relation $\models$ that holds between a state ω and a d$\mathcal{L}$ formula ϕ, written $\omega \models \phi$, whenever the formula ϕ is true in the state ω. Its definition will include, among other cases, the following:

$$\begin{array}{lll}
\omega \models P \land Q & \text{iff} & \omega \models P \text{ and } \omega \models Q \\
\omega \models \langle\alpha\rangle P & \text{iff} & \nu \models P \text{ for some state } \nu \text{ such that } (\omega,\nu) \in [\![\alpha]\!] \\
\omega \models [\alpha]P & \text{iff} & \nu \models P \text{ for all states } \nu \text{ such that } (\omega,\nu) \in [\![\alpha]\!]
\end{array}$$

The other way is to inductively define, for each d$\mathcal{L}$ formula ϕ, the set of states, written $[\![\phi]\!]$, in which ϕ is true. Its definition will include, among other cases, the following:

$$\begin{array}{lll}
[\![e \geq \tilde{e}]\!] & = & \{\omega : [\![e]\!]\omega \geq [\![\tilde{e}]\!]\omega\} \\
[\![P \land Q]\!] & = & [\![P]\!] \cap [\![Q]\!] \\
[\![\neg P]\!] & = & [\![P]\!]^{\complement} = \mathcal{S} \setminus [\![P]\!] \\
[\![\langle\alpha\rangle P]\!] & = & [\![\alpha]\!] \circ [\![P]\!] = \{\omega : \nu \in [\![P]\!] \text{ for some state } \nu \text{ such that } (\omega,\nu) \in [\![\alpha]\!]\} \\
[\![[\alpha]P]\!] & = & [\![\neg[\alpha]\neg P]\!] = \{\omega : \nu \in [\![P]\!] \text{ for all states } \nu \text{ such that } (\omega,\nu) \in [\![\alpha]\!]\} \\
[\![\exists x\, P]\!] & = & \{\omega : \nu \in [\![P]\!] \text{ for some state } \nu \text{ that agrees with } \omega \text{ except on } x\} \\
[\![\forall x\, P]\!] & = & \{\omega : \nu \in [\![P]\!] \text{ for all states } \nu \text{ that agree with } \omega \text{ except on } x\}
\end{array}$$

Prove that both styles of defining the semantics are equivalent. That is $\omega \models \phi$ iff $\omega \in [\![\phi]\!]$ for all states ω and all d$\mathcal{L}$ formulas ϕ.

Exercise 12. What would happen with the bouncing ball if $c < 0$? Consider a variation of the arguments in Sect. 11 where instead of the assumption in (21), you assume $c < 0$. Is the formula valid? What would happen with a bouncing ball of damping $c = 1$?

Exercise 13. We went from (23) to (24) by removing an `if-then-else`. Explain how this works and justify why it is okay to do this transformation. It is okay to focus only on this case, even though the argument is more general.

Exercise 14 (**). Sect. A used a mix of a systematic and ad-hoc approach for producing an intermediate condition that was based on solving and combining differential equations. Can you think of a more systematic rephrasing?

Exercise 15 (*). Find an intermediate condition for proving the first formula in (31). The proof of the resulting formulas is complicated significantly by the fact that assignments have not yet been discussed in this lecture. Can you find a way of proving the resulting formulas before the next lecture develops how to handle assignments?

Exercise 16 (***). Before looking at subsequent lectures: How could you prove the formula (29), which involves a differential equation?

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